PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 4, Pages 1139–1143
S 0002-9939(01)06380-8
Article electronically published on November 9, 2001

THE DIFFEOMORPHISM TYPE OF CERTAIN S^3-BUNDLES OVER S^4

MARC SANCHEZ AND FREDERICK WILHELM

(Communicated by Ralph Cohen)

ABSTRACT. In this note we show that the unit tangent bundle of S^4 is diffeomorphic to the total space of a certain principal S^3-bundle over S^4, solving a problem of James and Whitehead.

For more than 50 years it has been known that the S^3-bundles over S^4 are classified by $\mathbb{Z} \oplus \mathbb{Z}$. The bundle that corresponds to $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ is obtained by gluing two copies of $\mathbb{R}^4 \times S^3$ together via the diffeomorphism $g_{m,n} : (\mathbb{R}^4 \backslash \{0\}) \times S^3 \longrightarrow (\mathbb{R}^4 \backslash \{0\}) \times S^3$ given by

$$g_{m,n}(u, v) \longrightarrow \left(\frac{u}{|u|^2}, \frac{u^m v u^n}{|u|^{n+m}} \right),$$

where we have identified $\mathbb{R}^4$ with $\mathbb{H}$ and S^3 with $\{v \in \mathbb{H} \mid |v| = 1\}$ ([Hat], [Steen]). We will call the bundle obtained from $g_{m,n}$ "the bundle of type (m, n)", and we will denote it by $E_{m,n}$.

The problem of classifying the total spaces of these bundles up to homotopy, homeomorphism, and diffeomorphism type is still open. It has led to a revolution in topology that began with Milnor's discovery that most of the bundles of type $(m, -m+1)$ are exotic spheres [Mil].

Further motivation for this problem is provided by the many interesting metrics discovered on these spaces in [GromMey], [GrovZil], [PetWil], [Wil1], and [Wil2].

In 1953 James and Whitehead gave the homotopy classification for

(total space, fiber)

pairs, except that among the bundles whose third homology group is $\mathbb{Z}/2\mathbb{Z}$ they were only able to assert that there are at most 2 homotopy types. We will complete this classification here by proving

Theorem 1. *The total spaces of $E_{1,1}$ and $E_{2,0}$ are diffeomorphic, via a diffeomorphism that takes a fiber of $E_{1,1}$ to a fiber of $E_{2,0}$.*

The complete classification of these total spaces is given independently in [CrowEsc], where it will be shown that there is no orientation preserving homotopy equivalence between these two total spaces.

Received by the editors March 20, 2000.
1991 *Mathematics Subject Classification.* Primary 53C20.
Key words and phrases. Unit tangent bundle.
This work was partially suported by the NSF.

In 1965 Sasao gave the homotopy classification of a family of CW-complexes that represents all of the homotopy types of the total spaces of S^3-bundles over S^4. It is probably possible to obtain the actual homotopy classification of the total spaces from Sasao's work, although Sasao did not do the computation.

A proof of Theorem 1 was given implicitly in [PetWil], and had been known to us for three or four years. At the time that that paper was written we were not aware of the gap in the James/Whitehead classification so neither the theorem nor its proof can be found explicitly in [PetWil]. Because of the obvious importance of the result it seemed advisable to explain the details here, where in addition we will give a new proof.

The new proof is self-contained apart from two references to well-known results, and seems more natural. Unfortunately it does not yield an explicit diffeomorphism. The old proof gives an explicit diffeomorphism, but relies on references that are less well known, and it is harder to see where the argument is going. The new proof is given is section 1 and the details of the old proof are explained in section 2.

1. A free S^3-action on the unit tangent bundle

In this section we will give the new proof of Theorem 1.

First recall that the unit tangent bundle of S^4 is the bundle of type $(1,1)$. This is shown in Theorem 9.5 on page 99 of [Huse].

Next observe that the unit tangent bundle of S^4, US^4, admits a free S^3-action. To see this, view S^4 as a subset of $\mathbb{H}\oplus\mathbb{R}$ and let S^3 act on $\mathbb{H}$ by left quaternionic multiplication, and trivially on $\mathbb{R}$. The induced action on S^4 is not free. It fixes the points $(1,0)$ and $(0,-1)$, but is otherwise free. The differential of this action gives us the desired free S^3-action on US^4. To describe the action more explicitly, think of US^4 as

$$US^4 \equiv \{ (x,v) \in \mathbb{H}\times\mathbb{H} |\, |x|=1, |v|=1, \langle x,v\rangle = 0 \}.$$

Our free S^3-action is then given by left multiplication in each factor.

What is the quotient of this action? Let U_r be the set of vectors in the unit tangent bundle whose foot points lie in the metric sphere of radius r about $(0,1)$ and let $foot : US^4 \longrightarrow S^4$ be the foot point map. The action leaves the U_r's invariant and for each r in $(0,\pi)$ the quotient of U_r is diffeomorphic to S^3. To see this fix x in $foot(U_r)$ and observe that every vector whose foot point is x is in a unique orbit and that every orbit in U_r has a vector whose foot point is x. Thus U_r/S^3 is diffeomorphic to the set of vectors whose foot point is x, that is, to S^3.

On the other hand, the S^3-action is transitive on U_0 and U_π so the quotient of each of these is a point. It follows that US^4/S^3 is a smooth manifold that is homeomorphic to the suspension of S^3, that is, to S^4. Moreover, the differential structure is the standard one. This follows from the main theorem of [Cerf] and

Proposition 1. *US^4/S^3 is a twisted 4-sphere. That is, it is diffeomorphic to a smooth manifold obtained by gluing together two copies of the four disk with a diffeomorphism of S^3.*

Proof. We will find embeddings $\iota_1, \iota_{-1} : D^4 \longrightarrow US^4/S^3$ and $C : S^3\times[-1,1] \longrightarrow US^4/S^3$ so that

$$C\left(S^3\times[-1,1]\right)\cap\iota_{-1}\left(D^4\right) = C\left(S^3\times\{-1\}\right) = \iota_{-1}\left(\partial D^4\right),$$

$$C\left(S^3 \times [-1,1]\right) \cap \iota_1\left(D^4\right) = C\left(S^3 \times \{1\}\right) = \iota_1\left(\partial D^4\right), \text{ and}$$

$$\iota_{-1}\left(\partial D^4\right) \cap \iota_1\left(\partial D^4\right) = \emptyset.$$

From this it follows that US^4/S^3 is a twisted sphere. Our proof that US^4 is homeomorphic to S^4 gave us implicitly an embedding of $S^3 \times (-2,2) \longrightarrow US^4/S^3$ whose image is the complement of the two points U_0/S^3, U_π/S^3. C is obtained by restricting this embedding to $S^3 \times [-1,1]$, after reparameterizing the interval part appropriately.

Let g_r denote the restriction of the product metric on $S^4 \times \mathbb{R}^5$ to US^4, and let $q : US^4 \longrightarrow US^4/S^3$ be the quotient map. Both $foot$ and q are Riemannian submersions with respect to g_r.

To get ι_1 choose w_0 in U_0 and let G_{w_0} be the set of q-horizontal, normal geodesics emanating from w_0. Notice that G_{w_0} gives us a map $e_{w_0} : D^4 \longrightarrow US^4$, defined via exponentiation, that is an embedding on a sufficiently small neighborhood of 0. Since the geodesics in G_{w_0} are q-horizontal, $q \circ e_{w_0}$ is also an embedding on a sufficiently small neighborhood of 0. Set ι_1 equal to the restriction of $q \circ e_{w_0}$ to this neighborhood. To see that the images of ι_1 and C intersect in the appropriate way notice that U_0 is a fiber for $foot$ as well as for q. Thus the geodesics in G_{w_0} are $foot$-horizontal, and $foot \circ e_{w_0}$ is an embedding of a neighborhood of 0 that takes the ball of radius r about 0 to the ball of radius r about $(1,0)$. From this it follows that the images of ι_1 and C intersect in the desired manner, provided the "interval part" of C is adjusted in the appropriate way.

The map ι_{-1} is defined in the same way as ι_1 with the role of $(1,0)$ being played by $(-1,0)$. □

We now know that in addition to being the bundle of type $(1,1)$, the total space of the unit tangent bundle is a principal S^3-bundle over S^4; that is, a bundle of type $(m,0)$ or a bundle of type $(0,m)$. Among these bundles only four have the same homology as the bundle of type $(1,1)$, namely the bundles of type $(2,0)$, $(-2,0)$, $(0,2)$, and $(0,-2)$ ([PetWil], Proposition 8.2). Its not hard to see that the total spaces of these four principal bundles are mutually diffeomorphic, and hence that they are all diffeomorphic to the bundle of type $(1,1)$. To see this note that the gluing maps satisfy $g_{-2,0} = g_{2,0}^{-1}$ and $g_{0,-2} = g_{0,2}^{-1}$, so $E_{2,0} \cong E_{-2,0}$ and $E_{0,2} \cong E_{0,-2}$. Finally notice that the map $C : \mathbb{R}^4 \times S^3 \longrightarrow \mathbb{R}^4 \times S^3$ that is given by $C(u,v) = (\overline{u}, \overline{v})$ satisfies $C \circ g_{2,0} = g_{0,2} \circ C$. So $E_{2,0} \cong E_{0,2}$. Notice that the diffeomorphisms between the total spaces of the principal bundles take fibers to fibers.

Thus $q : US^4 \longrightarrow US^4/S^3$ is one of the bundles $E_{2,0}, E_{-2,0}, E_{0,2}$, or $E_{0,-2}$, and the identity map of US^4 is therefore a diffeomorphism from the total space of $E_{1,1}$ to the total space of $q : US^4 \longrightarrow US^4/S^3$. Since U_0 is a fiber of both bundles, our diffeomorphism takes a fiber to a fiber.

2. An explicit diffeomorphism between $E_{2,0}$ and $E_{1,1}$

Let $Sp(2)$ denote the group of 2×2-unitary matrices with quaternion entries, that is, those that satisfy $QQ^* = Q^*Q = id$, where Q^* is conjugate transpose. S^3 acts freely on $Sp(2)$ via left multiplication by the matrices

$$\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \; q \in S^3.$$

The quotient, $E_{2,0}$, admits two different submersions onto S^4, $p_{2,0}$ and $p_{1,1}$. It turns out that these are the S^3-bundles over S^4 of types $(2,0)$ and $(1,1)$ respectively. $p_{2,0}$ is the map given by

$$p_{2,0} : \text{orbit} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \widetilde{h} \begin{pmatrix} b \\ d \end{pmatrix},$$

where $\widetilde{h} : S^7 \longrightarrow S^4$ is the Hopf fibration given by left quaternionic multiplication. To see that $p_{2,0}$ is a principal S^3-bundle, just observe that the S^3-action on $Sp(2)$ given by right multiplication by

$$\begin{pmatrix} \overline{q} & 0 \\ 0 & 1 \end{pmatrix}$$

commutes with the left action above, so they combine to give an $S^3 \times S^3$-action (cf. [GromMey]). The $S^3 \times S^3$-action is free, so the right action induces a free S^3-action on $E_{2,0} = Sp(2)/S^3$. This action on $E_{2,0}$ clearly leaves the fibers of $p_{2,0}$ invariant, so $E_{2,0}$ is a principal S^3-bundle over S^4. It was shown in [Rig, p. 192] that it is in fact the bundle of type $(2,0)$.

The map $p_{1,1}$ is given by

$$p_{1,1} : \text{orbit} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \widetilde{h} \begin{pmatrix} c & d \end{pmatrix}.$$

It was shown in Proposition 8.1 on page 362 of [PetWil] that this is the bundle of type $(1,1)$.

There is also a Lie Groups proof that $(E_{2,0}, p_{1,1})$ is US^4. To see this recall that $SO(5)$ acts transitively on US^4 by differentiating the $SO(5)$-action on S^4. The isotropy of a unit tangent vector is $SO(3)$, so $US^4 = SO(5)/SO(3)$. Note that the embedding of $SO(3)$ in $SO(5)$ is just the composition of the standard embeddings $SO(3) \hookrightarrow SO(4) \hookrightarrow SO(5)$. Taking the double covers of $SO(3)$ and $SO(5)$ we get that $US^4 = Spin(5)/Spin(3)$, and the embedding of $Spin(3)$ in $Spin(5)$ is standard. Now recall that $Sp(2)$ and $Spin(5)$ are isomorphic Lie groups, and also that S^3 and $Spin(3)$ are isomorphic. It follows that there is *some* embedding of S^3 in $Sp(2)$ so that $Sp(2)/S^3$ is the unit tangent bundle.

To see that this embedding is the one we have chosen view S^7 as the set of 2×1 quaternion matrices of unit length, and note that $Sp(2)$ acts by left multiplication on S^7. It was shown in Proposition 4.1 on page 183 of [GluWarZil] that this left action of $Sp(2)$ is a group of symmetries of the Hopf fibration $h : S^7 \longrightarrow S^4$ that is given by *right* quaternionic multiplication. They also showed that the induced action on S^4 is the standard Z_2-ineffective $Spin(5)$-action. (Also giving a very nice proof that $Sp(2)$ and $Spin(5)$ are isomorphic.)

Thus to check that our embedding of S^3 in $Sp(2)$ is the standard one, we just need to check that the action that S^3 induces on S^4 via h is standard. This is easy to check using the explicit formula for h in [Wil1]. Namely

$$h \begin{pmatrix} a \\ c \end{pmatrix} = \left(a\bar{c}, \frac{1}{2} \left(|a|^2 - |c|^2 \right) \right).$$

Thus the identity map of $Sp(2)/S^3$ is a diffeomorphism between the total spaces of $(Sp(2)/S^3, p_{2,0})$ and $(Sp(2)/S^3, p_{1,1})$. Since

$$\begin{aligned} p_{2,0}^{-1}\left(\widetilde{h}(0,1)\right) &= \left\{ \text{orbit}\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \middle| q_1, q_2 \in S^3 \right\} \\ &= p_{1,1}^{-1}\left(\widetilde{h}(0,1)\right), \end{aligned}$$

we have constructed another diffeomorphism that takes a fiber to a fiber.

Acknowledgement

We are grateful to D. Crowley for making us aware of this problem and to D. Crowley, C. Escher, and W. Ziller for stimulating communications about these solutions.

References

[CrowEsc] D. Crowley and C. Escher, *A classification of S^3–bundles over S^4*, preprint.

[Cerf] J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* $(\Gamma_4 = 0)$, Lecture Notes in Mathematics, Vol. 53, Springer Verlag, Berlin, 1968. MR **37:**4824

[GluWarZil] H. Gluck, F. Warner, and W. Ziller, *The geometry of the Hopf fibrations* L'Enseignement Math. 32 (1986) 173-198. MR **88e:**53067

[GromMey] D. Gromoll and W. Meyer, *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. **100** (1974) 401-406. MR **51:**11347

[GrovZil] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Annals of Math. **152** (2000) 331-367. CMP 2001:03

[Hat] A. Hatcher, *A proof of the Smale Conjecture, $Diff(S^3) \simeq O(4)$*, Ann. of Math. **117** (1983) 553-607. MR **85c:**57008

[Huse] D. Husemoller, Fibre Bundles, 3rd edition, Springer–Verlag, 1994. MR **94k:**55001

[JamWhi] I. James and J. Whitehead, *The homotopy theory of sphere bundles over spheres (II)*, Proc. London Math. Soc. **5** (1955) 148-166. MR **16:**948d

[Mil] J. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Annals of Math. **64** (1956) 399-405. MR **18:**498d

[PetWil] P. Petersen and F. Wilhelm, *Examples of Riemannian manifolds with positive curvature almost everywhere*, Geometry and Topology **3** (1999) 331-367, http://www.maths.warwick.ac.uk/gt/GTVol3/paper14.abs.html. MR **2000g:**53030

[Rig] A. Rigas, *Some bundles of non-negative curvature*, Math. Ann. 232 (1978) 187-193. MR **57:**4188

[Steen] N. Steenrod, *Topology of Fiber Bundles*, Princeton Mathematical Series, Princeton University Press, 1951. MR **12:**522b

[Wil1] F. Wilhelm, *Exotic spheres with lots of positive curvatures*, J. Geom. Anal. **11** (2001) 161-186. CMP 2001:12

[Wil2] F. Wilhelm, *An exotic sphere with positive curvature almost everywhere*, J. Geom. Anal., to appear.

4243 Edgewood Place, Riverside, California 92506
E-mail address: marc.sanchez@usa.net

Department of Mathematics, University of California, Riverside, California 92521-0135
E-mail address: fred@math.ucr.edu