LOWER ESTIMATE
FOR THE INTEGRAL MEANS SPECTRUM FOR $p = -1$

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Abstract. In this paper we show that there exists a function $f$ bounded
and univalent in the unit disk, such that
$$\frac{\log \int |f'(re^{i\theta})|^p d\theta}{\log \frac{1}{1-r}} \leq C(1-r)^{-0.127},$$
$0 \leq r < 1$.

The aim of the paper is to obtain a new lower estimate for the integral means
spectrum
$$\beta(p) = \lim_{r \to 1^-} \frac{\log \int |f'(re^{i\theta})|^p d\theta}{\log \frac{1}{1-r}}$$
of bounded univalent functions in $D = \{|z| < 1\}$ for $p = -1$. Rohde [Roh89],
Pommerenke [Pom91] proved that there exists a bounded univalent function such that $\beta(-1) \geq 0.109$. Using Carleson-Jones ideas [CJ92], Kraetzer [Kra96] obtained numerical
evidence that for every $p \in [-2, 2]$ there exists a bounded univalent function for
which $\beta(p) \geq p^2/4$.

In this paper we analytically show that there exists a bounded univalent function $f$ for which $\beta(-1) \geq 0.127$.

Define the function
$$f(z) = z \exp \int_0^z \frac{e^{at} - 1}{t} dt, \quad a = 1.7646.$$ We shall prove that $f$ is univalent in $D = \{|z| < 1\}$ function. Since $f$ is a real
function it is enough to show that $L = \{f(e^{i\theta}), 0 < \theta < \pi\}$ is a simple curve and
that $L \cap \mathbb{R}$ is empty. It is useful to mention that
$$\frac{d}{d\theta} \log f(e^{i\theta}) = i \exp(ae^{i\theta}).$$

It follows from
$$\frac{d|f|}{d\theta} = -|f| e^{a \cos \theta} \sin(a \sin \theta) < 0, \quad 0 < \theta < \pi,$$ that $L$ is a simple curve.

Consider
$$\frac{d}{d\theta} \arg f(e^{i\theta}) = e^{a \cos \theta} \cos(a \sin \theta).$$
In our case, the equation \( \frac{d}{d \theta} \arg f = 0 \) is equivalent to the equation \( a \sin \theta = \pi/2 \) which has two roots \( \theta_1 < \theta_2 \) on \( (0, \pi) \). Now, it is clear that \( \text{Im} f(e^{i\theta_1}) > 0 \) implies \( L \cap R = \emptyset \). But this follows from a straightforward calculation.

Therefore, \( f \) is univalent and bounded in the unit disk \( D \). Hence the functions \( f_n(z) = f(z^n)^{1/n} \) are also bounded and univalent in \( D \). Note that

\[
f_n'(z) = \exp \left( a z^n + \frac{1}{n} \int_0^{z^n} \frac{e^{at} - 1}{t} \, dt \right).
\]

Put

\[
\Phi(z) = M^{1+1/q} \lim_{n \to \infty} g_n(z), \quad M = \exp \int_0^1 \frac{e^{at} - 1}{t} \, dt
\]

where \( g_0(z) = z, g_n(z) = g_{n-1}(M^{-1/q^{n-1}} f_{q^{n-1}}(z)), n = 1, 2, \ldots \).

Applying standard methods of geometric function theory it is easy to establish that the function \( \Phi \) is well defined, bounded, and univalent in \( D \). The idea of using compositions of univalent functions was first used by Pommerenke [Pom91]. At the present time it is a most effective method for constructing pathologic mappings.

We have

\[
\log \Phi'(z) = \sum_{k=0}^{\infty} \log f_n'(\phi_k(z)) = \sum_{k=0}^{\infty} \left( a \phi_k^{q^k}(z) + \frac{1}{q^k} \int_0^{\phi_k^{q^k}(z)} \frac{e^{at} - 1}{t} \, dt \right),
\]

where \( \phi_k(z) = M^{\frac{1}{q^{k+1}}} z + \ldots \) and \( |\phi_k| < 1, z \in D \). Therefore

\[
\left| \log \Phi'(z) - \sum_{k=0}^{\infty} a \phi_k^{q^k}(z) \right| \leq \text{const}, z \in D.
\]

Since the Taylor coefficients of \( \phi_k \) are positive then

\[
|\phi_k(z) - M^{\frac{1}{q^{k+1}}} z| \leq |z| \left( 1 - M^{\frac{1}{q^{k+1}}} \right)
\]

and

\[
|\phi_k^{q^k}(z) - M^{\frac{1}{q^{k+1}}} z^{q^k}| \leq |z|^{q^k} |z|^{q^{k+1}} \leq \frac{|z|^{q^k} \log M}{q - 1}.
\]

It is known [Pom91] that

\[
\sum_{k=1}^{\infty} r^{q^k} \leq \log \frac{1}{1 - r} / \log q + \text{const}.
\]

Thus,

\[
(1) \quad \left| \log \Phi'(z) - \sum_{k=0}^{\infty} a M^{-1/(q-1)} z^{q^k} \right| \leq \frac{a \log M}{(q - 1) \log q} \log \frac{1}{1 - r} + \text{const}, \quad r = |z|,
\]

and we can prove the following

**Theorem 1.**

\[
\int_0^{2\pi} |\Phi'(re^{i\theta})|^{-1} d\theta \geq \text{const}(1 - r)^{-0.127}.
\]
Proof. Define \( \log f'_0(z) = \sum_{k=1}^{\infty} aM^{-1/(q-1)} z^k \). Rohde [Roh89], [Pom91] proved that

\[
\int_0^{2\pi} |f'_0(re^{i\theta})|^{-1} d\theta \geq \text{Const}(1-r)^{-\alpha}
\]

where \( \alpha = \log I_0(aM^{-1/(q-1)})/\log q \) and

\[
I_0(x) = \sum_{\nu=0}^{\infty} \frac{x^{2\nu}}{2^{2\nu} \nu!} \text{ is a modified Bessel function.}
\]

Now, it follows from (1) that

\[
\int_0^{2\pi} |\Phi(re^{i\theta})|^{-1} d\theta \geq \text{const}(1-r)^{-\gamma}
\]

where

\[
\gamma = \frac{\log I_0(aM^{-1/(q-1)})}{\log q} - \frac{a \log M}{(q-1) \log q}.
\]

With the choice \( q = 69 \) we obtain our estimate. \( \square \)

Let us remark that the author [Kay01] used the Koebe function as a starting function for lower estimates when \( p \) is positive.

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References


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