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AN ISOPERIMETRIC COMPARISON THEOREM FOR SCHWARZSCHILD SPACE AND OTHER MANIFOLDS

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ABSTRACT. We give a very general isoperimetric comparison theorem which, as an important special case, gives hypotheses under which the spherically symmetric $(n-1)$-spheres of a spherically symmetric n-manifold are isoperimetric hypersurfaces, meaning that they minimize $(n-1)$-dimensional area among hypersurfaces enclosing the same n-volume. This result greatly generalizes the result of Bray (Ph.D. thesis, 1997), which proved that the spherically symmetric 2-spheres of 3-dimensional Schwarzschild space (which is defined to be a totally geodesic, space-like slice of the usual $(3+1)$-dimensional Schwarzschild metric) are isoperimetric. We also note that this Schwarzschild result has applications to the Penrose inequality in general relativity, as described by Bray.

1. INTRODUCTION

In [B1], isoperimetric surfaces were used to prove the Penrose inequality ([Pen], [HI], [B2]), an inequality concerning the mass of black holes in general relativity, in certain special cases. In particular, the approach needed the fact that the spherically symmetric spheres of 3-dimensional Schwarzschild space (Figure 1) minimize perimeter for a given volume. In this paper, we greatly generalize the techniques used in [B1] from the Schwarzschild case to twisted products of $\mathbf{R}$ with a compact Riemannian manifold Σ. The important special case that Σ is a round n-sphere yields results about the isoperimetric hypersurfaces of spherically symmetric manifolds.

Theorem 2.1 provides a very general comparison theorem yielding new examples from old examples of isoperimetric surfaces in manifolds. Corollary 2.3 deduces complete families of isoperimetric surfaces. Corollaries 2.4 and 2.5 specialize to surfaces of revolution. Finally Corollary 2.6 recovers Bray's theorem on Schwarzschild space.

The (3-dimensional) Schwarzschild metric is a good example to consider to motivate the isoperimetric theorems of this paper. Geometrically, the Schwarzschild metric is spherically symmetric and has zero scalar curvature. Physically, this metric corresponds to a totally geodesic, space-like slice of the usual $(3+1)$-dimensional Schwarzschild space (which we note represents a single, spherically symmetric black hole in vacuum). Conveniently, the 3-dimensional Schwarzschild metric is isometric

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to the spherically symmetric graph $\{w^2 = 8m|(x,y,z)| - 16m^2\} \subset \mathbf{R}^4$ (as pictured in Figure 1) and is also isometric to $\mathbf{R}^3 - \{0\}$ with metric $(1 + m/2|(x,y,z)|)^4\delta_{ij}$.

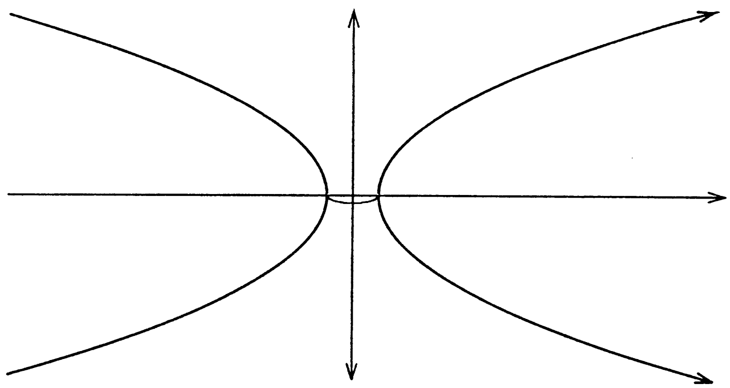

FIGURE 1. In Schwarzschild space, spheres of revolution are isoperimetric [B1, Fig. 1.1].

In Figure 1, it is clear that there is an area-minimizing sphere at the "neck" of the Schwarzschild manifold. However, it is a more difficult task to show that the other spherically symmetric spheres minimize area among homologous surfaces enclosing the same volume. This result follows as Corollary 2.6 to our main Theorem 2.1.

Except for some recent work on two-dimensional manifolds (see the survey [HHM]), there are very few manifolds for which the isoperimetric surfaces are known. There are some results on $\mathbf{R} \times \mathbf{H}^n$ by Hsiang and Hsiang [HH], on $\mathbf{R}P^3, \mathbf{S}^1 \times \mathbf{R}^2$, and $\mathbf{T}^2 \times \mathbf{R}$ by Ritoré and Ros [RR2], [RR1], [R1], on $\mathbf{R} \times \mathbf{S}^n$ by Pedrosa [P], on $\mathbf{S}^1 \times \mathbf{R}^n, \mathbf{S}^1 \times \mathbf{S}^2$, and $\mathbf{S}^1 \times \mathbf{H}^2$ by Pedrosa and Ritoré [PR], and more recently on certain cones $0 \mathbin{⨳} \Sigma$ by Morgan and Ritoré [MR] (see Remark after 2.3).

S. Montiel [Mon] provides some interesting related results on constant mean curvature hypersurfaces in warped products.

Open questions. Since our results apply only to relatively modest variations of known examples, many simple examples remain open. For example, are geodesic spheres about the origin isoperimetric in the paraboloid

$$\{w = x^2 + y^2 + z^2\} \subset \mathbf{R}^4?$$

2. Isoperimetric comparison theorems

We identify compact, n-dimensional surfaces without boundary which minimize area among homologous surfaces enclosing the same volume, i.e., such that the region bounded by their difference has net algebraic volume 0. The results apply in any class of surfaces: smooth submanifolds, piecewise smooth immersions, or the integral currents of geometric measure theory [M]. As described in the introduction, the main Theorem 2.1 has a number of interesting corollaries, including Corollary 2.6 on Schwarzschild space.

2.1. **Isoperimetric comparison theorem.** *For any subinterval $I \subset R$ and smooth, compact, n-dimensional Riemannian manifold Σ, consider smooth Riemannian manifolds M_0 ("the model") and M given as twisted products $I \times \Sigma$ with* *metrics* $dr^2 + \varphi_0(r)^2\, ds_\Sigma^2$ *and* $dr^2 + \varphi(r)^2\, ds_\Sigma^2$. *Fix* r_1 *in* I. *Let* A_0, A *denote the area of the slice* $\{r\} \times \Sigma$ *enclosing volume* V *with* $\{r_1\} \times \Sigma$, *which may be viewed either as a function of* r *or as a function of* V. *Let*

$$a = A(r_1)/A_0(r_1).$$

Suppose that

$$a \leq A(r)/A_0(r) \quad \text{for } r < r_1, \tag{1}$$

$$a \leq A(aV)/A_0(V) \leq 1 \quad \text{for } V > 0. \tag{2}$$

Then if $S = \{r_1\} \times \Sigma$ *minimizes area* [*uniquely*] *for net volume* 0 *with itself in* M_0, *then it also does so in* M. *Alternatively uniqueness follows if either inequality in* (2) *is strict.*

Proof. Consider the map $f : M \to M_0$ mapping slices homothetically to slices. Map S to itself. Continue the map outward, choosing the unique mapping which scales the volume form at each point by the factor $1/a$. By (2), the tangential stretch t lies between 1 and $a^{-1/n}$. Since the radial stretch is $1/at^n$, the area stretch is at most $1/a$. Indeed, if $t \geq 1/at^n$, the area stretch is at most $t^n \leq 1/a$, while if $t \leq 1/at^n$, the area stretch is at most $t^{n-1}/at^n = 1/at \leq 1/a$. Continue the map inward at unit radial velocity. By (1), this part of f stretches area by a factor at most $1/a$ and stretches volume at most $1/a$.

Consider an S' satisfying the volume constraint, with $\operatorname{area} S' \leq \operatorname{area} S$. Then $f(S')$, with no more volume inside $f(S)$ and the same amount outside, encloses no less net volume with no more perimeter. Since $f(S)$ is [uniquely] minimizing, these inequalities must all be equalities and S is [uniquely] minimizing. Alternatively if the first inequality in (2) is strict, then S' must lie in $\{r \leq r_1\}$ and be S. If the second inequality is strict, then for maximum area stretch S' must be entirely tangential and hence be S.

2.2.*Remark.* One may allow φ_0 to go to 0 (and M_0 to end) at some $r_2 < r_1$ before the lower end of I. In the construction of f in the proof, just map everything extra to r_2. For $r \leq r_2$, (1) holds trivially. Actually, it generally suffices to check (1) just for $r_3 < r < r_1$ for some $r_3 > r_2$, since in the proof the map f inward may accelerate to have $f(r_3) = r_2$ while still keeping the area and volume stretches less than $1/a$. See Remark 2.7.

Examples. Some models M_0 for which the slices are known to be minimizing are $\mathbf{R}^{n+1}, \mathbf{H}^{n+1}, \mathbf{S}^{n+1}$, the 2-dimensional surfaces of revolution of [MHH] and [R2], and the $(n+1)$-dimensional cones of [MR]. If you take $\varphi = a^{1/n}\varphi_0$ $(0 < a < 1)$ (giving M a conical singularity at the origin), it follows that all slices of M are minimizing. Perturbations, yielding more general manifolds with conical singularities, may be arranged to maintain the hypotheses for at least one minimizing slice.

2.3. **Corollary.** *Let* C *be the cone over a smooth, compact Riemannian manifold* Σ^n; *i.e.,* $C = [0, \infty) \times \Sigma$, *with metric* $ds^2 = dr^2 + r^2\, ds_\Sigma^2$. *Suppose that each slice* $C_r = \{x \in C : |x| = r\}$ [*uniquely*] *minimizes perimeter for fixed volume in* C. *Let* $M = I \times \Sigma$ *for any closed interval* $I \subset \mathbf{R}$ *with metric* $dr^2 + \varphi^2(r)\, ds_\Sigma^2$. *Suppose that*

$$\varphi' \text{ is nondecreasing for all } r \tag{1}$$

and that

$$0 \leq \varphi' \leq 1 \quad \textit{for all } r \geq r_0. \tag{2}$$

Then every slice $M_r = \{r\} \times \Sigma \subset M$ for $r \geq r_0$ [uniquely] minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0} (volume below r_0 counting as negative). Alternatively uniqueness follows if $\varphi' < 1$ for $r \geq r_0$.

Condition (1) *holds if and only if the radial sectional curvatures of M* (*which all equal $-\varphi''/\varphi$*) *are nonpositive.*

Remark. At least with Σ isometrically embedded in some round sphere $\mathbf{S}^N$, [MR] proves that if Σ is connected and $\mathrm{Ric}_\Sigma > n-1$, then the slices C_r are uniquely minimizing (unless $C = \mathbf{R}^{n+1}$). Incidentally, if $\mathrm{Ric}_\Sigma \geq n-1$ and $\varphi''^2 \leq 1$, then the tangential Ricci curvature of M is greater than or equal to $-\varphi''/\varphi$; for Σ the round unit sphere, $\varphi''^2 \leq 1$ is equivalent to the tangential Ricci curvature of C greater than or equal to $-\varphi''/\varphi$.

Remark. Conditions (1) and (2) are sharp in the sense that if Σ is a round sphere of radius $\delta < 1$, then C_r has nonnegative second variation for fixed volume if and only if $\varphi'^2 - \varphi\varphi'' \leq 1/\delta^2$. (See for example [R2, Lemma 1.6].)

Remark. By Remark 2.2, we may allow conical singularities with φ vanishing at the lower endpoint of I.

Remark. For $r < r_0$, condition (1) may be relaxed to

$$(\varphi(r_0) - \varphi(r))/(r_0 - r) \leq \varphi'(r_0). \tag{3}$$

Also, we do not need to assume φ differentiable: (1) may be replaced by "φ convex" and (2) may be replaced by a statement about finite difference quotients: "$0 \leq \Delta\varphi \leq \Delta r$ for $\Delta r \geq 0$."

Proof of Corollary 2.3. Fix r_1, $a^{1/n} = \varphi'(r_1)$. If $a = 0$, S is uniquely minimizing because projection onto S is distance nonincreasing; equality could hold only for round spheres in a portion of a round cylinder, among which only S encloses the prescribed volume.

So we may assume that $a > 0$. Let M_0 be the relevant portion of C, parametrized so that $A(r_1)/A_0(r_1) = a$ (see Remark 2.2). Now the two hypotheses imply the two hypotheses of Theorem 2.1. To verify 2.1(2), note that in terms of the areas A and volumes V of the slices M_r, since $A = |\Sigma|\varphi^n$ and $dV/dr = A$,

$$\varphi' = \frac{1}{n|\Sigma|^{1/n}} A^{1/n} \frac{dA}{dV} = \frac{1}{(n+1)|\Sigma|^{1/n}} \frac{dA^{(n+1)/n}}{dV}. \tag{4}$$

Hence $a \leq A(aV)/A_0(V) \leq a^{n/(n+1)} \leq 1$. Therefore, Theorem 2.1 applies and yields the conclusions of Corollary 2.3.

The special case of surfaces of revolution deserves individual mention:

2.4. **Corollary.** *Consider a surface of revolution M, i.e., $I \times \mathbf{S}^n$, for any interval $I \subset R$, with metric $dr^2 + \varphi^2(r)\, ds_n^2$, where ds_n is the metric on the round unit sphere. Suppose that*

$$\varphi' \textit{ is nondecreasing for all } r \tag{1}$$

and that

$$0 \leq \varphi' \leq 1 \quad \textit{for all } r \geq r_0. \tag{2}$$

Then every sphere of revolution S_r for $r \geq r_0$ minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0} (volume below r_0 counting as negative), uniquely if $\varphi'(r) < 1$.

Condition (1) *holds if and only if M has nonpositive radial Ricci curvature. For any r, condition* (2) *holds if and only if S_r has nonnegative mean curvature and M has nonnegative tangential sectional curvature (or equivalently for $n = 2$, S_r has nonnegative Hawking mass).*

(Hawking mass for $n = 2$ is defined as $(A/16\pi)^{1/2}(1 - \int H^2/4\pi)$, where the mean curvature H is the average of the principal curvatures.)

Proof. Most of the proof follows immediately from Corollary 2.3. To prove the final conditions equivalent to (2), note that mean curvature satisfies

$$H = (1/n)(dA/dV) = \varphi'/\varphi$$

and that tangential sectional curvature equals $(1 - \varphi'^2)/\varphi^2$ [Pet, p. 71].

Remark. S. Montiel [Mon, Cor. 5] shows under the weaker hypothesis

$$\varphi'^2 - \varphi\varphi'' \leq 1,$$

with $\varphi(0) = 0$, $\varphi'(0) = 1$, that a smooth, *star-shaped*, constant-mean-curvature hypersurface must be a sphere of revolution (or a translate if the ambient has constant curvature).

2.5. **Corollary.** *If a surface of revolution M has nonpositive radial Ricci curvature and nonnegative tangential sectional curvature, then every sphere of revolution is uniquely minimizing.*

Proof. Apply Corollary 2.4 to both ends of M if necessary.

2.6. **Corollary** ([B1, Thm. 8])**.** *In the Schwarzschild metric, every sphere of revolution is uniquely minimizing.*

2.7.*Remark.* Using the method of Remark 2.2, [B1, Thm. 10] shows that if M eventually coincides with Schwarzschild space, then spheres of revolution are eventually minimizing.

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