PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 5, Pages 1533–1537
S 0002-9939(01)06202-5
Article electronically published on October 5, 2001

L^p ESTIMATES ON FUNCTIONS OF MARKOV OPERATORS

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(Communicated by Christopher D. Sogge)

ABSTRACT. We prove L^p estimates for functions of Markov operators on a discrete measure space of superpolynomial volume growth.

Let X be a discrete, measurable space endowed with a measure dx and a measurable distance $d(.,.)$. Let us denote by $B(x,r)$ the ball of center x and radius r. If $|B(x,r)|$ is the dx-measure of $B(x,r)$, we assume that there exist $0<\alpha'\leq\alpha\leq 1$ and $\kappa,\kappa',c,c'>0$ such that

$$c'e^{\kappa' r^{\alpha'}} \leq |B(x,r)| \leq ce^{\kappa r^{\alpha}}, \quad \forall x\in X,\, r>0, \tag{1}$$

i.e. X has superpolynomial ($\alpha<1$) or exponential ($\alpha'=\alpha=1$) volume growth.

Let us consider a bounded symmetric Markov kernel $P(x,y)$ on X and let us set $P_0(x,y)=\delta_x(y)$, where δ_x is the Dirac mass at x, $P_1(x,y)=P(x,y)$ and $P_n(x,y)=\int P_{n-1}(x,z)P(z,y)dz$ for $n\geq 2$. We assume that there exists constants $c,\beta>0$ such that

$$P_n(x,y)\leq ce^{-\beta\frac{d(x,y)^2}{n}} \tag{2}$$

for any $x,y\in X$ and $n\in\mathbb{N}$.

Markov chains with transition kernels satisfying an estimate such as (2) were first studied by N.Th. Varopoulos [8]. T.K. Carne [3] proves (2) in the case when X is countable by improving the result of [8]. G. Alexopoulos [1] generalised this in the context of continuous groups.

In the presence of a group structure on X, translation invariant, symmetric Markov kernels are obtained by the convolution powers of a probability measure μ on X. In fact, if μ has a bounded symmetric density f with respect to the left invariant Haar measure dx, then the Markov kernel

$$P(x,y)=f(x^{-1}y)$$

is translation invariant and satisfies

$$P_n(x,y)=f^n(x^{-1}y)$$

where f^n is the $n-$convolution power of f.

Received by the editors September 10, 2000 and, in revised form, November 2, 2000.
1991 *Mathematics Subject Classification.* Primary 22E25, 22E30, 43A80.
Key words and phrases. Markov chains, multipliers.

Let $|x|$ be a word distance. Then by [8], [3] and [1], there exist a $\beta > 0$ such that

$$f^n(x^{-1}y) \leq ce^{-\beta \frac{|x^{-1}y|^2}{n}},$$

for any $x, y \in X$ and $n \in \mathbb{N}$.

It is worth mentioning that every locally compact group is at most of exponential volume growth. Further, in [5], Grigorchuck proved that there exist discrete finitely generated groups such that

$$ce^{r^\alpha} \leq |B(x,r)| \leq Ce^{r^{\alpha'}}$$

with $\alpha, \alpha' \in (0,1)$. In this case, for a class of symmetric and bounded probability densities f, one can prove that

$$f^n(x^{-1}y) \leq ce^{-\frac{n^{1/3}}{c} - \beta \frac{|x^{-1}y|^2}{n}}, \quad \forall x, y \in X;$$

see [6], Remark 1, p. 690.

If P is the Markov operator with kernel $P(x,y)$, then $I - P$ is symmetric, positive, bounded on L^2 and admits the spectral decomposition

$$I - P = \int_0^\infty \lambda dE_\lambda.$$

Also, for any bounded Borel function m on $\mathbb{R}$, by the spectral theorem we can define the operator

$$m(I-P) = \int_0^\infty m(\lambda) dE_\lambda$$

which is bounded on L^2.

Let us consider the following class $\mathcal{T}$ of Borel functions: $m \in \mathcal{T}$ iff its Fourier transform satisfies

$$|\hat{m}(t)| \leq ce^{-W|t|}, \quad \forall t \in \mathbb{R}, \tag{3}$$

for some $W > 0$. The class $\mathcal{T}$ is of the type of multipliers introduced in [4] and [7]. In fact, the class $\mathcal{F}_0(e^{-W|t|}, b)$, $b > 0$ ([7], p. 787), contains functions m which satisfy

$$\left|\hat{m}^{(k)}(t)\right| \leq c \left(\frac{k}{b}\right)^k e^{-W|t|}$$

for any t and $k \geq 0$.

We note that if m is smooth in the zone $\overline{\Omega}_W = \{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq W\}$ and holomorphic on Ω_W, then it belongs to $\mathcal{F}_0(e^{-W|t|}, b)$ for some $b > 0$, iff

$$|m(\lambda)| \leq c \left(\frac{k}{b}\right)^k (1 + |\lambda|)^{-k/2}$$

for any $\lambda \in \overline{\Omega}$ and $k \geq 0$ ([7], Lemma 5.5).

In [2], G. Alexopoulos proved an analog of the Mikhlin-Hörmander multiplier theorem for random walks on discrete groups of polynomial volume growth. In this article we prove the following analog of the main result of M. Taylor [7].

Theorem. *Let as assume that P_n satisfies* (2)*, $m \in \mathcal{T}$ and that either*

(i) *X is of superpolynomial volume growth but not exponential, i.e. assumption* (1) *is valid with $\alpha', \alpha \in (0,1)$,*

(ii) *X is of exponential volume growth and $\beta > \kappa + \delta$, $W\delta > \frac{\kappa}{2}$ where δ is the supremum of $\eta \in (0, e^{-1})$ such that $\eta \leq \max(\beta, e^{-\beta})$.*

Then $m(I-P)$ is bounded on L^p, $p \geq 1$.

The proof of the Theorem is based on the following lemmas.

For a fixed $y \in X$ we shall denote by $A_p(y)$, $p \in \mathbb{N}$, the shell $\{x : 2^{p/2} \leq d(x,y) \leq 2^{(p+1)/2}\}$. Let us also recall that the Dirac mass δ_y at y is in $L^2(X)$ if X is discrete.

Lemma 1. *There is $\eta \in (0, e^{-1})$ with $\eta \leq \max(\beta, e^{-\beta})$ such that for all $p \in \mathbb{N}$, $x \in A_p(y)$ and $|t| \leq \eta 2^{p/2}$,*

$$\left|e^{itP}(\delta_y)(x)\right| \leq ce^{-(\beta-\eta)d(x,y)}. \tag{4}$$

Proof. We have that

$$\begin{aligned} e^{itP}\delta_y(x) &= \sum_{n\geq 0} \frac{(it)^n}{n!} P^n \delta_y(x) \\ &= \sum_{n \leq 2^{p/2}} \frac{(it)^n}{n!} P^n(x,y) + \sum_{n > 2^{p/2}} \frac{(it)^n}{n!} P^n(x,y) = I_1 + I_2. \end{aligned}$$

It follows from (2) that for $x \in A_p(y)$ and $|t| \leq \eta 2^{p/2}$

$$\begin{aligned} |I_1| &\leq c \sum_{n \leq 2^{p/2}} \frac{(\eta 2^{p/2})^n}{n!} e^{-\beta 2^{p/2}} \\ &\leq ce^{-\beta 2^{p/2}} e^{\eta 2^{p/2}} = ce^{-(\beta-\eta)2^{p/2}}. \end{aligned}$$

By (2) and Stirling's formula $n! \sim \sqrt{n}\left(\frac{n}{e}\right)^n$, we get that

$$\begin{aligned} |I_2| &\leq c \sum_{n > 2^{p/2}} \frac{(\eta 2^{p/2})^n}{n!} \leq c \sum_{n > 2^{p/2}} \left(\frac{\eta 2^{p/2} e}{n}\right)^n \frac{1}{\sqrt{n}} \\ &\leq c \frac{1}{2^{(p-1)/4}} \sum_{n > 2^{p/2}} (\eta e)^n \leq c \frac{1}{2^{(p-1)/4}} (\eta e)^{2^{p/2}} \end{aligned}$$

provided that $\eta e < 1$.

Now,

$$(\eta e)^{2^{p/2}} = e^{2^{p/2}\log(\eta e)} \leq e^{-(\beta-\eta)2^{p/2}}$$

provided that

$$\log(\eta e) \leq -(\beta - \eta)$$

which holds true if

$$\eta e \leq e^{\eta} e^{-\beta} \text{ or } \eta \leq e^{-\beta}$$

since $\eta < 1$. ☐

Since

$$m(I-P) = \int_{\mathbb{R}} \hat{m}(t) e^{it(I-P)} dt$$

the kernel $K(x,y)$ of the operator $m(I-P)$ is given by

$$\begin{aligned} K(x,y) &= \int_{\mathbb{R}} \hat{m}(t) e^{it(I-P)} \delta_y(x) dt \\ &= \int_{|t|\leq \eta 2^{p/2}} + \int_{|t|\geq \eta 2^{p/2}} = K_0(x,y) + K_\infty(x,y). \end{aligned}$$

We have the following:

Lemma 2. *Let η be as in the Theorem. Then there is $c>0$ such that*

$$\int_X |K_0(x,y)|\, dx \leq c \sum_{p\geq 0} e^{-(\beta-\eta)2^{p/2}} 2^{p/2} |A_p(y)|. \tag{5}$$

Also, for any $\varepsilon>0$, there is $c>0$ such that

$$\int_X |K_\infty(x,y)|\, dx \leq c \sum_{p\geq 0} e^{-\eta(W-\varepsilon)2^{p/2}} |A_p(y)|^{1/2}. \tag{6}$$

Proof. It follows from (4) and (3) that for $x \in A_p(y)$

$$\begin{aligned} |K_0(x,y)| &\leq c \int_{|t|\leq \eta 2^{p/2}} |\hat{m}(t)|\, e^{-(\beta-\eta)d(x,y)} dt \\ &\leq c e^{-(\beta-\eta)2^{p/2}} \eta 2^{p/2}. \end{aligned}$$

Thus, by using (1)

$$\begin{aligned} \int_X |K_0(x,y)|\, dx &= \sum_{p\geq 0} \int_{A_p(y)} |K_0(x,y)|\, dx \\ &\leq c \sum_{p\geq 0} e^{-(\beta-\eta)2^{p/2}} \eta 2^{p/2} \int_{A_p(y)} dx \\ &\leq c \sum_{p\geq 0} e^{-(\beta-\eta)2^{p/2}} 2^{p/2} |A_p(y)|. \end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality, (1), (2), the decay of $\hat{m}(t)$ for t large and the fact that the Dirac mass $\delta_y \in L^2(X)$, we get that

$$\begin{aligned} &\int_{A_p(y)} |K_\infty(x,y)|\, dx \leq \int_{A_p(y)} \left(\int_{|t|\geq \eta 2^{p/2}} \left| \hat{m}(t) e^{it(I-P)} \delta_y(x) \right| dt \right) dx \\ &\leq c \int_{|t|\geq \eta 2^{p/2}} |\hat{m}(t)|\, dt \int_{A_p(y)} \left| e^{itP} \delta_y(x) \right| dx \\ &\leq c \int_{|t|\geq \eta 2^{p/2}} |\hat{m}(t)|\, |A_p(y)|^{1/2} \left\| e^{itP} \right\|_2 \|\delta_y\|_2\, dt \\ &\leq c\, |A_p(y)|^{1/2} \int_{|t|\geq \eta 2^{p/2}} |\hat{m}(t)|\, dt \\ &\leq c\, |A_p(y)|^{1/2} e^{-\eta(W-\varepsilon)2^{p/2}} \int_{|t|\geq \eta 2^{p/2}} e^{-\varepsilon|t|} dt \\ &\leq c\, |A_p(y)|^{1/2} e^{-\eta(W-\varepsilon)2^{p/2}}. \end{aligned}$$

Therefore,

$$\int_X |K_\infty(x,y)|\,dx = \sum_{p\geq 0}\int_{A_p(y)} |K_\infty(x,y)|\,dx$$
$$\leq c\sum_{p\geq 0} c\,|A_p(y)|^{1/2}\,e^{-\eta(W-\varepsilon)2^{p/2}}.$$

Proof of the Theorem. (i) Let us assume that (1) holds for $\alpha', \alpha \in (0,1)$. Then by (5) and (6) we get that

$$\int_X |K(x,y)|\,dx$$
$$\leq c\sum_{p\geq 0} e^{-(\beta-\eta)2^{p/2}}2^{p/2}\,|A_p(y)| + c\sum_{p\geq 0} e^{-\eta(W-\varepsilon)2^{p/2}}\,|A_p(y)|^{1/2}$$
$$\leq c\sum_{p\geq 0} e^{-(\beta-\eta)2^{p/2}}2^{p/2}e^{\kappa(2^{p/2})^\alpha} + \sum_{p\geq 0} e^{-\eta(W-\varepsilon)2^{p/2}}e^{\frac{\kappa}{2}(2^{p/2})^\alpha} < \infty.$$

This implies that $m(I-P)$ is bounded on $L^\infty(X)$. As already mentioned, by the spectral theorem $m(I-P)$ is bounded on $L^2(X)$ and by interpolation and duality we obtain the boundedness of $m(I-P)$ on $L^p(X)$, $p\geq 1$.

(ii) Similarly, if $|B(x,r)|\leq ce^{\kappa r}$, then

$$\int_X |K(x,y)|\,dx$$
$$\leq c\sum_{p\geq 0} e^{-(\beta-\eta)2^{p/2}}2^{p/2}e^{\kappa 2^{p/2}} + \sum_{p\geq 0} e^{-\eta(W-\varepsilon)2^{p/2}}e^{\frac{\kappa}{2}2^{p/2}} < \infty,$$

provided that $\beta > \kappa+\eta$ and $\eta W > \kappa/2$ and the boundedness of $m(I-P)$ on $L^\infty(X)$ follows.

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