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GENERIC, ALMOST PRIMITIVE AND TEST ELEMENTS OF FREE LIE ALGEBRAS

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ABSTRACT. We construct a series of generic elements of free Lie algebras. New almost primitive and test elements were found. We present an example of an almost primitive element which is not generic.

1. INTRODUCTION

Let $\mathfrak{U}$ be a variety of groups defined by the set of laws $\mathfrak{V}$, F the free group, and let $\mathfrak{V}(F)$ be the verbal subgroup defined by $\mathfrak{V}$. An element g of F is said to be $\mathfrak{U}$-generic if $g \in \mathfrak{V}(F)$ but $g \notin \mathfrak{V}(G)$ for every proper subgroup G of F.

In the articles [5], [6]–[9], [18, 20, 22, 23, 26] generic elements of free groups were studied. In particular, series of generic elements of free groups were constructed. In [5] it was shown that for a variety with solvable word problem there is an algorithm to recognize generic elements.

A test element in a free group F is an element f with the property that if $\varphi(f) = f$ for an endomorphism φ of F, then φ must be an automorphism. An almost primitive element (APE) is an element of a free group F which is not primitive in F but which is primitive in any proper subgroup of F containing it. There are test elements of free groups which are not generic [5]. In general, generic does not imply APE [8]. At the same time, in [8] it was proved that if $\mathfrak{U}$ is a nontrivial variety defined by the set of laws $\mathfrak{V}$ and $w \in \mathfrak{V}(F)$, and w is APE of F, then w is $\mathfrak{U}$-generic and hence test element in F.

The origins of the theory of Lie algebras lie in the papers (published in the 1930s) of P. Hall, W. Magnus and E. Witt which were devoted to the structure of free and nilpotent groups. Many results for free Lie algebras are similar to the corresponding results for free groups. But at the same time the proof techniques in free Lie algebras are different from the free group techniques, and a series of results in the theory of free groups has no analog for free Lie algebras (for the details we refer to the monographs [1, 2, 15, 16]).

In this article we consider generic elements, almost primitive and test elements in free Lie algebras. In previous papers many basic results from group theory have been carried over successfully to this context (see [10, 11, 12, 14]). Mirroring the

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situation in free groups we can give examples of APE's which are not generic and generic elements which are not APE's. As in the situation with free groups (see [5]) since free Lie algebras of finite rank are Hopfian, generic elements are test elements.

2. Definitions

In this section we give basic definitions for free Lie algebras. Let $\mathfrak{U}$ be a variety of Lie algebras over a field K defined by the set of identities $\mathfrak{V}$, $L = L(X)$ the free Lie algebra with the set X of free generators, and let $\mathfrak{V}(L)$ be the verbal ideal of L defined by $\mathfrak{V}$. An element f of L is said to be $\mathfrak{U}$-generic if $f \in \mathfrak{V}(L)$ but $f \notin \mathfrak{V}(H)$ for every proper subalgebra H of L. Let $\mathfrak{A}$ be the variety of Abelian Lie algebras. An element u of the free Lie algebra L is $\mathfrak{A}$-generic if $u \in [L, L]$, and if $u \in [H, H]$ for some subalgebra H of L, then $H = L$.

Let X be a finite set. For $u \in L$, by $\mathrm{d}(u)$ we denote the usual degree of u. Consider a weight function $\mu\colon X \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Let $\Gamma(X)$ be the free groupoid of nonassociative monomials in the alphabet X, $S(X)$ the free semigroup of associative words in X, and $\tilde{}\colon \Gamma(X) \to S(X)$ the bracket removing homomorphism. We set $\mu(x_1 \cdots x_n) = \sum_{i=1}^n \mu(x_i)$ for $x_1, \dots, x_n \in X$, and $\mu(u) = \mu(\tilde{u})$ for $u \in \Gamma(X)$. If $\mu(x) = 1$ for all $x \in X$, then μ is just the usual degree, $\mu = \mathrm{d}$. If $a \in L(X)$, $a = \sum \alpha_i a_i$, $0 \neq \alpha_i \in K$, a_i are basic monomials, $a_j \neq a_s$ with $j \neq s$, then we set $\mu(a) = \max_i\{\mu(\widetilde{a_i})\}$. By $\widehat{a}$ we denote the leading part of a: $\widehat{a} = \sum_{j, \mu(a_j) = \mu(a)} \alpha_j a_j$. We denote the leading part of an element a with respect to the usual degree function by a°.

A subset M of $L = L(X)$ is said to be independent if M is a set of free generators of the subalgebra of L generated by M. A subset $M = \{a_i\}$ of nonzero elements of L is called reduced if for any i the leading part $\widehat{a_i}$ of the element a_i does not belong to the subalgebra of L generated by the set $\{\widehat{a_j} \mid j \neq i\}$.

Let $S = \{s_\alpha \mid \alpha \in I\}$ be a subset of L. A mapping $\theta\colon S \to S' \subseteq L$ is an elementary transformation of S if either θ is a nondegenerate linear transformation of S, or there is $\beta \in I$ such that $\theta(s_\alpha) = s_\alpha$ for all $\alpha \in I$, $\alpha \neq \beta$, and

$$\theta(s_\beta) = s_\beta + f(\{s_\alpha \mid \alpha \neq \beta\}).$$

One can transform any finite set of elements of the algebra $L = L(X)$ to a reduced set using a finite number of elementary transformations and possibly cancelling zero elements, and every reduced subset of the algebra L is an independent subset. Moreover, using Kurosh's method, one can construct a reduced set of generators for any subalgebra of the algebra L. Hence any subalgebra of L is free. These results were obtained by A. I. Shirshov [19] and E. Witt [25]. In [4] P. M. Cohn showed that the automorphism group of L is generated by elementary automorphisms. For more properties of free Lie algebras we refer to monographs [1, 2, 15, 16].

An element u of $L(X)$ is said to be primitive if it is an element of some set of free generators of the algebra $L(X)$. Matrix characterization of primitive systems of elements of free Lie algebras were obtained by A. A. Mikhalev and A. A Zolotykh [13, 15]. An element u of $L(X)$ is said to be a test element if for any endomorphism φ of $L(X)$ it follows from $\varphi(u) = u$ that φ is an automorphism of $L(X)$. In the articles [5, 20, 21, 24] one can find characterizations of test elements of free groups. A characterization of test elements of free Lie algebras was obtained by A. A. Mikhalev, J.-T. Yu and A. A. Zolotykh in [10, 11, 14].

An almost primitive element (APE) of the free Lie algebra $L = L(X)$ is an element which is not primitive in L, but which is primitive in any proper subalgebra of L containing it. In [12] a series of APE of free algebras was constructed.

3. Generic elements

For $a \in L(X)$ by $\operatorname{ad} a$ and $\operatorname{Ad} a$ we denote the operators of left and right multiplication, respectively, $(\operatorname{ad} a)(b) = [a, b]$ and $(b)\operatorname{Ad}(a) = [b, a]$ for $b \in L(X)$.

Proposition 3.1. *Let K be a field, $\operatorname{char} K \neq 2$, $X = \{x, y, z\}$. Then the element $u = [x, y] + (x)(\operatorname{Ad} z)^k$ $(k \geq 2)$ is an APE of the free Lie algebra $L(X)$ over K. It is also a test element of $L(X)$.*

Proof. Suppose that the element u belongs to a finitely generated subalgebra H of $L(X)$. Let $x > y > z$, and let $\{h_1, \dots, h_m\}$ be a reduced set of free generators of H. Then the leading part u° of u is a polynomial in $h_1^\circ, \dots, h_m^\circ$.

It is clear that $u^\circ = (x)(\operatorname{Ad} z)^k$. If in the expression of u° we have a nonzero linear summand h_i° for some i, then $u^\circ = \alpha h_i^\circ + f(\{h_j^\circ \mid j \neq i\})$, $0 \neq \alpha \in K$; therefore $u = \alpha h_i + f'(\{h_j \mid j \neq i\})$, and u is a primitive element of H. Otherwise $z \in H$.

Now we consider the weight function μ given by $\mu(x) = 1$, $\mu(y) = N > k$, $\mu(z) = 1$. Let $\{h'_1, \dots, h'_m\}$ be a reduced set of free generators of the subalgebra H with respect to the weight function μ. We have $\widehat{u} = [x, y]$, where $\widehat{u}$ is a polynomial in $\widehat{h'_1}, \dots, \widehat{h'_m}$. If $\widehat{u}$ has a nonzero linear summand $\widehat{h_j}$ for some j, then

$$\widehat{u} = \beta \widehat{h_j} + g(\{\widehat{h_i} \mid i \neq j\}),\ 0 \neq \beta \in K;$$

therefore $u = \beta h_j + g'(\{h_i \mid i \neq j\})$, and u is a primitive element of H. Otherwise $x \in H$.

If $x, z \in H$, then $u' = u - (x)(\operatorname{Ad} z)^k = [x, y] \in H$. If $(u')^\circ$ has a linear summand in the expression in $h_1^\circ, \dots, h_m^\circ$, then u is a primitive element of H. Otherwise $y \in H$, and $H = L(X)$. Thus u is an APE of $L(X)$. Since the component of degree one of u is zero, by Lemma 3.3 of [12] which says that an APE with zero component of degree one is a test element, u is a test element of $L(X)$. □

Proposition 3.2. *The element*

$$u = (x)(\operatorname{Ad} y)^k + (x)(\operatorname{Ad} z)^l, \text{ where } k \text{ and } l \text{ are positive integers,}$$

is an APE of $L(x, y, z)$ if and only if $\min\{k, l\} = 1$ and $\max\{k, l\} \geq 2$.

Proof. If $k = l = 1$, then $u = [x, y + z]$, and u is not a primitive element of the proper subalgebra of $L(x, y, z)$ generated by x and $y + z$.

Suppose that $k, l > 1$. Let $u_1 = (x)(\operatorname{Ad} y)^{k-1}$, $u_2 = (x)(\operatorname{Ad} z)^{l-1}$, $u_3 = y$, and $u_4 = z$. Then $U = \{u_1, u_2, u_3, u_4\}$ is a reduced set, $u = [u_1, u_2] + [u_3, u_4]$, $H = L(u_1, u_2, u_3, u_4) \neq L(x, y, z)$, and u is not a primitive element of H. Hence u is not an APE of $L(x, y, z)$. Using Proposition 3.1 we complete the proof. □

Remark that the element u considered above with $k, l > 1$ is not a $\mathfrak{A}$-generic element of $L(x, y, z)$.

Now we consider elements

$$u_{k,l}(x, y) = (\operatorname{ad} x)^k(y) + (x)(\operatorname{Ad} y)^l$$

of the free Lie algebra $L(x, y)$. If $k = l = 2$, then $u_{2,2}(x, y) = [[x, y], x - y]$, and this element is not an APE of $L(x, y)$. If $\operatorname{char} K = p > 2$, then

$$u_{p,p}(x, y) = [x + y, (\operatorname{ad}^{p-1}(x + y))(y - x)],$$

and $u_{p,p}(x, y)$ is not an APE of $L(x, y)$.

Proposition 3.3. *Let $k, l \geq 2$, $k \neq l$. Then the element $u_{k,l}(x, y) = (\operatorname{ad} x)^k(y) + (x)(\operatorname{Ad} y)^l$ is an APE of $L(x, y)$.*

Proof. We may suppose that $k > l$. In this case $u^{\circ}_{k,l}(x, y) = (\operatorname{ad} x)^k(y)$. Suppose that the element u belongs to a finitely generated subalgebra H of $L(X)$. Let $\{h_1, \dots, h_m\}$ be a reduced set of free generators of H. Then the leading part u° of u is a polynomial in $h^{\circ}_1, \dots, h^{\circ}_m$. As in the proof of Proposition 3.1, if u is not a primitive element of H, then $x \in H$. In this case we consider the weight function μ given by $\mu(x) = 1$, $\mu(y) = N > k$. Let $\{h'_1, \dots, h'_m\}$ be a reduced set of free generators of the subalgebra H with respect to the weight function μ. We have $\widehat{u} = (x)(\operatorname{Ad} y)^l$. Again, either u is a primitive element of H, or $y \in H$. If $y \in H$, then $H = L(x, y)$. This completes the proof. □

It is clear that an element u of $[L, L]$ is $\mathfrak{A}$-generic if and only if $u = \sum \alpha_i [f_i, g_i]$, where $\alpha_i \in K$ and the elements $\{f_i, g_i\}$ generate the algebra $L = L(X)$. Therefore, if $u = \sum_{i=1}^{s} \alpha_i [u_i, v_i]$, where u_i, v_i are monomials, $\mathrm{d}(u_i), \mathrm{d}(v_i) > 1$, $\alpha_i \in K$, then u is not a $\mathfrak{A}$-generic element of $L(X)$.

Proposition 3.4. *Let u be an APE of $L = L(X)$, $u \in [L, L]$. Then u is a $\mathfrak{A}$-generic element of L.*

Proof. Assume that $u \in [H, H]$ for some proper subalgebra H of L. Since u is an APE of L, u is a primitive element of H. But any free generating set of H form a linear basis of H modulo $[H, H]$. This contradiction proves the statement. □

Proposition 3.5 ([12]). *Let $L(X)$ be a free product of two of its proper subalgebras A and B, $L = A * B$. Also let a and b be homogeneous APE of A and B relatively to some generalized degree functions μ_1 and μ_2 of A and B, respectively (μ_1 and μ_2 could be different). Then $a + b$ is an APE of L.*

Note that if a free group F is a free product of its proper subgroups A and B, $F = A * B$, and a and b are APE of A and B, respectively, then ab is APE in F [3, 8, 18].

Theorem 3.6. *Let K be a field,* $\operatorname{char} K \neq 2$, *$X = \{x_1, \dots, x_n\}$, and let $L(X)$ be the free Lie algebra over K.*

1) *If $n = 2m$, then*

$$[x_1, x_2] + \dots + [x_{2m-1}, x_{2m}]$$

is a $\mathfrak{A}$-generic element of $L(X)$;

2) *If $n = 2m$, k_i, l_i, $i = 1, \dots, m$, are integers, $k_i, l_i \geq 2$, $k_i \neq l_i$, then*

$$u_{k_1,l_1}(x_1, x_2) + \dots + u_{k_m,l_m}(x_{2m-1}, x_{2m})$$

is a $\mathfrak{A}$-generic element of $L(X)$;

3) *If $n = 2m + 1$ and $l \geq 2$, then*

$$[x_1, x_2] + (x_1)(\operatorname{Ad} x_3)^l + [x_4, x_5] + \dots + [x_{2m}, x_{2m+1}]$$

is a $\mathfrak{A}$-generic element of $L(X)$;

4) *If* $n = 3m$, $l_1, \dots, l_m \geq 2$, *then*

$$[x_1, x_2] + (x_1)(\mathrm{Ad}\, x_3)^{l_1} + \dots [x_{3m-2}, x_{3m-1}] + (x_{3m-2})(\mathrm{Ad}\, x_{3m})^{l_m}$$

is a $\mathfrak{A}$*-generic element of* $L(X)$.

Proof. Since $k_i, l_i \geq 2$, the elements $u_{k_i,l_i}(x_{2i-1}, x_{2i})$ are μ-homogeneous, where the weight function is given by $\mu(x_{2i-1}) = l_i - 1$ and $\mu(x_{2i}) = k_i - 1$. The statements of the theorem follow from Propositions 3.1, 3.3, 3.4, and 3.5. □

Note that if n is even, then $[x_1, x_2] \cdot [x_3, x_4] \cdots [x_{n-1}, x_n]$ is a $\mathfrak{A}$-generic element of the free group on $x_1, \dots, x_n$ [6, 17, 22].

Let $X = \{x, y\}$, $x > y$, $L = L(x, y)$, $L^{(1)} = [L, L]$, and $L^{(2)} = [L^{(1)}, L^{(1)}]$. The monomials

$$((x)\mathrm{Ad}^{\,i}(y))\mathrm{Ad}^{\,j}(x),\ i \geq 1, j \geq 0, \tag{1}$$

form a set of free generators of the free Lie algebra $L^{(1)}$ (see [1]).

Lemma 3.7. *Let* ℓ *be a nonzero linear combination of* x *and* y, $L = L(x, y)$. *Then:*

1) *if* $[g, \ell] \in L^{(2)}$ *for a homogeneous element* g *of* L *of degree* 4, *then* $g = 0$;
2) *if* $[[[x, y], x], \ell] = [f, \ell_1]$ *for some homogeneous element* $f \in L$ *of degree* 3 *and a linear element* ℓ_1 *of* L, *then either* $[\ell, \ell_1] = 0$ *or* $[\ell_1, x] = 0$.

Proof. 1) We may assume that $\ell = x$. The element g can be written in the left-normed form

$$g = \alpha_1[x, y, y, y] + \alpha_2[x, y, y, x] + \alpha_3[x, y, x, x].$$

Hence

$$[g, l] = \alpha_1[x, y, y, y, x] + \alpha_2[x, y, y, x, x] + \alpha_3[x, y, x, x, x] \in L^{(2)}.$$

Since the elements (1) are linear independent modulo $L^{(2)}$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and $g = 0$.

2) Let

$$\begin{array}{rclcrcl} \ell &=& \alpha_1 x + \alpha_2 y, && (\alpha_1, \alpha_2) &\neq& (0,0), \\ \ell_1 &=& \beta_1 x + \beta_2 y, && (\beta_1, \beta_2) &\neq& (0,0), \\ f &=& \gamma_1[[x, y], x] + \gamma_2[[x, y], y], && (\gamma_1, \gamma_2) &\neq& (0,0). \end{array}$$

Then the equality $[x, y, x, \ell] = [f, \ell_1]$ is equivalent to the system of equations

$$\alpha_1 = \gamma_1\beta_1,\ \alpha_2 = \gamma_2\beta_1 + \gamma_1\beta_2,\ \gamma_2\beta_2 = 0.$$

If $\beta_2 = 0$, then $\ell_1 = \beta_1 x$, and if $\gamma_2 = 0$, then $\ell = \gamma\ell_1$. □

Theorem 3.8. *The element*

$$u = [x, y] + [[x, y], x] + [[[x, y], x], [x, y]]$$

is a $\mathfrak{A}$*-generic element of the free Lie algebra* $L = L(x, y)$, *but at the same time it is not an APE of this algebra.*

Proof. The element u belongs to the proper subalgebra of L generated by $[x, y]$ and $[[x, y], x]$. It is clear that u is not a primitive element of this subalgebra. Therefore u is not an APE of L.

Suppose that u is not an $\mathfrak{A}$-generic element of L. Then there is a proper subalgebra H of L such that $u \in [H, H]$. By H° we denote the subalgebra of leading terms of elements of H. Since H is generated by some reduced set, if $f \in [H, H]$, then

$f^\circ \in [H^\circ, H^\circ]$, and $u^\circ = [[[x,y],x],[x,y]] \in [H^\circ, H^\circ]$. Hence u is a linear combination of homogeneous elements of degree 5 of the form $[f,g]$, where $f, g \in H^\circ$. We may assume that $\mathrm{d}(f) > \mathrm{d}(g)$. If $\mathrm{d}(g) = 1$, then g is a linear element of L. If H° contain two linearly independent linear elements, then $H = L$. Therefore, we may suppose that $u^\circ = [f,[x,y]] + [g,\ell]$, where ℓ is a linear element. Then $[g,\ell] \in L^{(2)}$. By Lemma 3.7, $[g,\ell] = 0$. Hence $f = [[x,y],x]$, and $[[x,y],x],[x,y] \in H^\circ$, and H contains elements of the form $a = [[x,y],x] + \ell_1$, $b = [x,y] + \ell_2$, where ℓ_1 and ℓ_2 are linear elements. Consider the element

$$w = u - [a,b] = [x,y] + [[x,y], x + \ell_1] - [x,y,x,\ell_2] \in [H,H].$$

Suppose that $\ell_2 = 0$. If $x + \ell_1 \neq 0$, then $w^\circ = [[x,y], x+\ell_1] \in [H^\circ, H^\circ]$. It follows that $x + \ell_1 \in H$. Therefore $w - w^\circ = [x,y] \in [H,H]$, and $H = L$. If $x + \ell_1 = 0$, then $[x,y] \in [H,H]$, and again $H = L$.

If $\ell_2 \neq 0$, then

$$w^\circ = -[x,y,x,\ell_2] \in [H^\circ, H^\circ].$$

This is possible only if $[x,y,x,\ell_2] = [g,\ell]$, where ℓ is a nonzero linear element of H, and g is a homogeneous element of H° of degree 3. By Lemma 3.7, either $\ell = \alpha\ell_2$ or $\ell = \alpha x$.

If $\ell_2 \in H$, then $[x,y] = b - \ell_2 \in H$. As in the case $\ell_2 = 0$, we get $H = L$. Let $\ell = x \in H$. Then we may assume that $a = [[x,y],x] + \beta y$, $b = [x,y] + \gamma y \in H$, and $0 \neq \gamma \in K$. Therefore

$$a - [b,x] - \gamma b = (\beta - \gamma^2) y \in H.$$

Since $H \neq L$, $\beta = \gamma^2$, and

$$w = u - [a,b] = [x,y] + [[x,y], x + \gamma^2 y] - \gamma[x,y,x,y] \in [H,H].$$

It follows that $w^\circ = -\gamma[x,y,x,y] \in [H^\circ, H^\circ]$. The element $[x,y,x,y]$ has the unique such presentation, $[x,y,x,y] = [x,y,y,x]$. Therefore $[[x,y],y] \in H^\circ$, and $c = [[x,y],y] + \delta y \in H$. Consider the element

$$w_1 = w + \gamma[c,x] = (1 - \gamma\delta)[x,y] + [[x,y], \alpha + \gamma^2 y] \in [H,H].$$

We have $w_1^\circ = [[x,y], x + \gamma^2 y] \in [H^\circ, H^\circ]$. Therefore $x + \gamma^2 y \in H$. Since $\gamma \neq 0$, $H = L$.

We showed that in all cases $H = L$. Thus u is a $\mathfrak{A}$-generic element of L. □

Let $\mathfrak{N}_m$ be the variety of nilpotent Lie algebras of degree not less than m. This variety is given by the identity $[x_1, x_2, \ldots, x_m] = 0$, where $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$, $[x_1, \ldots, x_m] = [[x_1, \ldots, x_{m-1}], x_m]$.

Theorem 3.9. *Let $X = \{x_1, \ldots, x_n\}$. The element $u = [x_1, \ldots, x_n]$ is an $\mathfrak{N}_n$-generic element of the free Lie algebra $L(X)$.*

Proof. Let H be a subalgebra of $L(X)$, and $\{h_1, \ldots, h_m\}$ a reduced set of free generators of H. Consider the lower central series of H: $\gamma_2(H) = [H,H]$ and $\gamma_{i+1}(H) = [\gamma_i(H), H]$. Suppose that the element u belongs to $\gamma_n(H)$. Then

$$u = \sum_{i=1}^{k} [g_1^{(i)}, \ldots, g_n^{(i)}],$$

where $g_1^{(i)}, \ldots, g_n^{(i)} \in H$, $1 \leq i \leq k$. It follows that the element $u^\circ = u$ is a Lie polynomial of degree n in $h_1^\circ, \ldots, h_m^\circ$. It is possible only if u depends on

$h_{i_1}, \ldots, h_{i_n}$, $l(h_{i_1}) = \ldots = l(h_{i_n}) = 1$. But in this case $H = L$. Thus u is an $\mathfrak{N}_n$-generic element. □

For free groups the statement of Theorem 3.9 was proved by V. Shpilrain [20]. Remark that $[x_1, x_2, x_3]$ is an $\mathfrak{N}_3$-generic element of $L(x_1, x_2, x_3)$, but it is not an APE.

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