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ON THE FAILURE OF THE FACTORIZATION CONDITION FOR NON-DEGENERATE FOURIER INTEGRAL OPERATORS

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Abstract. In this paper we give examples of polynomial phase functions for which the factorization condition of Seeger, Sogge and Stein (Ann. Math. **134** (1991)) fails. The corresponding Fourier integral operators turn out to be still continuous in L^p. We also give examples of the failure of the factorization condition for translation invariant operators. In this setting the frequency space must be at least 5-dimensional, which shows that the examples are optimal. We briefly discuss the stationary phase method for the corresponding operators.

Let X and Y be open subsets of $\mathbb{R}^n$. A Fourier integral operator $T \in I^\mu(X,Y;\Lambda)$ is an operator which can be locally written in the form

$$Tu(x) = \int_Y \int_{\mathbb{R}}^n e^{i\Phi(x,y,\theta)} a(x,y,\theta) u(y) d\theta dy,$$

where $a \in S^\mu(X,Y,\mathbb{R}^n)$ is a symbol of order m, i.e. a smooth function with the property that

$$|\partial_x^\beta \partial_\xi^\alpha a(x,y,\theta)| \leq C(1+|\xi|)^{\mu-|\alpha|},$$

locally uniformly in x,y for all multi-indices α and β. The canonical relation Λ is a conic Lagrangian submanifold of the cotangent bundle $T^*(X\times Y)\backslash 0$ with the symplectic form $\sigma_X \bigoplus -\sigma_Y$, where σ_X and σ_Y are the canonical symplectic forms in T^*X and T^*Y respectively. Let $\pi_{X\times Y}$ be the canonical projection from $T^*(X\times Y)$ to $X\times Y$. The canonical relation Λ can be locally parametrized as the set of points

$$\Lambda_\Phi = \{(x,y,d_x\Phi,d_y\Phi) : d_\theta\Phi(x,y,\theta)=0\}.$$

We will assume that Λ is a local canonical graph, which means that $\partial_y\partial_\theta\Phi$ is a non-degenerate matrix. The regularity properties of Fourier integral operators are related to the geometric properties of Λ. Let Λ satisfy the smooth factorization condition. This means that for every $\lambda = (x_0,y_0,\xi_0,\eta_0) \in \Lambda$ there is a conic neighborhood Λ_0 of λ_0 in Λ and a smooth map $\pi_{\lambda_0} : \Lambda_0 \to \Lambda$ homogeneous of degree 0 such that $\operatorname{rank} d\pi_{\lambda_0} \equiv n+k$ and $\pi_{X\times Y}|_{\Lambda_0} = \pi_{X\times Y}\circ\pi_{\Lambda_0}$, for some k. Under this condition it was shown in [7] that operators $T \in I^\mu(X,Y;\Lambda)$ are bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(Y)$, provided that $1<p<\infty$ and $\mu \leq -k|1/p-1/2|$. It was also shown in [4] that the order $-k|1/p-1/2|$ is optimal; that is, for $\mu > -k|1/p-1/2|$ elliptic operators $T \in I^\mu(X,Y;\Lambda)$ are not bounded in L^p. For general properties of

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operators with ranks k and their relation to the singularity theory of affine fibrations we refer to [5] for the real valued phase functions, and to [6] for the complex valued phase functions, respectively. For the backgrounds on the L^p theory of Fourier integral operators we refer to [8], [9], and a survey [5] for smaller ranks k.

In this paper we will consider a case for which the factorization condition fails. We give an example of an operator for which the factorization condition fails but the L^p result holds. Such a family of phase functions was suggested in [3]. Let X, Y be open subsets of $\mathbb{R}^3$ and define Φ by

$$\Phi(x,y,\xi) = \langle x-y,\xi\rangle - \frac{1}{\xi_3}(y_1\xi_1 + y_2\xi_2)^2 \tag{1}$$

in the cone $|(\xi_1,\xi_2)| \le C|\xi_3|$ for some $C>0$. The factorization condition clearly fails for this phase function. The maximal rank of $d\pi_{X\times Y}|_\Lambda$ equals 4, so $k=1$ and it follows from [4] that the best order for the L^p continuity can be $-|1/p-1/2|$.

Theorem 1. *Let $T \in I^\mu(\mathbb{R}^3,\mathbb{R}^3;\Lambda_\Phi)$ with Λ_Φ defined by* (1). *Then T is bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(Y)$, provided that $1<p<\infty$ and $\mu \le -|1/p-1/2|$.*

This statement can be generalized to higher dimensions, but this is not the purpose of the paper. Our point is to present operators for which the factorization condition fails but L^p estimates are still valid.

We will give a brief proof of this result based on the technique and notations of [7]. The operator T is defined by

$$Tu(x) = \int\int e^{i\Phi(x,y,\xi)} b(x,y,\xi) u(y) d\xi dy \tag{2}$$

and the support of its symbol $b(x,y,\xi)$ away from $\xi_3 = 0$. The set $\Sigma = \pi_{X\times Y}(\Lambda_\Phi) \subset \mathbb{R}^3 \times \mathbb{R}^3$ can be represented by the set of points $(\nabla_\xi \phi(y,\xi), y)$ parametrized by ξ, where

$$\phi(y,\xi) = \langle y,\xi\rangle + \frac{1}{\xi_3}(y_1\xi_1 + y_2\xi_2)^2.$$

In a neighborhood of $x=y$ the set Σ_y after a choice $\sigma = (y_1\xi_1 + y_2\xi_2)/\xi_3$ can be parametrized by

$$\Sigma_y = \{(y_1 + 2y_1\sigma, y_2 + 2y_2\sigma, y_3 - \sigma^2)\}.$$

By the analytic interpolation technique it is sufficient to check that operators in (2) are bounded from the Hardy space H^1 to L^1 when the order of b is $-1/2$ (see [9] or [7] for details). From the atomic decomposition of Hardy space, it is sufficient to check $||Ta_Q||_{L^1} \le C$ for any atom a_Q with C independent of a_Q and a cube Q with a small sidelength. Recall that a_Q is supported in a cube Q and satisfies $|a_Q| \le |Q|^{-1}$ almost everywhere as well as the cancellation property $\int a_Q(x)dx = 0$.

From the atomic decomposition we need to consider only the atoms a_Q with Q containing the points where the rank of $\phi''_{\xi\xi}$ drops, because otherwise T is conormal in Q. We also assume $|Q| \le 1$. The singularities of Σ occur at the points $y = (0,0,y_3)$. For $y_1^2 + y_2^2 \ne 0$ we denote by N^y a tubular neighborhood of Σ_y with width $|Q|^{2/3}$. Then $|N^y| \le c|Q|^{2/3}$. Now, for y with $y_1 = y_2 = 0$ the singular set Σ_y is a point, but we enlarge it by taking a limit of Σ_y with $y_1^2 + y_2^2 \to 0$, and the limit is a straight ray from y. For these y we define N^y in a similar way as a tubular neighborhood of this ray with width $|Q|^{2/3}$. Finally, we define

$$N_Q = \bigcup_{y \in Q} N^y.$$

The size of N_Q is $|N_Q| \le C|Q|^{2/3}$. Now we want to estimate the L^1 norm of Ta_Q on N_Q. By Cauchy-Schwartz inequality we have

$$||Ta_Q||_{L^1(N_Q)} \le C|Q|^{1/3}||Ta_Q||_{L^2(N_Q)}.$$

Since T is of order $-1/2$, the operator $T(I-\Delta)^{1/4}$ is bounded on L^2, and hence we get

$$||Ta_Q||_{L^2} \le C||(I-\Delta)^{-1/4}a_Q||_{L^2} \le C||a_Q||_{p_n},$$

the last inequality following from the Hardy-Littlewood-Sobolev inequality with $p_n = 3/2$. Now, using $||a_Q||_\infty \le |Q|^{-1}$, we obtain $||a_Q||_{3/2} \le |Q|^{-1}|Q|^{2/3}$, so that

$$||Ta_Q||_{L^1(N_Q)} \le C|Q|^{1/3}|Q|^{-1}|Q|^{2/3} \le C.$$

Next we want to estimate in $L^1(\mathbb{R}^3 \backslash N_Q)$.

1. First we decompose T into a finite sum

$$T = \sum_{l=1}^{L} T_l, \tag{3}$$

so that the fibers of the Lagrangian for each T_l are close to each other. Let $\{R_l\}_1^L$ be a decomposition of (y_1, y_2)-space $\mathbb{R}^2$ into sectors of equal angle $2\pi/L$ and all the lines starting from zero. Let α_l be a partition of unity, homogeneous of degree 0 and related to $R_l \cap S^1$. Define $T_l = T \circ \alpha_l$, where α_l means multiplication by it. Then (3) holds and it is enough to make estimates for some T_l. In view of this decomposition we will assume further that the symbol $b(x, y, \xi)$ of T in (2) is supported in some R_l with respect to y_1 and y_2. The set of $(0, 0, y_3)$ is of measure zero, so we can exclude it from the decomposition.

2. Now we make a dyadic decomposition in ξ-space. Let $\beta \in C_0^\infty((1/2, 2))$ satisfy $\sum_{-\infty}^{\infty} \beta(2^{-k}s) = 1$, $s > 0$. We define

$$b_\lambda(x, y, \xi) = \beta(|\xi|/\lambda) b(x, y, \xi)$$

and

$$T_\lambda u(x) = \int\int e^{i\Phi(x,y,\xi)} b_\lambda(x, y, \xi) u(y) d\xi dy.$$

The corresponding dyadic decomposition of T is now

$$T = \sum_{k \ge 1} T_{2^k}. \tag{4}$$

3. We will also need a further angular decomposition of T_λ. In order to accomplish it we make a partition of the unit sphere in ξ-space, related to the smooth factorization property. Let Γ be a narrow cone in ξ-space, containing the support of b. For each $y \in \overset{\circ}{R_l}$ there exists an r-dimensional submanifold $S_r(y)$, $r = 1$, of $S^2 \cap \Gamma$, such that $S^2 \cap \Gamma$ is parametrized by $\bar{\xi} = \bar{\xi}_y(u, v)$ for (u, v) in a bounded open set $U \times V$ near $(0, 0) \in \mathbb{R} \times \mathbb{R}$. Furthermore,

$$\bar{\xi}_y(u, v) \in S_r(y) \Leftrightarrow v = 0$$

and

$$\nabla_\xi \phi\left(y, \bar{\xi}_y(u,v)\right) = \nabla_\xi \phi\left(y, \bar{\xi}_y(u,0)\right),$$

so that v defines a parametrization of the fibers. The set $S_r(y)$ depends smoothly on $y \in \overset{\circ}{R}_l$ but has unbounded variation as $y_1^2 + y_2^2 \to 0$. However, the image set of $S_r(y)$ can be made small by choosing large L.

This implies that U is bounded uniformly with respect to $y \in \overset{\circ}{R}_l$ and for any $\lambda = 2^k$, $k > 0$ we can choose u_λ^ν, $\nu = 1, 2, \ldots, N(\lambda)$, such that $|u_\lambda^\nu - u_\lambda^{\nu'}| \geq C_0 \lambda^{-1/2}$ for $\nu \neq \nu'$, and such that U is covered by balls with center u_λ^ν and radius $C_1 \lambda^{-1/2}$. Note that $N(\lambda) = O(\lambda^{1/2})$.

4. We introduce homogeneous partitions of unity of $\mathbb{R}^3 \backslash 0$ that depend on the scale λ of the dyadic decomposition. First let $\bar{\chi}_\lambda^\nu$ be a smooth partition of unity in U, satisfying $||D_u^\gamma \bar{\chi}_\lambda^\nu||_\infty = O(\lambda^{|\gamma|/2})$, and having the natural support properties associated to the partition u_λ^ν, namely $\bar{\chi}_\lambda^\nu(u_\lambda^\nu) = 1$ and $\bar{\chi}_\lambda^\nu(u) = 0$ if $|u - u_\lambda^\nu| \geq C\lambda^{-1/2}$. Then we define a corresponding partition of unity on Γ by $\chi_\lambda^\nu(s\bar{\xi}_y(u,v)) = \bar{\chi}_\lambda^\nu(u)$, $s > 0$.

The idea behind this decomposition is that the χ_λ^ν have the largest possible angular support so that $\xi \to \phi$ behaves like a linear function on supp b_λ^ν, where T_λ^ν is an operator with kernel

$$K_\lambda^\nu(x,u) = \int e^{i[\langle x,\xi\rangle - \phi(y,\xi)]} b_\lambda^\nu(x,y,\xi) d\xi, \; b_\lambda^\nu(x,y,\xi) = \chi_\lambda^\nu(\xi) b_\lambda(x,y,\xi).$$

5. On the support of $b_\lambda^\nu(x,y,\xi)$ the idea is to replace the function $\phi(y,\xi)$ by its linear approximation $\langle \nabla_\xi \phi(y, \bar{\xi}_y(u_\lambda^\nu, 0), \xi\rangle$. We define

$$r_\lambda^\nu(y,\xi) = \phi(y,\xi) - \langle \nabla_\xi \phi(y, \bar{\xi}_y(u_\lambda^\nu, 0), \xi\rangle.$$

Then for $N \geq 1$ and ξ in the support of $b_\lambda^\nu(x,y,\xi)$ the following holds:

$$|(\langle \nabla_\xi, \bar{\xi}_y(u_\lambda^\nu, 0)\rangle)^N r_\lambda^\nu(y,\xi)| \leq C_N \lambda^{-1} |\xi|^{1-N}, \tag{5}$$

$$D_\xi^\alpha r_\lambda^\nu(y,\xi)| \leq C_N \min\{\lambda^{-1/2}, |\xi|^{1-N}\}, \; |\alpha| = N. \tag{6}$$

Note that the term $|\xi|^{1-N}$ corresponds to the homogeneous behavior of $r_\lambda^\nu(y,\xi)$. So we need to show it for $\xi \in S^2 \cap$ supp χ_λ^ν. In view of Euler formula $r_\lambda^\nu(y, \bar{\xi}_y(u_\lambda^\nu, 0)) = 0$ and $\nabla_\xi r_\lambda^\nu(y, \bar{\xi}_y(u_\lambda^\nu, 0)) = 0$, so that the Taylor expansion of $r_\lambda^\nu(y,\xi)$ around $\bar{\xi}_y(u_\lambda^\nu, 0)$ implies (5) since $|\xi - \bar{\xi}_y(u_\lambda^\nu, 0)| \leq C\lambda^{-1/2}$ for $\xi \in S^2 \cap$ supp χ_λ^ν. Similarly, for (6) we observe that $|\xi|^{1-N} \leq C\lambda^{-1/2}$ if $N > 1$ and $\xi \in$ supp b_λ^ν and $\nabla_\xi r_\lambda^\nu(y,\xi) = \nabla_\xi \phi_\lambda^\nu(y,\xi) - \nabla_\xi \phi_\lambda^\nu(y, \bar{\xi}_y(u_\lambda^\nu, 0)) = O(\lambda^{-1/2})$.

6. Finally, we define

$$\tilde{b}_\lambda^\nu(x,y,\xi) = e^{ir_\lambda^\nu(y,\xi)} b_\lambda^\nu(x,y,\xi).$$

By a rotation, we assume that for every $\xi \in \Gamma$ there is the splitting $\xi = (\xi', \xi'') \in \mathbb{R} \times \mathbb{R}^2$, such that ξ'' is normal to $S_r(y)$ at $\bar{\xi}_y(u_k^\nu, 0)$. The stationary phase partial integrations are performed with a selfadjoint operator

$$L_\lambda^\nu = (I - \lambda\langle \nabla'_\xi, \nabla'_\xi\rangle)(I - \lambda^2 \langle \nabla''_\xi, \nabla''_\xi\rangle).$$

The estimates for $\bar{\chi}_\lambda^\nu$ and the fact that $b(x,y,\xi)$ is of order $-1/2$ imply that

$$|(L_\lambda^\nu)^N b_\lambda^\nu(x,y,\xi)| \leq C_N \lambda^{-1/2}.$$

Furthermore, the same estimate holds for $\tilde{b}^\nu_\lambda(x,y,\xi)$ instead of $b^\nu_\lambda(x,y,\xi)$ in view of (5) and (6). Integration by parts gives

$$K^\nu_\lambda(x,y) = H^\nu_{N,\lambda}(x,y)\int e^{i\langle x-\nabla_\xi\phi(y,\bar{\xi}_y(u^\nu_\lambda,0),\xi\rangle}(L^\nu_\lambda)^N\tilde{b}^\nu_\lambda(x,y,\xi)d\xi, \tag{7}$$

where

$$H^\nu_{N,\lambda}(x,y) = (1+\lambda|(x-\nabla_\xi\phi(y,\bar{\xi}_y(u^\nu_\lambda,0))'|^2)^{-N}(1+\lambda^2|(x-\nabla_\xi\phi(y,\bar{\xi}_y(u^\nu_\lambda,0))''|^2)^{-N}.$$

The estimate for $|(L^\nu_\lambda)^N\tilde{b}^\nu_\lambda(x,y,\xi)|$ and the fact that the support in the integral in (7) has volume $O(\lambda^{1/2}\lambda^2)$ imply that

$$|K^\nu_\lambda(x,y)| \le C_N\lambda^2 H^\nu_{N,\lambda}(x,y).$$

7. For fixed value of y this implies

$$\int_{\mathbb{R}^3}|K^\nu_\lambda(x,y)|dx \le C\lambda^2\lambda^{-1/2}\lambda^{-2} = C\lambda^{-1/2}.$$

The number of kernels $K^\nu_\lambda(x,y)$ was $N(\lambda)=O(\lambda^{1/2})$, so that we obtain

$$\int_{\mathbb{R}^3}|K_\lambda(x,y)|dx \le C.$$

Similarly, with the decompositions above, the rest of the proof is standard as in [7].

The statement of Theorem 1 can be clearly generalized to higher dimensions. The point of this paper is, however, to show that the optimal L^p estimates hold even in some cases when the factorization condition fails. Let us now give an example of a translation invariant operator for which the factorization condition fails. It is an interesting problem to investigate the L^p properties of such a Fourier integral operator.

By the invariant wave front we mean the wave front of a translation invariant distributional kernel $K(x,y)=K(x-y)$ which corresponds to the convolution operators. Let

$$\Phi(x,y,\xi) = \langle x,\xi\rangle - \phi(y,\xi) \tag{8}$$

be a non-degenerate phase function, smooth in y and positively homogeneous of degree one in ξ, where x,y,ξ are in some open subsets of $\mathbb{R}^n$, and $\det\phi''_{y\xi}\neq 0$. Let U be open in $\mathbb{R}^n\times\mathbb{R}^n$ and let $k=\max_{(y,\xi)\in U}\operatorname{rank}\phi''_{\xi\xi}(y,\xi)$. Denote by $U^{(k)}$ the set of $(y,\xi)\in U$ such that the rank is equal to k. By the implicit function theorem the mapping $\kappa: U^{(k)}\ni(y,\xi)\mapsto\ker\phi''_{\xi\xi}(y,\xi)\in\mathbb{G}_{n-k}(\mathbb{R}^n)$ is smooth. The factorization condition then means that κ extends smoothly from $U^{(k)}$ to U. Since Φ is homogeneous, the factorization condition always holds when $k=0$ or $k=n-1$. In terms of the phase function, translation invariance means that $\phi(y,\xi)=\langle y,\xi\rangle - H(\xi)$, where H is smooth and positively homogeneous of degree one. We will write $\kappa(\xi)$ for $\kappa(y,\xi)$. In [5], we have shown that if H is analytic, the smallest dimension when the factorization condition may fail, is $n=5$. Moreover, the factorization condition holds if $k\le 2$. There is a general explanation for it based on the fact that $\nabla\phi$ is constant along $\kappa(\xi)$ ([5]).

Thus, the smallest dimension when it may fail is $n=5$ with $k=3$. Let $m,l\ge 2$ and consider the function

$$\Phi(x,y,\xi) = \langle x-y,\xi\rangle + \xi_1\xi_2^l\xi_5^{-l} + (\xi_3\xi_5-\xi_2\xi_4)^m\xi_5^{1-2m} \tag{9}$$

in a conic neighborhood of $\xi_5 = 1$. At $\xi_5 = 1$, we get

$$\kappa(\xi) = \text{ span } \langle (\frac{l}{m} \frac{(\xi_3 - \xi_2 \xi_4)^{m-1}}{\xi_2^{l-1}}, 0, \xi_2, 1) \rangle.$$

Therefore, at $\xi_2 = \xi_3 = 0$ and $\xi_5 = 1$, κ is discontinuous and the factorization condition fails. There is also an obvious generalization of (9) to higher dimensions (see [5]).

It is an interesting problem to verify the regularity properties of the corresponding Fourier integral operators $T \in I^\mu(\mathbb{R}^n, \mathbb{R}^n; \Lambda')$, where $\Lambda = \Lambda_\Phi$ is the graph of a canonical symplectic transformation from $T^*\mathbb{R}^n$ to $T^*\mathbb{R}^n$ (see [2], [7], [6] for the notation). If the singular support of T is not smooth, the factorization condition may still hold. In general, it is shown in [4] that if T is elliptic and continuous from L^p_{comp} to L^p_{loc}, $1 < p < \infty$, then $\mu \leq -k|1/p - 1/2|$. A standard argument for such negative results is based on the stationary phase method which is essentially contained in [1]. Namely, one observes that if $P \in \Psi^{-s}(\mathbb{R}^n)$ is an elliptic properly supported pseudodifferential operator of order $-s$ and type $(1, 0)$, then $f = P\delta \in L^p(\mathbb{R}^n)$ when $s > n(1 - 1/p)$ and δ is a standard δ-function. It is sufficient to consider $1 < p < 2$ since the rest follows by taking adjoints. One can analyze $Tf = (T \circ P)\delta$ explicitly to see that μ has to be $\leq -k|1/p - 1/2|$ if T is L^p continuous. It is possible to check in a number of cases that if the phase function of an elliptic operator T is given by (9), then $\mu = -k|1/p - 1/2|$ implies $Tf \in L^p$.

Therefore, since the stationary phase test with functions with point singularities fails for T, it is reasonable to conjecture that if Φ is given by (9), the operators of order $-3|1/p - 1/2|$ are L^p continuous. Note that by the complex interpolation method this would follow from the boundedness from the Hardy space $H^1(\mathbb{R}^5)$ to $L^1(\mathbb{R}^5)$ of operators of order $-3/2$ with phase function (9).

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