MINIMAL LAGRANGIAN SUBMANIFOLDS
WITH CONSTANT SECTIONAL CURVATURE
IN INDEFINITE COMPLEX SPACE FORMS

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ABSTRACT. We study minimal Lagrangian immersions from an indefinite real space form $M_s^n(c)$ into an indefinite complex space form $\tilde{\mathbb{M}}_s^n(4\tilde{c})$. Provided that $c \neq \tilde{c}$, we show that $M_s^n(c)$ has to be flat and we obtain an explicit description of the immersion. In the case when the metric is positive definite or Lorentzian, this result was respectively obtained by Ejiri (1982) and by Kriele and the author (1999). In the case that $c = \tilde{c}$, this theorem is no longer true; see for instance the examples discovered by Chen and the author (accepted for publication in the Tôhoku Mathematical Journal).

1. INTRODUCTION

Whereas Lagrangian submanifolds of positive definite complex space forms are widely studied (see a.o. [6] and the references contained therein), not much is known about Lagrangian submanifolds of indefinite complex space forms. As far as we know most results about submanifolds of indefinite complex space forms (see for example [1], [8], [9], [10] and [11]) deal with complex submanifolds. In this paper we study minimal Lagrangian submanifolds $M_s^n(c)$ with constant sectional curvature of indefinite complex space forms $\tilde{\mathbb{M}}_s^n(4\tilde{c})$. Provided $c \neq \tilde{c}$, we obtain a complete classification and show amongst others that M has to be flat. The corresponding theorems for Lagrangian immersions in Riemannian or Lorentzian complex space forms were obtained respectively by Ejiri in [4] and Kriele and the author in [5].

The paper is organized as follows. In Section 2 we recall the basic models of indefinite complex space forms and we give the basic formulas for Lagrangian submanifolds. In particular, we notice that the basic formulas are similar to those of affine hyperspheres with constant sectional curvature. This similarity allows us to apply the results of [12] to obtain an intrinsic characterization of these submanifolds and a special local frame. In Section 3, we start by recalling a result of Reckziegel [7] which allows us, using the Hopf fibration, to consider the horizontal lifts of these immersions into indefinite real space forms. We also present some examples of flat indefinite Lagrangian submanifolds. We then apply a coordinate change, motivated

by the explicit form of these examples. Using these new coordinates, together with Reckziegel's result, the classification follows easily. The main difference with the approach of [4] and [5] is the use of the uniqueness part of the solution to a linear algebra problem obtained in [12] together with the explicit form of an example in order to change the coordinates before integrating.

2. Indefinite complex space forms and their Lagrangian submanifolds

In this section, we briefly recall some facts about indefinite complex space forms. For more details, we refer the reader to [2]. Let $\tilde{\mathbb{M}}_s^n(4\tilde{c})$ be an indefinite complex space form of complex dimension n and complex index s. The complex index is defined as the (complex) dimension of the largest complex negative definite vector subspace of the tangent space. The curvature tensor $\tilde{R}$ of $\tilde{\mathbb{M}}_s^n(4\tilde{c})$ is given by

$$\tilde{R}(X,Y)Z = \tilde{c}\big(\langle Y,Z \rangle X - \langle X,Z \rangle Y + \langle JY,Z \rangle JX - \langle JX,Z \rangle JY + 2 \langle X,JY \rangle JZ \big),$$

where J denotes the complex structure. We refer to [2] for the construction of the standard models of indefinite complex space forms $\mathbb{C}P_s^n(4\tilde{c})$, when $\tilde{c} > 0$, $\mathbb{C}H_s^n(4\tilde{c})$, when $\tilde{c} < 0$ and $\mathbb{C}_s^n$. For our purposes it is sufficient to know that there exist pseudo-Riemannian submersions, called Hopf fibrations,

$$\Pi : S_{2s}^{2n+1}(\tilde{c}) \to \mathbb{C}P_s^n(4\tilde{c}) : z \mapsto z \cdot \mathbb{C}^\star$$

if $\tilde{c} > 0$ and if $\tilde{c} < 0$ by

$$\Pi : H_{2s+1}^{2n+1}(\tilde{c}) \to \mathbb{C}H_s^n(4\tilde{c}) : z \mapsto z \cdot \mathbb{C}^\star,$$

where

$$S_{2s}^{2n+1}(\tilde{c}) = \{z \in \mathbb{C}^{n+1} | b_{s,n+1}(z,z) = \tfrac{1}{\tilde{c}}\},$$
$$H_{2s+1}^{2n+1}(\tilde{c}) = \{z \in \mathbb{C}^{n+1} | b_{s+1,n+1}(z,z) = \tfrac{1}{\tilde{c}}\}$$

and $b_{p,q}$ is the standard Hermitian form with index p on $\mathbb{C}^q$. For our convenience, we will assume that we have chosen an orthonormal basis such that the first p odd terms appear with a minus sign.

In [2] it is shown that locally any indefinite complex space form is holomorphically isometric to either $\mathbb{C}_s^n$, $\mathbb{C}P_s^n(4\tilde{c})$, or $\mathbb{C}H_s^n(4\tilde{c})$. We remark that, by replacing the metric $\langle .,. \rangle$ by $-\langle .,. \rangle$, we have that $\mathbb{C}H_s^n(4\tilde{c})$ is holomorphically anti-isometric with $\mathbb{C}P_{n-s}^n(-4\tilde{c})$. For that purpose, we may assume that $n - 2s \geq 0$ and if $n - 2s = 0$, we only need to consider $\mathbb{C}_s^n$ and $\mathbb{C}P_s^n(4\tilde{c})$.

Next, we consider Lagrangian submanifolds. A submanifold M of a Kähler manifold is Lagrangian if and only if J interchanges the tangent and the normal space. Hence a Lagrangian submanifold of an indefinite complex space form of index s has real index s. From now on let $\tilde{\mathbb{M}}_s^n(4\tilde{c})$ be an indefinite complex space form of holomorphic sectional curvature $4\tilde{c}$. We denote by $\tilde{\nabla}$ the Levi Civita connection of the metric $\langle .,. \rangle$ on $\tilde{\mathbb{M}}_s^n(4\tilde{c})$. Then, the formulas of Gauss and Weingarten are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\tilde{\nabla}_X JY = -A_{JY} X + \nabla_X^\perp JY,$$

defining the induced connection ∇, the second fundamental form h, the Weingarten operator A and the normal connection $\nabla^\perp$. Since J is parallel, we deduce as in the Riemannian case that

$$\nabla_X^\perp JY = J\nabla_X Y,$$
$$A_{JY} X = -Jh(X, Y).$$

We now introduce a tensor K on M by $K(X, Y) = K_X Y = -Jh(X, Y) = A_{JY} X$. It follows from the above equations that $\langle K(X, Y), Z \rangle$ is totally symmetric. A straightforward computation shows that the equations of Gauss, Codazzi and Ricci for Lagrangian submanifolds are equivalent to

(2.1) $$R(X, Y)Z = \tilde{c}(\langle Y, Z \rangle X - \langle X, Z \rangle Y) + [K_X, K_Y]Z,$$
(2.2) $$(\nabla_X K)(Y, Z) = (\nabla_Y K)(X, Z).$$

A Lagrangian submanifold M_s^n with constant sectional curvature c is minimal if and only if

(2.3) $$\operatorname{trace} K_X = 0$$

for every tangent vector field X. The Gauss and Codazzi equations reduce to

(2.4) $$[K_X, K_Y]Z = a(\langle Y, Z \rangle X - \langle X, Z \rangle Y),$$
(2.5) $$(\nabla_X K)(Y, Z) = (\nabla_Y K)(X, Z),$$

where $a = c - \tilde{c}$.

Remark that, since ∇ is the Levi Civita connection of the metric $\langle ., . \rangle$, this is exactly the situation which also appears in affine differential geometry when one investigates affine hyperspheres with constant sectional curvature; see [12]. Therefore, recalling the results of [12], we obtain in particular the following:

Theorem 2.1. *Let a be a nonzero number, let n be the dimension and let s be a nonnegative integer number with $n - 2s \geq 0$. Then:*

  (i) *If $n - 2s > 1$ and a is negative, then there does not exist a metric $\langle ., . \rangle$ (with signature s) and a tensor K, which is totally symmetric with respect to $\langle ., . \rangle$, on $\mathbb{R}^n$ which satisfy (2.3) and (2.4).*
  (ii) *Otherwise, solutions $\langle ., . \rangle$ (with index s) and K do exist. Moreover any other solution $(\tilde{h}, \tilde{K})$ is obtained by*

$$\tilde{h}(x, y) = \langle Ax, Ay \rangle,$$
$$\tilde{K}(x, y) = A^{-1} K(Ax, Ay),$$

  where A is an arbitrary linear automorphism of $\mathbb{R}^n$.

Theorem 2.2. *Let $\phi : M_s^n(c) \to \tilde{M}_s^n(4\tilde{c})$ be a minimal Lagrangian isometric immersion. Suppose that $n - 2s \geq 0$ and that $c \neq \tilde{c}$. Then, if $n - 2s > 1$, we must have that $c < \tilde{c}$. Moreover, let $p \in M_s^n(c)$. Then there exist vector fields $\{U_1, V_1, \ldots, U_s, V_s, E_1, \ldots, E_r\}$, defined on a neighborhood of p, where $r = n - 2s$ such that $\langle E_k, E_\ell \rangle = \delta_{k\ell}$, $\langle U_m, V_j \rangle = \delta_{mj}$, where $j, m = 1, \ldots, s$ and $k, \ell = 1, \ldots, r$ and all other components of the metric vanish. Assuming moreover that $\ell < k$ and*

$m < j$, we have that K is given by

U_j, V_j and E_k are eigenvectors of K_{U_m} with eigenvalue λ_m,

U_j, V_j and E_k are eigenvectors of K_{V_m} with eigenvalue $\frac{(n-2m+2)}{2}\lambda_m^2$,

$K(U_j, U_j) = -\frac{n-2j}{2}\lambda_j U_j + V_j$,

$K(U_j, V_j) = \sum_{m=1}^{j-1}(\lambda_m V_m + \frac{(n-2m+2)}{2}\lambda_m^2 U_m) - \frac{(n-2j)(n-2j+2)}{4}\lambda_j^2 U_j - \frac{(n-2j)}{2}\lambda_j V_j$,

$K(V_j, V_j) = \frac{(n-2j+2)^3}{8}\lambda_j^3 U_j - \frac{1}{4}(n-2j)(n-2j+2)\lambda_j^2 V_j$,

$K(E_k, E_\ell) = -\frac{\mu_\ell}{r-\ell}E_k$,

$K(E_k, E_k) = \sum_{j=1}^{s}(\lambda_j V_j + \frac{(n-2j+2)}{2}\lambda_j^2 U_j) - \sum_{\ell=1}^{k-1}\frac{\mu_\ell}{r-\ell}E_\ell + \mu_k E_k$,

where the μ_k and λ_j are constants determined by

$$(2.6) \qquad \lambda_j^3 = -\frac{2a(n+1)}{(n-2j+2)(n-2j+1)(n-2j+3)},$$

$$(2.7) \qquad \mu_j^2 = -\frac{a(n+1)(r-j)^2}{(r-j+1)(r-j+2)}.$$

Using the fact that K is a Codazzi tensor with respect to ∇, (2.5), it is then straightforward; see also [12] to show the following:

Lemma 2.3. *Let $\{U_1, V_1, \ldots, U_s, V_s, W_1, \ldots, W_r\}$ be the frame constructed before. Then all connection coefficients (with respect to ∇) vanish. In particular M has flat metric, i.e. $c = 0$.*

3. Classification results

Using the results of the previous section, we already obtain the following corollaries:

Corollary 3.1. *Let M_s^n be a minimal Lagrangian submanifold with constant sectional curvature c of $\mathbb{C}H_s^n(4\tilde{c})$, $n - 2s > 1$, where $\tilde{c} < 0$. Then $c = \tilde{c}$.*

Proof. Suppose that $c \neq \tilde{c}$. In that case, $a \neq 0$. Since $n - 2s > 1$, it follows from the previous theorem (2.7) that a has to be negative. Since, from the previous lemma, we know that our Lagrangian submanifold has to be flat, we find that $\tilde{c}$ is positive. This is a contradiction. $\qquad\square$

Of course, also in the flat case, a similar theorem can be obtained:

Corollary 3.2. *Let M_s^n be a minimal Lagrangian submanifold with constant sectional curvature c of $\mathbb{C}_s^n$. Then $c = 0$.*

In general, applying the results of the previous chapters, a classification result can be proved. However, in order to obtain explicit equations, we first recall some basic facts from [7] which relate Lagrangian submanifolds of respectively $\mathbb{C}P_s^n(4\tilde{c})$ and $\mathbb{C}H_s^n(4\tilde{c})$ to horizontal immersions in respectively $S_{2s}^{2n+1}(\tilde{c})$ and $H_{2s+1}^{2n+1}(\tilde{c})$. Here a horizontal immersion $f\colon M_s^n \to S_{2s}^{2n+1}(\tilde{c})$ (respectively, $f\colon M_s^n \to H_{2s+1}^{2n+1}(\tilde{c})$) is an immersion which satisfies i$f(p) \perp f_*(T_p M_s^n)$ for all $p \in M_s^n$, where i $= \sqrt{-1}$.

Theorem 3.3 ([7]). *Let $\tilde{c} > 0$ and let $\Pi : S^{2n+1}_{2s}(\tilde{c}) \to \mathbb{C}P^n_s(4\tilde{c})$ be the Hopf fibration. If $f : M^n_s \to S^{2n+1}_{2s}(\tilde{c})$ is a horizontal immersion, then $F = \Pi \circ f : M^n_s \to \mathbb{C}P^n_s(4\tilde{c})$ is a Lagrangian immersion.*

Conversely, let M^n_s be a simply connected manifold and let $F : M^n_s \to \mathbb{C}P^n_s(4\tilde{c})$ be a Lagrangian immersion. Then there exist a 1-parameter family of horizontal lifts $f : M^n_s \to S^{2n+1}_{2s}(\tilde{c})$ such that $F = \Pi \circ f$. Any two such lifts f_1 and f_2 are related by $f_1 = e^{i\theta} f_2$, where θ is a constant.

The analogous statement for $\tilde{c} < 0$ also holds if one replaces $S^{2n+1}_{2s}(\tilde{c})$ by $H^{2n+1}_{2s+1}(\tilde{c})$ and $\mathbb{C}P^n_s(4\tilde{c})$ by $\mathbb{C}H^n_s(4\tilde{c})$. Remark that both immersions have the same induced metric and also that the second fundamental forms of both immersions are closely related. For more details see [7].

Before continuing with the classification, we now want to use the above theorem to present two classes of indefinite flat Lagrangian submanifolds.

Example 3.4. We define an immersion Φ of $\mathbb{R}^n$ into $S^{2n+1}_{2s}(\tilde{c})$ by

$$\Phi(u_1, v_1, \ldots, u_s, v_s, w_1, \ldots, w_r) = \tfrac{\alpha}{\sqrt{n+1}}(\sqrt{2}e^{iu_1}\sinh v_1, \sqrt{2}e^{iu_1}\cosh v_1, \ldots,$$
$$\sqrt{2}e^{iu_s}\sinh v_s, \sqrt{2}e^{iu_s}\cosh v_s, e^{iw_1}, \ldots, e^{iw_r}, e^{i\beta}),$$

where $\alpha^2 = \frac{1}{\tilde{c}}$, $r = 2n - s$ and $\beta = -2\sum_{j=1}^{s} u_j - \sum_{k=1}^{r} w_k$. Clearly, remembering that the first s odd terms have to be taken with a minus sign, $\langle \Phi, \Phi \rangle = \alpha^2$. We also have that

$$\Phi_{u_1} = \tfrac{\alpha}{\sqrt{n+1}}(i\sqrt{2}e^{iu_1}\sinh v_1, i\sqrt{2}e^{iu_1}\cosh v_1, 0, \ldots, 0, -2ie^{i\beta}),$$
$$\Phi_{v_1} = \tfrac{\alpha}{\sqrt{n+1}}(\sqrt{2}e^{iu_1}\cosh v_1, \sqrt{2}e^{iu_1}\sinh v_1, 0, \ldots, 0, 0),$$
$$\Phi_{w_1} = \tfrac{\alpha}{\sqrt{n+1}}(0, \ldots, 0, ie^{iw_1}, 0, \ldots, 0, -ie^{i\beta}).$$

From the above equations, it follows straightforwardly that Φ is horizontal and thus $\Pi \circ \Phi$ is a Lagrangian immersion in $\mathbb{C}P^n_s(4\tilde{c})$. We now introduce a vector field η by

$$\eta = \tfrac{\alpha}{\sqrt{n+1}}(0, \ldots, 0, e^{i\beta}).$$

It then follows that

$$\eta = \tfrac{1}{n+1}(\Phi + i\sum_{j=1}^{s} \Phi_{u_j} + i\sum_{k=1}^{r} \Phi_{w_k}).$$

Using this, we obtain for the second derivatives of the immersion the following formulas:

$$\begin{aligned}
\Phi_{v_j v_j} &= -i\Phi_{u_j} + 2\eta, & \Phi_{u_j u_j} &= i\Phi_{u_j} - 6\eta, \\
\Phi_{u_j v_j} &= i\Phi_{v_j}, & \Phi_{u_j u_m} &= -4\eta, \\
\Phi_{u_j w_k} &= -2\eta, & \Phi_{w_k w_k} &= i\Phi_{w_k} - 2\eta, \\
\Phi_{w_k w_\ell} &= -\eta, & \Phi_{v_j v_m} &= 0, \\
\Phi_{v_j w_k} &= 0, & \Phi_{v_j u_m} &= 0.
\end{aligned}$$

Since the above expressions do not contain any tangential terms, we obtain that all connection coefficients vanish and hence Φ (and also $\Pi \circ \Phi$) is flat. Using the above formulas, together with the fact that $A_{iX}Y = -ih(X, Y)$, where h and A

denote respectively the second fundamental form and the Weingarten operator of the immersion in the indefinite sphere, we also get that

$$\text{trace } A_{i\Phi_{v_j}} = 0,$$
$$\text{trace } A_{i\Phi_{u_j}} = 1 - \frac{6}{n+1} + 1 - \frac{4(s-1)}{n+1} - 2\frac{r}{n+1} = 0,$$
$$\text{trace } A_{i\Phi_{w_k}} = -\frac{2s}{n+1} + 1 - \frac{2}{n+1} - \frac{r-1}{n+1} = 0,$$

implying that the immersion is minimal in the indefinite sphere. We also obtain, using [7], that the tensor K and the metric are respectively given by

$$K(\partial_{v_j}, \partial_{v_j}) = -\partial_{u_j} + 2\tilde{\eta}, \qquad K(\partial_{u_j}, \partial_{u_j}) = \partial_{u_j} - 6\tilde{\eta},$$
$$K(\partial_{u_j}, \partial_{v_j}) = \partial_{v_j}, \qquad K(\partial_{u_j}, \partial_{u_m}) = -4\tilde{\eta},$$
$$K(\partial_{u_j}, \partial_{w_k}) = -2\tilde{\eta}, \qquad K(\partial_{w_k}, \partial_{w_k}) = \partial_{w_k} - 2\tilde{\eta},$$
$$K(\partial_{w_k}, \partial_{w_\ell}) = -\tilde{\eta}, \qquad K(\partial_{v_j}, \partial_{v_m}) = 0,$$
$$K(\partial_{v_j}, \partial_{w_k}) = 0, \qquad K(\partial_{v_j}, \partial_{u_m}) = 0,$$

where $j \neq m$ and $k \neq \ell$ and $\tilde{\eta}$ is defined by

$$\tilde{\eta} = \frac{1}{n+1}\left(\sum_{j=1}^{s} \partial_{u_j} + \sum_{k=1}^{r} \partial_{w_k}\right)$$

and

$$\langle \partial_{v_j}, \partial_{v_j} \rangle = -\frac{2\alpha^2}{(n+1)}, \qquad \langle \partial_{u_j}, \partial_{u_j} \rangle = \frac{6\alpha^2}{(n+1)},$$
$$\langle \partial_{u_j}, \partial_{v_j} \rangle = 0, \qquad \langle \partial_{u_j}, \partial_{u_m} \rangle = \frac{4\alpha^2}{(n+1)},$$
$$\langle \partial_{u_j}, \partial_{w_k} \rangle = \frac{2\alpha^2}{(n+1)}, \qquad \langle \partial_{w_k}, \partial_{w_k} \rangle = \frac{2\alpha^2}{(n+1)},$$
$$\langle \partial_{w_k}, \partial_{w_\ell} \rangle = \frac{\alpha^2}{(n+1)}, \qquad \langle \partial_{v_j}, \partial_{v_m} \rangle = 0,$$
$$\langle \partial_{v_j}, \partial_{w_k} \rangle = 0, \qquad \langle \partial_{v_j}, \partial_{u_m} \rangle = 0.$$

By a straightforward computation or by using the fact that Φ is horizontal and thus $\Pi \circ \Phi$ is Lagrangian, we see that the index of the above metric equals s.

Example 3.5. For the next example, we assume that $n - 2s = 1$ and we define an immersion Ψ of $\mathbb{R}^n$ into $H_{2s}^{2n+1}(\tilde{c})$ by

$$\Psi(u_1, v_1, \ldots, u_s, v_s, w) = \frac{\alpha}{\sqrt{s+1}}(e^{iu_1} \cosh v_1, e^{iu_1} \sinh v_1, \ldots, e^{iu_s} \cosh v_s,$$
$$e^{iu_s} \sinh v_s, e^{-i\beta} \cosh w, e^{-i\beta} \sinh w),$$

where $\alpha^2 = -\frac{1}{\tilde{c}}$ and $\beta = \sum_{j=1}^{s} u_j$. Clearly, remembering that the first $s + 1$ odd terms have to be taken with a minus sign, $\langle \Psi, \Psi \rangle = -\alpha^2$. That Ψ is horizontal, minimal and flat now follows in a similar way as in the previous example.

Then, if we denote by $\mathbb{M}_s^n(4\tilde{c}) = \mathbb{C}P_s^n(4\tilde{c})$ if $\tilde{c} > 0$ and $\mathbb{M}_s^n(4\tilde{c}) = \mathbb{C}H_s^n(4\tilde{c})$ if $\tilde{c} < 0$ we have the following result:

Theorem 3.6. *Let $\phi : M_s^n(c) \to \mathbb{M}_s^n(4\tilde{c})$ be a minimal isometric Lagrangian immersion into $\mathbb{M}_s^n(4\tilde{c})$, $n - 2s \geq 0$. Suppose that $c \neq \tilde{c}$. Then $c = 0$, and if $n - 2s > 1$, then $\tilde{c} > 0$. Moreover if $\tilde{c} > 0$, we get that ϕ is locally congruent with the immersion $\Pi \circ \Phi$, as defined in Example 3.4, and if $\tilde{c} < 0$ and $n - 2s = 1$, ϕ is locally congruent with the immersion $\Pi \circ \Psi$, as defined in Example 3.5.*

Proof. We already know that $c = 0$. Next we assume that $n - 2s > 1$ and $\tilde{c} > 0$. Now, let V be an n-dimensional vector space. Then, on V we can define an operator K (resp. $\tilde{K}$) and a metric h (resp. $\tilde{h}$) as in Theorem 2.2 (resp. Example 3.4). However, by the uniqueness theorem, we know that there exists a linear transformation such that

$$\tilde{K}(x, y) = A^{-1}K(Ax, Ay),$$
$$\tilde{h}(x, y) = h(Ax, Ay).$$

This means that the expression of $\tilde{K}$ and $\tilde{h}$ with respect to a basis $\{Ae_1, \ldots, Ae_n\}$ is the same as the expression of K and h with respect to the basis $\{e_1, \ldots, e_n\}$.

Now taking a point p and the basis constructed in Theorem 2.2, we then know from the above that there exists a transformation of this basis, with a matrix B, such that the components of the metric and the difference tensor with respect to this new basis are as in Example 3.4. Since all components are constant, we can take the same matrix B at every point. Since for the old basis all connection coefficients vanish, the same is true for the new basis. Consequently there exist coordinates $u_1, v_1, \ldots, u_s, v_s, w_1, \ldots, w_k$ on M_s^n such that the Hopf lift of the immersion of M_s^n into $\mathbb{M}_s^n(4\tilde{c})$, which we denote by x, is characterized by the following system of differential equations:

$$\begin{aligned}
x_{v_j v_j} &= -ix_{u_j} + 2\eta, & x_{u_j u_j} &= ix_{u_j} - 6\eta, \\
x_{u_j v_j} &= ix_{v_j}, & x_{u_j u_m} &= -4\eta, \\
x_{u_j w_k} &= -2\eta, & x_{w_k w_k} &= ix_{w_k} - 2\eta, \\
x_{w_k w_\ell} &= -\eta, & x_{v_j v_m} &= 0, \\
x_{v_j w_k} &= 0, & x_{v_j u_m} &= 0,
\end{aligned}$$

where $\eta = \frac{1}{n+1}(x + i\sum_{m=1}^{s} x_{u_m} + i\sum_{k=\ell}^{r} x_{w_\ell})$. It now follows that

$$\begin{aligned}
\eta_{v_j} &= \tfrac{1}{n+1}(x_{v_j} + ix_{u_j v_j}) = 0, \\
\eta_{u_j} &= \tfrac{1}{n+1}(-6i\eta - 4i(s-1)\eta - 2ir\eta) = -2i\eta, \\
\eta_{w_k} &= \tfrac{1}{n+1}(-2si\eta - 2i\eta - (r-1)i\eta) = -i\eta.
\end{aligned}$$

Consequently there exists a constant vector C_{n+1} such that

$$\eta = C_{n+1}e^{-2i\sum_{j=1}^{s} u_j - i\sum_{k=1}^{r} w_k}.$$

The system of differential equations for x can now be rewritten as

$$\begin{aligned}
(x - \eta)_{v_j v_j} &= -i(x - \eta)_{u_j}, & (x - \eta)_{u_j u_j} &= i(x - \eta)_{u_j}, \\
(x - \eta)_{u_j v_j} &= i(x - \eta)_{v_j}, & (x - \eta)_{u_j u_m} &= 0, \\
(x - \eta)_{u_j w_k} &= 0, & (x - \eta)_{w_k w_k} &= i(x - \eta)_{w_k}, \\
(x - \eta)_{w_k w_\ell} &= 0, & (x - \eta)_{v_j v_m} &= 0, \\
(x - \eta)_{v_j w_k} &= 0, & (x - \eta)_{v_j u_m} &= 0.
\end{aligned}$$

It now follows that there exist constants $C_1, D_1, \ldots, C_s, D_s, C_{2s+1}, \ldots, C_n$ such that

$$x - \eta = \sum_{j=1}^{s}(C_j e^{iu_j} \sinh v_j + D_j e^{iu_j} \cosh v_j) + \sum_{k=1}^{r} C_{2s+k}e^{iw_k}.$$

Since ix is normal to the immersion (horizontally) and $\langle x, x \rangle = \alpha^2 = \frac{1}{c}$, it now follows that

$$\langle C_j, C_j \rangle = -\langle D_j, D_j \rangle,$$
$$\langle D_j, D_j \rangle - 2\langle C_{n+1}, C_{n+1} \rangle = 0,$$
$$\langle C_{2s+k}, C_{2s+k} \rangle - \langle C_{n+1}, C_{n+1} \rangle = 0,$$
$$\sum_{j=1}^{s} \langle D_j, D_j \rangle + \sum_{k=1}^{r} \langle C_{2s+k}, C_{2s+k} \rangle + \langle C_{n+1}, C_{n+1} \rangle = \alpha^2,$$

and all other inner products of the form $\langle C_., iD_. \rangle, \langle C_., D_. \rangle, \langle C_., iC_. \rangle, \ldots$ vanish. Hence ϕ is locally congruent with the immersion Φ described in Example 3.4. In case that $\tilde{c} < 0$ and $n - 2s = 1$ we proceed in a similar fashion to obtain that ϕ is congruent with the immersion Ψ as described in Example 3.5. $\square$

Remark that although the basis described in Example 3.4 and Example 3.5 is extremely convenient for integration and obtaining the explicit formulas for the immersion, it is not particular suited to prove for example Lemma 2.3. For that purpose it is easier to use the basis mentioned in Theorem 2.2.

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