

## EVALUATIONS OF INITIAL IDEALS AND CASTELNUOVO-MUMFORD REGULARITY

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ABSTRACT. This paper characterizes the Castelnuovo-Mumford regularity by evaluating the initial ideal with respect to the reverse lexicographic order.

### 1. INTRODUCTION

Let  $S = k[x_1, \dots, x_n]$  be a polynomial ring over a field  $k$  of arbitrary characteristic. Let  $I \subset S$  be an arbitrary homogeneous ideal and

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I \longrightarrow 0$$

a graded minimal free resolution of  $S/I$ . Write  $b_i$  for the maximum degree of the generators of  $F_i$ . The *Castelnuovo-Mumford regularity*

$$\text{reg}(S/I) := \max\{b_i - i \mid i = 0, \dots, p\}$$

is a measure for the complexity of  $I$  in computational problems [EG], [BM], [V]. One can use Buchsberger's syzygy algorithm to compute  $\text{reg}(S/I)$ . However, such a computation is often very big. Theoretically, if  $\text{char}(k) = 0$ ,  $\text{reg}(S/I)$  is equal to the largest degree of the generators of the generic initial ideal of  $I$  with respect to the reverse lexicographic order [BS]. But it is difficult to know when an initial ideal is generic. Therefore, it would be of interest to have other methods for the computation of  $\text{reg}(S/I)$ .

The aim of this paper is to present a simple method for the computation of  $\text{reg}(S/I)$  which is based only on evaluations of  $\text{in}(I)$ , where  $\text{in}(I)$  denotes the initial ideal of  $I$  with respect to the reverse lexicographic order. We are inspired by a recent paper of Bermejo and Gimenez [BG] which gives such a method for the computation of the Castelnuovo-Mumford regularity of projective curves.

Let  $d = \dim S/I$ . For  $i = 0, \dots, d$  put  $S_i = k[x_1, \dots, x_{n-i}]$ . Let  $J_i$  be the ideal of  $S_i$  obtained from  $\text{in}(I)$  by the evaluation  $x_{n-i+1} = \cdots = x_n = 0$ . Let  $\tilde{J}_i$  denote the ideal of  $S_i$  obtained from  $J_i$  by the evaluation  $x_{n-i} = 1$ . These ideals can be easily computed from the generators of  $\text{in}(I)$ . In fact, if  $\text{in}(I) = (f_1, \dots, f_s)$ , where  $f_1, \dots, f_s$  are monomials in  $S$ , then  $J_i$  is generated by the monomials  $f_j$  not divided

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by any of the variables  $x_{n-i+1}, \dots, x_n$  and  $\tilde{J}_i$  by those monomials obtained from the latter by setting  $x_{n-i} = 1$ . Put

$$c_i(I) := \sup\{r \mid (\tilde{J}_i/J_i)_r \neq 0\},$$

with  $c_i(I) = -\infty$  if  $\tilde{J}_i = J_i$  and

$$r(I) := \sup\{r \mid (S_d/J_d)_r \neq 0\}.$$

We can express  $\text{reg}(S/I)$  in terms of these numbers as follows. Assume that  $c_i(I) < \infty$  for  $i = 0, \dots, d-1$ . Then

$$\text{reg}(S/I) = \max\{c_0(I), \dots, c_{d-1}(I), r(I)\}.$$

The assumption  $c_i(I) < \infty$  for  $i = 0, \dots, d-1$  is satisfied for a sufficiently general choice of the variables. If  $I$  is the defining saturated ideal of a projective (not necessarily reduced) curve, this assumption is automatically satisfied if  $k[x_{n-1}, x_n]$  is a Noether normalization of  $S/I$ . In this case,  $c_0(I) = -\infty$  and  $\text{reg}(S/I) = \max\{c_1(I), r(I)\}$ . From this formula we can easily deduce the results of Bermejo and Gimenez.

Similarly we can compute the *partial regularities*  $\ell\text{-reg}(S/I) := \max\{b_i - i \mid i \geq \ell\}$ ,  $\ell > 0$ , which were recently introduced by Bayer, Charalambous and Popescu [BCP] (see also Aramova and Herzog [AH]). These regularities can be defined in terms of local cohomology. Let  $\mathfrak{m}$  denote the maximal homogeneous ideal of  $S$ . Let  $H_{\mathfrak{m}}^i(S/I)$  denote the  $i$ th local cohomology module of  $S/I$  with respect to  $\mathfrak{m}$  and set  $a_i(S/I) = \max\{r \mid H_{\mathfrak{m}}^i(S/I)_r \neq 0\}$  with  $a_i(S/I) = -\infty$  if  $H_{\mathfrak{m}}^i(S/I) = 0$ . For  $t \geq 0$  we define  $\text{reg}_t(S/I) := \max\{a_i(S/I) + i \mid i = 0, \dots, t\}$ . Then  $\text{reg}_t(S/I) = (n-t)\text{-reg}(S/I)$  [T2]. Under the assumption  $c_i(I) < \infty$  for  $i = 0, \dots, t$  we obtain the following formula:

$$\text{reg}_t(S/I) = \max\{c_i(I) \mid i = 0, \dots, t\}.$$

The numbers  $c_i(I)$  also allow us to determine the place at which  $\text{reg}(S/I)$  is attained in the minimal free resolution of  $S/I$ . In fact,  $\text{reg}(S/I) = b_t - t$  if  $c_t(I) = \max\{c_i(I) \mid i = 0, \dots, d\}$ . Moreover,  $r(I)$  can be used to estimate the reduction number of  $S/I$  which is another measure for the complexity of  $I$  [V].

It turns out that the numbers  $c_i(I)$  and  $r(I)$  can be described combinatorially in terms of the lattice vectors of the generators of  $\text{in}(I)$  (see Propositions 4.1–4.3 for details). These descriptions together with the above formulae give an effective method for the computation of  $\text{reg}(S/I)$  and  $\text{reg}_t(S/I)$ . From this we can derive the estimation

$$\text{reg}_t(S/I) \leq \max\{\deg g_i - n + i \mid i = 0, \dots, t\},$$

where  $g_i$  is the least common multiple of the minimal generators of  $\text{in}(I)$  which are not divided by any of the variables  $x_{n-i+1}, \dots, x_n$ .

This paper is organized as follows. In Section 2 we prepare some facts on the Castelnuovo–Mumford regularity. In Section 3 we prove the above formulae for  $\text{reg}(S/I)$  and  $\text{reg}_t(S/I)$ . The combinatorial descriptions of  $c_i(I)$  and  $r(I)$  are given in Section 4. Section 5 deals with the case of projective curves.

## 2. FILTER-REGULAR SEQUENCE OF LINEAR FORMS

We shall keep the notations of the preceding section. Let  $\mathbf{z} = z_1, \dots, z_{t+1}$  be a sequence of homogeneous elements of  $S$ ,  $t \geq 0$ . We call  $\mathbf{z}$  a *filter-regular sequence* for  $S/I$  if  $z_{i+1} \notin \mathfrak{p}$  for any associated prime  $\mathfrak{p} \neq \mathfrak{m}$  of  $(I, z_1, \dots, z_i)$ ,  $i = 0, \dots, t$ .

This notion was introduced in order to characterize generalized Cohen-Macaulay rings [STC]. Recall that  $S/I$  is a generalized Cohen-Macaulay ring if and only if  $I$  is equidimensional and  $(R/I)_{\mathfrak{p}}$  is a Cohen-Macaulay ring for every prime ideal  $\mathfrak{p} \neq \mathfrak{m}$ . This condition is satisfied if  $I$  is the defining ideal of a projective curve. We call  $\mathbf{z}$  a homogeneous system of parameters for  $S/I$  if  $t + 1 = d$  and  $(I, z_1, \dots, z_d)$  is an  $\mathfrak{m}$ -primary ideal. It is known that every homogeneous system of parameters for  $S/I$  is a filter-regular sequence if  $S/I$  is a generalized Cohen-Macaulay ring. In general, a homogeneous system of parameters need not be a filter-regular sequence. However, if  $k$  is an infinite field, any ideal which is primary to the maximal graded ideal and which is generated by linear forms can be generated by a homogeneous filter-regular sequence (proof of [T1, Lemma 3.1]).

For  $i = 0, \dots, t$  we put

$$a_{\mathbf{z}}^i(S/I) := \sup\{r \mid [(I, z_1, \dots, z_i) : z_{i+1}]_r \neq (I, z_1, \dots, z_i)_r\},$$

with  $a_{\mathbf{z}}^i(S/I) = -\infty$  if  $(I, z_1, \dots, z_i) : z_{i+1} = (I, z_1, \dots, z_i)$ . These invariants can be  $\infty$  and they are a measure for how far  $\mathbf{z}$  is from being a regular sequence in  $S/I$ . It can be shown that  $\mathbf{z}$  is a filter-regular sequence for  $S/I$  if and only if  $a_{\mathbf{z}}^i(S/I) < \infty$  for  $i = 0, \dots, t$  [T1, Lemma 2.1]. Note that our definition of  $a_{\mathbf{z}}^i(S/I)$  is one less than that in [T1]. There is the following close relationship between these numbers and the partial regularity of  $S/I$ .

**Theorem 2.1** ([T1, Proposition 2.2]). *Let  $\mathbf{z}$  be a filter-regular sequence of linear forms for  $S/I$ . Then*

$$\text{reg}_t(S/I) = \max\{a_{\mathbf{z}}^i(S/I) \mid i = 0, \dots, t\}.$$

We will use the following characterization of  $a_{\mathbf{z}}^i(S/I)$ .

**Lemma 2.2.**  $a_{\mathbf{z}}^i(S/I) = \max\{r \mid [\bigcup_{m \geq 1} (I, z_1, \dots, z_i) : z_{i+1}^m]_r \neq (I, z_1, \dots, z_i)_r\}$ .

*Proof.* Put  $r_0 = \max\{r \mid [\bigcup_{m \geq 1} (I, z_1, \dots, z_i) : z_{i+1}^m]_r \neq (I, z_1, \dots, z_i)_r\}$ . By definition,  $a_{\mathbf{z}}^i(S/I) \leq r_0$ . Conversely, if  $y$  is an element of  $\bigcup_{m \geq 1} (I, z_1, \dots, z_i) : z_{i+1}^m]_{r_0}$ , then

$$yz_{i+1} \in [\bigcup_{m \geq 1} (I, z_1, \dots, z_i) : z_{i+1}^m]_{r_0+1} = (I, z_1, \dots, z_i)_{r_0+1}.$$

Hence  $y \in [(I, z_1, \dots, z_i) : z_{i+1}]_{r_0}$ . This implies  $r_0 \leq a_{\mathbf{z}}^i(S/I)$ . So we get  $r_0 = a_{\mathbf{z}}^i(S/I)$ . □

Since  $\text{reg}(S/I) = \text{reg}_d(S/I)$ , to compute  $\text{reg}(S/I)$  we need a filter-regular sequence of linear forms of length  $d + 1$ . But that can be avoided by the following observation.

**Lemma 2.3.** *Let  $\mathbf{z} = z_1, \dots, z_d$  be a filter-regular sequence for  $S/I$ ,  $d = \dim(S/I)$ . Then  $\mathbf{z}$  is a system of parameters for  $S/I$ .*

*Proof.* Let  $\mathfrak{p}$  be an arbitrary associated prime  $\mathfrak{p}$  of  $(I, z_1, \dots, z_i)$  with  $\dim S/\mathfrak{p} = d - i$ ,  $i = 0, \dots, d - 1$ . Then  $\mathfrak{p} \neq \mathfrak{m}$  because  $\dim S/\mathfrak{p} > 0$ . By the definition of a filter-regular sequence,  $z_{i+1} \notin \mathfrak{p}$ . Hence  $\mathbf{z}$  is a homogeneous system of parameters for  $S/I$ . □

If  $\mathbf{z}$  is a homogeneous system of parameters for  $S/I$ , then  $S/(I, z_1, \dots, z_d)$  is of finite length. Hence  $(S/(I, z_1, \dots, z_d))_r = 0$  for  $r$  large enough. Following [NR] we call

$$r_{\mathbf{z}}(S/I) := \max\{r \mid (S/(I, z_1, \dots, z_d))_r \neq 0\}$$

the *reduction number* of  $S/I$  with respect to  $\mathbf{z}$ . It is equal to the maximum degree of the generators of  $S/I$  as a module over  $k[z_1, \dots, z_d]$  [V]. Note that the minimum of  $r_{\mathbf{z}}(S/I)$  is called the reduction number of  $S/I$ .

**Theorem 2.4** ([BS, Theorem 1.10], [T1, Corollary 3.3]). *Let  $\mathbf{z}$  be a filter-regular sequence of  $d$  linear forms for  $S/I$ . Then*

$$\text{reg}(S/I) = \max\{a_{\mathbf{z}}^0(S/I), \dots, a_{\mathbf{z}}^{d-1}(S/I), r_{\mathbf{z}}(S/I)\}.$$

*Remark.* Theorem 2.4 was proved in [BS] under an additional condition on the maximum degree of the generators of  $I$ .

### 3. EVALUATIONS OF THE INITIAL IDEAL

Let  $c_i(I)$ ,  $i = 0, \dots, d$ , and  $r(I)$  be the invariants defined in Section 1 by means of evaluations of  $\text{in}(I)$ , where  $\text{in}(I)$  is the initial ideal of  $I$  with respect to the reverse lexicographic order. We will use the results of Section 2 to express  $\text{reg}_t(S/I)$  and  $\text{reg}(S/I)$  in terms of  $c_i(I)$  and  $r(I)$ .

**Lemma 3.1.** *For  $\mathbf{z} = x_n, \dots, x_{n-t}$  and  $i = 0, \dots, t$  we have*

$$a_{\mathbf{z}}^i(S/I) = c_i(I).$$

*Proof.* By [BS, Lemma (2.2)],  $[(I, x_n, \dots, x_{n-i+1}) : x_{n-i}]_r = (I, x_n, \dots, x_{n-i+1})_r$  if and only if  $[(\text{in}(I), x_n, \dots, x_{n-i+1}) : x_{n-i}]_r = (\text{in}(I), x_n, \dots, x_{n-i+1})_r$  for all  $r \geq 0$ . Therefore

$$a_{\mathbf{z}}^i(S/I) = a_{\mathbf{z}}^i(S/\text{in}(I)).$$

By Lemma 2.2 we get

$$\begin{aligned} a_{\mathbf{z}}^i(S/\text{in}(I)) &= \sup\{r \mid [ \bigcup_{m \geq 1} (\text{in}(I), x_n, \dots, x_{n-i+1}) : x_{n-i}^m ]_r \\ &\quad \neq (\text{in}(I), x_n, \dots, x_{n-i+1})_r \}. \end{aligned}$$

Note that  $J_i$  is the ideal of  $S_i = k[x_1, \dots, x_{n-i}]$  obtained from  $\text{in}(I)$  by the evaluation  $x_{n-i+1} = \dots = x_n = 0$  and that this evaluation corresponds to the canonical isomorphism  $S/(x_{n-i+1}, \dots, x_n) \cong S_i$ . Then we may rewrite the above formula as

$$a_{\mathbf{z}}^i(S/\text{in}(I)) = \sup\{r \mid [ \bigcup_{m \geq 1} J_i : x_{n-i}^m ]_r \neq (J_i)_r \}.$$

Since  $J_i$  is a monomial ideal,  $\bigcup_{m \geq 1} J_i : x_{n-i}^m$  is generated by the monomials  $g$  in the variables  $x_1, \dots, x_{n-i-1}$  for which there exists an integer  $m \geq 1$  such that  $gx_{n-i}^m \in J_i$ . Such a monomial  $g$  is determined by the condition  $g \in \tilde{J}_i$ . Hence

$$a_{\mathbf{z}}^i(S/\text{in}(I)) = \sup\{r \mid (\tilde{J}_i)_r \neq (J_i)_r \} = c_i(I).$$

□

As a consequence of Lemma 3.1 we can use the invariants  $c_i(I)$  to check when  $x_n, \dots, x_{n-t}$  is a regular resp. filter-regular sequence for  $S/I$ .

**Corollary 3.2.**  *$x_{n-i}$  is a non-zerodivisor in  $S/(I, x_n, \dots, x_{n-i+1})$  if and only if  $c_i(I) = -\infty$ .*

*Proof.* By definition,  $a_{\mathbf{z}}^i(S/I) = -\infty$  if and only if  $x_{n-i}$  is a non-zerodivisor in  $S/(I, x_n, \dots, x_{n-i+1})$ . Hence the conclusion follows from Lemma 3.1. □

**Corollary 3.3.** *Let  $\mathbf{z} = x_n, \dots, x_{n-t}$ . Then  $\mathbf{z}$  is a filter-regular sequence for  $S/I$  if and only if  $c_i(I) < \infty$  for  $i = 0, \dots, t$ .*

*Proof.* It is known that  $\mathbf{z}$  is a filter-regular sequence for  $S/I$  if and only if  $a_{\mathbf{z}}^i(S/I) < \infty$  for  $i = 0, \dots, t$  [T1, Lemma 2.1]. □

Now we can characterize  $\text{reg}_t(S/I)$  as follows.

**Theorem 3.4.** *Assume that  $c_i(I) < \infty$  for  $i = 0, \dots, t$ . Then*

$$\text{reg}_t(S/I) = \max\{c_i(I) \mid i = 0, \dots, t\}.$$

*Proof.* This follows from Theorem 2.1, Lemma 3.1 and Corollary 3.3. □

We can also give a characterization of  $\text{reg}(S/I)$  which involves  $r(I)$ .

**Lemma 3.5.** *Assume that  $c_i(I) < \infty$  for  $i = 0, \dots, d - 1$ . Then*

$$r_{\mathbf{z}}(S/I) = r(I).$$

*Proof.* By Corollary 3.3,  $\mathbf{z} = x_n, \dots, x_{n-d+1}$  is a filter-regular sequence for  $S/I$ . By Lemma 2.3 and [T2, Theorem 4.1], this implies that  $\mathbf{z}$  is a homogeneous system of parameters for  $S/\text{in}(I)$  with

$$r_{\mathbf{z}}(S/I) = r_{\mathbf{z}}(S/\text{in}(I)).$$

Note that  $S/(x_{n-d+1}, \dots, x_n) \cong S_d$  and that  $J_d$  is the ideal obtained from  $\text{in}(I)$  by the evaluation  $x_{n-d+1} = \dots = x_n = 0$ . Then

$$\begin{aligned} r_{\mathbf{z}}(S/\text{in}(I)) &= \max\{r \mid (S/(\text{in}(I), x_n, \dots, x_{n-d+1}))_r \neq 0\} \\ &= \max\{r \mid (S_d/J_d)_r \neq 0\} \\ &= r(I). \end{aligned}$$

□

**Theorem 3.6.** *Assume that  $c_i(I) < \infty$  for  $i = 0, \dots, d - 1$ . Then*

$$\text{reg}(S/I) = \max\{c_0(I), \dots, c_{d-1}(I), r(I)\}.$$

*Proof.* This follows from Theorem 2.4, Lemma 3.1, Corollary 3.3 and Lemma 3.5. □

#### 4. COMBINATORIAL DESCRIPTION

First, we want to show that the condition  $c_i(I) < \infty$  can be easily checked in terms of the lattice vectors of the generators of  $\text{in}(I)$ . Let  $\mathcal{B}$  be the (finite) set of monomials which minimally generates  $\text{in}(I)$ . We set

$$E_i := \{v \in \mathbb{N}^{n-i} \mid x^v \in \mathcal{B}\},$$

where  $x^v = x_1^{\varepsilon_1} \cdots x_s^{\varepsilon_s}$  if  $v = (\varepsilon_1, \dots, \varepsilon_s)$ . For  $j = 1, \dots, n - i$  we denote by  $p_j$  the projection from  $\mathbb{N}^{n-i}$  to  $\mathbb{N}^{n-i-1}$  which deletes the  $j$ th coordinate. For two lattice vectors  $a = (\alpha_1, \dots, \alpha_s)$  and  $b = (\beta_1, \dots, \beta_s)$  of the same size we say  $a \geq b$  if  $\alpha_j \geq \beta_j$  for  $j = 1, \dots, s$ .

**Lemma 4.1.**  *$c_i(I) < \infty$  if and only if for every element  $a \in p_{n-i}(E_i) \setminus E_{i+1}$  there are elements  $b_j \in E_{i+1}$  such that  $p_j(a) \geq p_j(b_j)$ ,  $j = 1, \dots, n - i - 1$ .*

*Proof.* Recall that  $c_i(I) = \sup\{r \mid (\tilde{J}_i/J_i)_r \neq 0\}$ . Then  $c_i(I) < \infty$  if and only if  $\tilde{J}_i/J_i$  is of finite length. By the definition of  $J_i$  and  $\tilde{J}_i$ , the latter condition is equivalent to the existence of a number  $r$  such that  $x_j^r \tilde{J}_i \subseteq J_i$  for  $j = 1, \dots, n-i$ . It is clear that  $J_i$  is generated by the monomials  $x^v$  with  $v \in E_i$ . From this it follows that  $\tilde{J}_i$  is generated by  $J_i$  and the monomials  $x^a$  with  $a \in p_{n-i}(E_i) \setminus E_{i+1}$ . For such a monomial  $x^a$  we can always find a number  $r$  such that  $x_{n-i}^r x^a \in J_i$ . For  $j < n-i$ ,  $x_j^r x^a \in J_i$  if and only if  $x_j^r x^a$  is divided by a generator  $x^{b_j}$  of  $J_i$ . Since  $x_j^r x^a$  does not contain  $x_{n-i}, \dots, x_n$ , so does  $x^{b_j}$ . Hence  $b_j \in E_{i+1}$ . Setting  $x_j = 1$  we see that  $x_j^r x^a$  is divided by  $x^{b_j}$  for some number  $r$  if and only if  $p_j(a) \geq p_j(b_j)$ .  $\square$

If  $c_i(I) = \infty$ , we should make a random linear transformation of the variables  $x_1, \dots, x_{n-i}$  and test the condition  $c_i(I) < \infty$  again. By Lemma 3.1 the linear transformation does not change the invariants  $c_j(I)$  for  $j < i$ . Moreover, instead of  $\text{in}(I)$  we only need to compute the smaller initial ideal  $\text{in}(I_i)$ , where  $I_i$  denotes the ideal of  $S_i$  obtained from  $I$  by the evaluation  $x_{n-i+1} = \dots = x_n = 0$ . Let  $\mathcal{B}_i$  be the set of monomials which minimally generates  $\text{in}(I_i)$ . It is easy to see that  $\mathcal{B}_i$  is the set of the monomials of  $\mathcal{B}$  which are not divided by  $x_{n-i+1}, \dots, x_n$ . From this it follows that  $E_j = \{v \in \mathbb{N}^{n-j} \mid x^v \in \mathcal{B}_i\}$  for  $j \leq i$ . Thus, we can use this formula to compute  $E_j$  and to check the condition  $c_j(I) < \infty$  for  $j \leq i$ . Once we know  $c_i(I) < \infty$  we can proceed to compute  $c_i(I)$ .

In the lattice  $\mathbb{N}^{n-i}$  we delete the shadow of  $E_i$ , that is, the set of elements  $a$  for which there is  $v \in E_i$  with  $v \leq a$ . The remaining lattice has the shape of a staircase and we will denote by  $F_i$  the set of its corners. It is easy to see that  $F_i$  is the set of the elements of the form  $a = \max(v_1, \dots, v_{n-i}) - (1, \dots, 1)$  with  $a \not\leq v$  for any element  $v \in E_i$ , where  $v_1, \dots, v_{n-i}$  is a family of  $n-i$  elements of  $E_i$  for which the  $j$ th coordinate of  $v_j$  is greater than the  $j$ th coordinate of  $v_h$  for all  $h \neq j$ ,  $j = 1, \dots, n-i$ , and  $\max(v_1, \dots, v_{n-i})$  denotes the element whose coordinates are the maxima of the corresponding coordinates of  $v_1, \dots, v_{n-i}$ . If  $a = (\alpha_1, \dots, \alpha_{n-i})$ , we set

$$|a| := \alpha_1 + \dots + \alpha_{n-i}.$$

**Proposition 4.2.** *Assume that  $c_i(I) < \infty$ . Then  $c_i(I) = -\infty$  if  $F_i = \emptyset$  and  $c_i(I) = \max_{a \in F_i} |a|$  if  $F_i \neq \emptyset$ .*

*Proof.* Let  $a$  be an arbitrary element of  $F_i$ . Then  $a = \max(v_1, \dots, v_{n-i}) - (1, \dots, 1)$  for some family  $v_1, \dots, v_{n-i}$  of  $S_i$ . Let  $v_j = (\varepsilon_{j1}, \dots, \varepsilon_{jn-i})$ ,  $j = 1, \dots, n-i$ . Then  $a = (\varepsilon_{11} - 1, \dots, \varepsilon_{n-in-i} - 1)$ . Since  $\varepsilon_{jj} > \varepsilon_{hj}$  for  $h \neq j$ , we get  $a \geq (\varepsilon_{n-i1}, \dots, \varepsilon_{n-in-i-1}, 0)$ . Therefore,  $x^a$  is divided by the monomial obtained from  $x^{v_{n-i}}$  by setting  $x_{n-i} = 1$ . Note that  $J_i$  is generated by the monomials  $x^v$  with  $x_v \in E_i$ . Since  $v_{n-i} \in E_i$ , we have  $x^{v_{n-i}} \in J_i$ , whence  $x^a \in \tilde{J}_i$ . On the other hand,  $x^a \notin J_i$  because  $a \not\leq v$  for any element  $v \in E_i$ . Since  $|a| = \deg x^a$ , this implies  $(\tilde{J}_i/J_i)_{|a|} \neq 0$ . Hence  $|a| \leq c_i(I)$ . So we obtain  $\max_{a \in F_i} |a| \leq c_i(I)$  if  $F_i \neq \emptyset$ .

To prove the converse inequality we assume that  $\tilde{J}_i/J_i \neq 0$ . Since  $c_i(I) < \infty$ , there is a monomial  $x^b \in \tilde{J}_i \setminus J_i$  such that  $\deg x^b = c_i(I)$ . Since  $x^b \notin J_i$ ,  $b \not\leq v$  for any element  $v \in E_i$ . For  $j = 1, \dots, n-i$  we have  $x_j x^b \in J_i$  because  $\deg x_j x^b = c_i(I) + 1$ . Therefore,  $x_j x^b$  is divided by some monomial  $x^{v_j}$  with  $v_j \in E_i$ . Let  $b = (\beta_1, \dots, \beta_{n-i})$  and  $v_j = (\varepsilon_{j1}, \dots, \varepsilon_{jn-i})$ . Then  $\beta_h \geq \varepsilon_{jh}$  for  $h \neq j$  and  $\beta_j + 1 \geq \varepsilon_{jj}$ .

Since  $b \not\leq v_j$ , we must have  $\beta_j < \varepsilon_{jj}$ , hence  $\beta_j = \varepsilon_{jj} - 1$ . It follows that  $\varepsilon_{jj} = \beta_j + 1 > \varepsilon_{hj}$  for all  $h \neq j$ . Thus, the family  $v_1, \dots, v_{n-i}$  belongs to  $\mathcal{S}_i$  and  $b = \max(v_1, \dots, v_{n-i}) - (1, \dots, 1)$ . So we have proved that  $b \in F_i$ . Hence  $c_i(I) = \deg x^b = |b| \leq \max_{a \in F_i} |a|$ .

The above argument also shows that  $F_i \neq \emptyset$  if  $\tilde{J}_i \neq J_i$ . So  $c_i(I) = -\infty$  if  $F_i = \emptyset$ . □

By Corollary 3.3, if  $c_i(I) < \infty$  for  $i = 0, \dots, d - 1$ , then  $\mathbf{z} = x_n, \dots, x_{n-d+1}$  is a filter-regular sequence for  $S/I$ . By Lemma 2.3 and Lemma 3.5, that implies  $r(I) = r_{\mathbf{z}}(S/I) < \infty$ . In this case, we have the following description of  $r(I)$ .

**Proposition 4.3.** *Assume that  $r(I) < \infty$ . Then  $r(I) = \max_{a \in F_d} |a|$ .*

*Proof.* This can be proved similarly to the proof of Lemma 4.2. □

Combining the above results with Theorem 3.4 and Theorem 3.6 we get a simple method to compute  $\text{reg}_t(S/I)$  and  $\text{reg}(S/I)$ . We will illustrate the above method by an example at the end of the next section. Moreover, we get the following estimation for  $\text{reg}_t(S/I)$ .

**Corollary 4.4.** *Let  $x_n, \dots, x_{n-t}$  be a filter-regular sequence for  $S/I$ . Let  $g_i$  denote the least common multiple of the minimal generators of  $\text{in}(I)$  which are not divided by any of the variables  $x_{n-i+1}, \dots, x_n$ . Then*

$$\text{reg}_i(S/I) \leq \max\{\deg g_i - n + i \mid i = 0, \dots, t\}.$$

*Proof.* By Corollary 3.3, the assumption implies that  $c_i(I) < \infty$  for  $i = 0, \dots, t$ . Thus, combining Theorem 3.4 and Lemma 4.2 we get

$$\text{reg}_i(S/I) \leq \max\{|a| \mid a \in F_i, i = 0, \dots, t\}.$$

It is easily seen from the definition of  $F_i$  that  $\max_{a \in F_i} |a| \leq \deg g_i - n + i, i = 0, \dots, t$ , hence the conclusion. □

*Remark.* Bruns and Herzog [BH, Theorem 3.1(a)], resp. Hoa and Trung [HT, Theorem 3.1], proved that for any monomial ideal  $I$ ,  $\text{reg}(S/I) \leq \deg f - 1$ , resp.  $\deg f - \text{ht } I$ , where  $f$  is the least common multiple of the minimal generators of  $I$ . Note that the mentioned result of Bruns and Herzog is valid for multigraded modules.

### 5. THE CASE OF PROJECTIVE CURVES

Let  $I_C \subset k[x_1, \dots, x_n]$  be the defining saturated ideal of a (not necessarily reduced) projective curve  $C \subset \mathbb{P}^{n-1}$ ,  $n \geq 3$ . We will assume that  $k[x_{n-1}, x_n] \hookrightarrow S/I_C$  is a Noether normalization of  $S/I_C$ . In this case, Theorem 3.6 can be reformulated as follows.

**Proposition 5.1.**  $\text{reg}(S/I_C) = \max\{c_1(I_C), r(I_C)\}$ .

*Proof.* By the above assumption  $S/I_C$  is a generalized Cohen-Macaulay ring of positive depth and  $x_n, x_{n-1}$  is a homogeneous system of parameters for  $S/I_C$ . Therefore,  $x_n, x_{n-1}$  is a filter-regular sequence for  $S/I_C$ . In particular,  $x_n$  is a non-zero-divisor in  $S/I_C$ . By Lemma 3.2,  $c_0(I_C) = -\infty$ . Hence the conclusion follows from Theorem 3.6. □

Since  $S/I_C$  has positive depth, the graded minimal free resolution of  $S/I_C$  ends at most at the  $(n - 1)$ th place:

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I_C \longrightarrow 0.$$

From Theorem 3.4 we obtain the following information on the shifts of  $F_{n-1}$ . Note that  $F_{n-1} = 0$  if  $S/I_C$  is a Cohen-Macaulay ring or, in other words, if  $C$  is an arithmetically Cohen-Macaulay curve.

**Proposition 5.2.** *If  $C$  is not an arithmetically Cohen-Macaulay curve,  $c_1(I_C) + n - 1$  is the maximum degree of the generators of  $F_{n-1}$ .*

*Proof.* Let  $b_{n-1}$  be the maximum degree of the generators of  $F_{n-1}$ . As we have seen in the introduction,  $b_{n-1} - n + 1 = (n - 1)\text{-reg}(S/I_C) = \text{reg}_1(S/I_C)$ . By Theorem 3.4,  $\text{reg}_1(S/I_C) = \max\{c_0(I_C), c_1(I_C)\} = c_1(I_C)$  because  $c_0(I_C) = -\infty$ . So we obtain  $b_{n-1} = c_1(I_C) + n - 1$ . □

Now we shall see that Proposition 5.1 contains all main results of Bermejo and Gimenez in [BG]. It should be noted that they did not use strong results such as Theorem 2.4. We follow the notations of [BG].

Let  $E := \{a \in \mathbb{N}^{n-2} \mid x^a \in \text{in}(I_C)\}$  and denote by  $H(E)$  the smallest integer  $r$  such that  $a \in E$  if  $|a| = r$ .

**Corollary 5.3** ([BG, Theorem 2.4]). *Assume that  $C$  is an arithmetically Cohen-Macaulay curve. Then  $\text{reg}(S/I_C) = H(E) - 1$ .*

*Proof.* Since  $x_n, x_{n-1}$  is a regular sequence in  $S/I_C$ , we have  $c_1(I_C) = -\infty$  by Corollary 3.2. By Proposition 5.1 this implies  $\text{reg}(S/I_C) = r(I_C)$ . But

$$r(I_C) = \sup\{r \mid (S_2/J_2)_r \neq 0\} = H(E) - 1$$

because  $J_2$  is generated by the monomials  $x^a, a \in E$ . □

Let  $I_0$  be the ideal in  $S$  generated by the polynomials obtained from  $I_C$  by the evaluation  $x_{n-1} = x_n = 0$ . Then  $S/I_0$  is a two-dimensional Cohen-Macaulay ring. Let  $\tilde{I}$  denote the ideal in  $S$  generated by the monomials obtained from  $\text{in}(I_C)$  by the evaluation  $x_{n-1} = x_n = 1$ . Let

$$F := \{a \in \mathbb{N}^{n-2} \mid x^a \in \tilde{I} \setminus \text{in}(I_0)\}.$$

For every vector  $a \in F$  let

$$E_a := \{(\mu, \nu) \in \mathbb{N}^2 \mid x^a x_{n-1}^\mu x_n^\nu \in \text{in}(I_C)\}.$$

Let  $\mathfrak{R} := \bigcup_{a \in F} \{a \times [\mathbb{N}^2 \setminus E_a]\}$  and denote by  $H(\mathfrak{R})$  the smallest integer  $r$  such that the number of the elements  $b \in \mathfrak{R}$  with  $|b| = s$  becomes a constant for  $s \geq r$ .

**Corollary 5.4** ([BG, Theorem 2.7]).  $\text{reg}(S/I_C) = \max\{\text{reg}(S/I_0), H(\mathfrak{R})\}$ .

*Proof.* As in the proof of Corollary 5.3 we have  $\text{reg}(S/I_0) = r(I_0)$ . But  $r(I_0) = r(I_C)$  because  $\text{in}(I_0)$  is the ideal generated by the monomials obtained from  $\text{in}(I_C)$  by the evaluation  $x_{n-1} = x_n = 0$ . Thus,

$$\text{reg}(S/I_0) = r(I_C).$$

It has been observed in [BG] that the number of the elements  $b \in \mathfrak{R}$  with  $|b| = s$  is the difference  $H_{S/I_C}(s) - H_{S/\tilde{I}}(s) = H_{S/\text{in}(I_C)}(s) - H_{S/\tilde{I}}(s) = H_{\tilde{I}/\text{in}(I_C)}(s)$ , where  $H_E(s)$  denotes the Hilbert function of a graded  $S$ -module  $E$ . Since  $x_n$  is a non-zero-divisor in  $S/\text{in}(I_C)$ ,  $H(\mathfrak{R})+1$  is the least integer  $r$  such that  $H_{(\tilde{I}, x_n)/(\text{in}(I_C), x_n)}(s)$



$= 0$  for  $s \geq r$ . On the other hand, since  $\text{in}(I_C)$  is generated by monomials which do not contain  $x_n$  and since  $J_1$  is the ideal in  $k[x_1, \dots, x_{n-1}]$  obtained from  $\text{in}(I_C)$  by the evaluation  $x_n = 0$ , we have  $\text{in}(I_C) = J_1 S$  and  $\tilde{I} = \tilde{J}_1 S$ , whence  $(\tilde{I}, x_n)/(\text{in}(I_C), x_n) \cong \tilde{J}_1/J_1$ . Note that  $c_1(I_C) = \max\{r \mid (\tilde{J}_1/J_1)_r \neq 0\}$  with  $c_1(I_C) = -\infty$  if  $\tilde{J}_1 = J_1$ . Then

$$H(\mathfrak{R}) = \max\{0, c_1(I_C)\}.$$

Thus, applying Proposition 5.1 we obtain  $\text{reg}(S/I_C) = \max\{\text{reg}(S/I_0), H(\mathfrak{R})\}$ .  $\square$

**Example.** Let  $C \subset \mathbb{P}^3$  be the monomial curve  $(t^\alpha s^\beta : t^\beta s^\alpha : s^{\alpha+\beta} : t^{\alpha+\beta})$ ,  $\alpha > \beta > 0$ , g.c.d.  $(\alpha, \beta) = 1$ . It is known that the defining ideal  $I_C \subset k[x_1, x_2, x_3, x_4]$  is generated by the quadric  $x_1 x_2 - x_3 x_4$  and the forms  $x_1^{\beta+r} x_3^{\alpha-\beta-r} - x_2^{\alpha-r} x_4^r$ ,  $r = 0, \dots, \alpha - \beta$ , and that this is a Gröbner basis of  $I_C$  for the reverse lexicographic order with  $x_1 > x_2 > x_3 > x_4$  [CM, Théorème 3.9]. Therefore,

$$\text{in}(I_C) = (x_1 x_2, x_2^\alpha, x_1^{\beta+1} x_3^{\alpha-\beta-1}, x_1^{\beta+2} x_3^{\alpha-\beta-2}, \dots, x_1^\alpha).$$

Using the notations of Section 3 we have

$$\begin{aligned} E_1 &= \{(1, 1, 0), (0, \alpha, 0), (\beta + 1, 0, \alpha - \beta - 1), (\beta + 2, 0, \alpha - \beta - 2), \dots, (\alpha, 0, 0)\}, \\ E_2 &= \{(1, 1), (0, \alpha), (\alpha, 0)\}. \end{aligned}$$

From this it follows that

$$\begin{aligned} F_1 &= \{(\beta + 1, 0, \alpha - \beta - 2), (\beta + 2, 0, \alpha - \beta - 3), \dots, (\alpha - 1, 0, 0)\}, \\ F_2 &= \{(0, \alpha - 1), (\alpha - 1, 0)\}. \end{aligned}$$

By Proposition 4.2,  $c_1(I_C) = \alpha - 1$  if  $\alpha - \beta \geq 2$  ( $c_1(I_C) = -\infty$  if  $\alpha - \beta = 1$ ) and  $r(I_C) = \alpha - 1$  by Proposition 4.3. Applying Proposition 5.1 we obtain  $\text{reg}(S/I_C) = \alpha - 1$ .

The direct computation of the invariant  $H(\mathfrak{R})$  is more complicated than that of  $c_1(I_C)$ . First, we should interpret  $F$  as the set of the elements of the form  $a \in \mathbb{N}^2$  such that  $a \geq b$  for some elements  $b \in p(E_1)$  but  $a \not\geq c$  for any element  $c \in E_2$ . Then we get

$$F = \{(\beta + 1, 0), (\beta + 2, 0), \dots, (\alpha - 1, 0)\}.$$

For all  $\varepsilon = \beta + 1, \dots, \alpha - 1$  we verify that  $E_{(\varepsilon, 0)} = (\alpha - \varepsilon, 0) + \mathbb{N}^2$ . It follows that

$$\mathfrak{R} = \{(\varepsilon, 0, \mu, \nu) \in \mathbb{N}^4 \mid \varepsilon = \beta + 1, \dots, \alpha - 1; \mu \leq \alpha - \varepsilon - 1\}.$$

If  $\alpha - \beta = 1$ , we have  $\mathfrak{R} = \emptyset$ , hence  $H(\mathfrak{R}) = 0$ . If  $\alpha - \beta \geq 2$ , we can check that  $H(\mathfrak{R}) = \alpha - 1$ .

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