EVALUATIONS OF INITIAL IDEALS
AND CASTELNUOVO-MUMFORD REGULARITY

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Abstract. This paper characterizes the Castelnuovo-Mumford regularity by evaluating the initial ideal with respect to the reverse lexicographic order.

1. Introduction

Let \( S = k[x_1, \ldots, x_n] \) be a polynomial ring over a field \( k \) of arbitrary characteristic. Let \( I \subset S \) be an arbitrary homogeneous ideal and

\[
0 \rightarrow F_p \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I \rightarrow 0
\]

a graded minimal free resolution of \( S/I \). Write \( b_i \) for the maximum degree of the generators of \( F_i \). The Castelnuovo-Mumford regularity

\[
\text{reg}(S/I) := \max\{b_i - i \mid i = 0, \ldots, p\}
\]

is a measure for the complexity of \( I \) in computational problems \([EG], [BM], [V]\). One can use Buchsberger’s syzygy algorithm to compute \( \text{reg}(S/I) \). However, such a computation is often very big. Theoretically, if \( \text{char}(k) = 0 \), \( \text{reg}(S/I) \) is equal to the largest degree of the generators of the generic initial ideal of \( I \) with respect to the reverse lexicographic order \([BS]\). But it is difficult to know when an initial ideal is generic. Therefore, it would be of interest to have other methods for the computation of \( \text{reg}(S/I) \).

The aim of this paper is to present a simple method for the computation of \( \text{reg}(S/I) \) which is based only on evaluations of \( \text{in}(I) \), where \( \text{in}(I) \) denotes the initial ideal of \( I \) with respect to the reverse lexicographic order. We are inspired by a recent paper of Bermejo and Gimenez \([BG]\) which gives such a method for the computation of the Castelnuovo-Mumford regularity of projective curves.

Let \( d = \dim S/I \). For \( i = 0, \ldots, d \) put \( S_i = k[x_1, \ldots, x_{n-i}] \). Let \( J_i \) be the ideal of \( S_i \) obtained from \( \text{in}(I) \) by the evaluation \( x_{n-i+1} = \cdots = x_n = 0 \). Let \( J_i \) denote the ideal of \( S_i \) obtained from \( J_i \) by the evaluation \( x_{n-i} = 1 \). These ideals can be easily computed from the generators of \( \text{in}(I) \). In fact, if \( \text{in}(I) = (f_1, \ldots, f_s) \), where \( f_1, \ldots, f_s \) are monomials in \( S \), then \( J_i \) is generated by the monomials \( f_j \) not divided...
by any of the variables $x_{n-i+1}, \ldots, x_n$ and $\tilde{J}_i$ by those monomials obtained from the latter by setting $x_{n-i} = 1$. Put
\[ c_i(I) := \sup\{r | (\tilde{J}_i/J_i)_r \neq 0\}, \]
with $c_i(I) = -\infty$ if $\tilde{J}_i = J_i$ and
\[ r(I) := \sup\{r | (S_d/J_d)_r \neq 0\}. \]

We can express $\text{reg}(S/I)$ in terms of these numbers as follows. Assume that $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$. Then
\[ \text{reg}(S/I) = \max\{c_0(I), \ldots, c_{d-1}(I), r(I)\}. \]
The assumption $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$ is satisfied for a sufficiently general choice of the variables. If $I$ is the defining saturated ideal of a projective (not necessarily reduced) curve, this assumption is automatically satisfied if $k[x_{n-1}, x_n]$ is a Noether normalization of $S/I$. In this case, $c_0(I) = -\infty$ and $\text{reg}(S/I) = \max\{c_1(I), r(I)\}$. From this formula we can easily deduce the results of Bermejo and Gimenez.

Similarly we can compute the partial regularities $\ell - \text{reg}(S/I) := \max\{b_i - i | i \geq \ell\}$, $\ell > 0$, which were recently introduced by Bayer, Charalambous and Popescu [BCP] (see also Aramova and Herzog [AH]). These regularities can be defined in terms of local cohomology. Let $\mathfrak{m}$ denote the maximal homogeneous ideal of $S$. Let $H_{\mathfrak{m}}^i(S/I)$ denote the $i$th local cohomology module of $S/I$ with respect to $\mathfrak{m}$ and set $a_i(S/I) = \max\{r | H_{\mathfrak{m}}^i(S/I)_r \neq 0\}$ with $a_i(S/I) = -\infty$ if $H_{\mathfrak{m}}^i(S/I) = 0$. For $t \geq 0$ we define
\[ \text{reg}_t(S/I) := \max\{a_i(S/I) + i | i = 0, \ldots, t\}. \]
Then $\text{reg}_t(S/I) = (n - t) - \text{reg}(S/I)$ [T2]. Under the assumption $c_i(I) < \infty$ for $i = 0, \ldots, t$ we obtain the following formula:
\[ \text{reg}_t(S/I) = \max\{c_i(I) | i = 0, \ldots, t\}. \]

The numbers $c_i(I)$ also allow us to determine the place at which $\text{reg}(S/I)$ is attained in the minimal free resolution of $S/I$. In fact, $\text{reg}(S/I) = b_t - t$ if $c_t(I) = \max\{c_i(I) | i = 0, \ldots, t\}$. Moreover, $r(I)$ can be used to estimate the reduction number of $S/I$ which is another measure for the complexity of $I$ [V].

It turns out that the numbers $c_i(I)$ and $r(I)$ can be described combinatorially in terms of the lattice vectors of the generators of $\text{in}(I)$ (see Propositions 4.1–4.3 for details). These descriptions together with the above formulae give an effective method for the computation of $\text{reg}(S/I)$ and $\text{reg}_t(S/I)$. From this we can derive the estimation
\[ \text{reg}_t(S/I) \leq \max\{\deg g_i - n + i | i = 0, \ldots, t\}, \]
where $g_i$ is the least common multiple of the minimal generators of $\text{in}(I)$ which are not divided by any of the variables $x_{n-i+1}, \ldots, x_n$.

This paper is organized as follows. In Section 2 we prepare some facts on the Castelnuovo-Mumford regularity. In Section 3 we prove the above formulae for $\text{reg}(S/I)$ and $\text{reg}_t(S/I)$. The combinatorial descriptions of $c_i(I)$ and $r(I)$ are given in Section 4. Section 5 deals with the case of projective curves.

2. Filter-regular sequence of linear forms

We shall keep the notations of the preceding section. Let $z = z_1, \ldots, z_{t+1}$ be a sequence of homogeneous elements of $S$, $t \geq 0$. We call $z$ a filter-regular sequence for $S/I$ if $z_{i+1} \notin p$ for any associated prime $p \neq \mathfrak{m}$ of $(I, z_1, \ldots, z_i)$, $i = 0, \ldots, t$. 

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This notion was introduced in order to characterize generalized Cohen-Macaulay rings [STC]. Recall that \( S/I \) is a generalized Cohen-Macaulay ring if and only if \( I \) is equidimensional and \( (R/I)_p \) is a Cohen-Macaulay ring for every prime ideal \( p \neq m \). This condition is satisfied if \( I \) is the defining ideal of a projective curve. We call \( z \) a homogeneous system of parameters for \( S/I \) if \( t + 1 = d \) and \( (I, z_1, \ldots, z_d) \) is an \( m \)-primary ideal. It is known that every homogeneous system of parameters for \( S/I \) is a filter-regular sequence if \( S/I \) is a generalized Cohen-Macaulay ring. In general, a homogeneous system of parameters need not be a filter-regular sequence. However, if \( k \) is an infinite field, any ideal which is primary to the maximal graded ideal and which is generated by linear forms can be generated by a homogeneous filter-regular sequence (proof of [T1] Lemma 3.1).

For \( i = 0, \ldots, t \) we put
\[
a^i_z(S/I) := \sup \{ r \mid (I, z_1, \ldots, z_i) : z_{i+1}^r \neq (I, z_1, \ldots, z_i)_r \},
\]
with \( a^i_z(S/I) = -\infty \) if \( (I, z_1, \ldots, z_i) : z_{i+1} = (I, z_1, \ldots, z_i) \). These invariants can be \( \infty \) and they are a measure for how far \( z \) is from being a regular sequence in \( S/I \). It can be shown that \( z \) is a filter-regular sequence for \( S/I \) if and only if \( a^i_z(S/I) < \infty \) for \( i = 0, \ldots, t \) [T1] Lemma 2.1. Note that our definition of \( a^i_z(S/I) \) is one less than that in [T1]. There is the following close relationship between these numbers and the partial regularity of \( S/I \).

**Theorem 2.1** ([T1] Proposition 2.2). Let \( z \) be a filter-regular sequence of linear forms for \( S/I \). Then
\[
\text{reg}_z(S/I) = \max \{ a^i_z(S/I) \mid i = 0, \ldots, t \}.
\]

We will use the following characterization of \( a^i_z(S/I) \).

**Lemma 2.2.** \( a^i_z(S/I) = \max \{ r \mid \bigcup_{m \geq 1} (I, z_1, \ldots, z_i) : z_{i+1}^m \neq (I, z_1, \ldots, z_i)_r \} \).

**Proof.** Put \( r_0 = \max \{ r \mid \bigcup_{m \geq 1} (I, z_1, \ldots, z_i) : z_{i+1}^m \neq (I, z_1, \ldots, z_i)_r \} \). By definition, \( a^i_z(S/I) \leq r_0 \). Conversely, if \( y \) is an element of \( \bigcup_{m \geq 1} (I, z_1, \ldots, z_i) : z_{i+1}^m \), then
\[
yz_{i+1} \in \bigcup_{m \geq 1} (I, z_1, \ldots, z_i) : z_{i+1}^m \]_r = (I, z_1, \ldots, z_i)_r + 1.
\] Hence \( y \in (I, z_1, \ldots, z_i) \). This implies \( r_0 \leq a^i_z(S/I) \). So we get \( r_0 = a^i_z(S/I) \).

Since \( \text{reg}(S/I) = \text{reg}_z(S/I) \), to compute \( \text{reg}(S/I) \) we need a filter-regular sequence of linear forms of length \( d + 1 \). But that can be avoided by the following observation.

**Lemma 2.3.** Let \( z = z_1, \ldots, z_d \) be a filter-regular sequence for \( S/I \), \( d = \dim(S/I) \). Then \( z \) is a system of parameters for \( S/I \).

**Proof.** Let \( p \) be an arbitrary associated prime \( p \) of \( (I, z_1, \ldots, z_i) \) with \( \dim(S/p) = d - i \), \( i = 0, \ldots, d - 1 \). Then \( p \neq m \) because \( \dim(S/p) > 0 \). By the definition of a filter-regular sequence, \( z_{i+1} \notin p \). Hence \( z \) is a homogeneous system of parameters for \( S/I \).

If \( z \) is a homogeneous system of parameters for \( S/I \), then \( S/(I, z_1, \ldots, z_d) \) is of finite length. Hence \( (S/(I, z_1, \ldots, z_d))_r = 0 \) for \( r \) large enough. Following [NR] we call
\[
r^r_z(S/I) := \max \{ r \mid (S/(I, z_1, \ldots, z_d))_r \neq 0 \}
\]
the reduction number of $S/I$ with respect to $z$. It is equal to the maximum degree of the generators of $S/I$ as a module over $k[z_1, \ldots, z_d]$. Note that the minimum of $r_z(S/I)$ is called the reduction number of $S/I$.

**Theorem 2.4** ([BS, Theorem 1.10], [T1, Corollary 3.3]). Let $z$ be a filter-regular sequence of $d$ linear forms for $S/I$. Then

$$\text{reg}(S/I) = \max\{a_z^0(S/I), \ldots, a_z^{d-1}(S/I), r_z(S/I)\}.$$ 

**Remark.** Theorem 2.4 was proved in [BS] under an additional condition on the maximum degree of the generators of $I$.

3. Evaluations of the Initial Ideal

Let $c_i(I)$, $i = 0, \ldots, d$, and $r(I)$ be the invariants defined in Section 1 by means of evaluations of $\text{in}(I)$, where $\text{in}(I)$ is the initial ideal of $I$ with respect to the reverse lexicographic order. We will use the results of Section 2 to express $\text{reg}(S/I)$ and $\text{reg}(S/I)$ in terms of $c_i(I)$ and $r(I)$.

**Lemma 3.1.** For $z = x_n, \ldots, x_{n-t}$ and $i = 0, \ldots, t$ we have

$$a_z^i(S/I) = c_i(I).$$

**Proof.** By [BS, Lemma 2.2], \([I, x_n, \ldots, x_{n-i}] : x_{n-i+1}\) = \((I, x_n, \ldots, x_{n-i+1})\) if and only if \([(\text{in}(I), x_n, \ldots, x_{n-i+1}) : x_{n-i}] = (\text{in}(I), x_n, \ldots, x_{n-i+1})\) for all $r \geq 0$. Therefore

$$a_z^i(S/I) = a_z^i(S/\text{in}(I)).$$

By Lemma 2.2 we get

$$a_z^i(S/\text{in}(I)) = \sup\{r \mid \bigcup_{m \geq 1} (\text{in}(I), x_n, \ldots, x_{n-i+1}) : x_{n-i}^m \neq \text{in}(I), x_n, \ldots, x_{n-i+1})\}. $$

Note that $J_i$ is the ideal of $S_i = k[x_1, \ldots, x_n]$ obtained from $\text{in}(I)$ by the evaluation $x_{n-i+1} = \cdots = x_n = 0$ and that this evaluation corresponds to the canonical isomorphism $S/(x_{n-i+1}, \ldots, x_n) \cong S_i$. Then we may rewrite the above formula as

$$a_z^i(S/\text{in}(I)) = \sup\{r \mid \bigcup_{m \geq 1} J_i : x_{n-i}^m \neq (J_i)\}. $$

Since $J_i$ is a monomial ideal, $\bigcup_{m \geq 1} J_i : x_{n-i}^m$ is generated by the monomials $g$ in the variables $x_1, \ldots, x_{n-i-1}$ for which there exists an integer $m \geq 1$ such that $gx_{n-i}^m \in J_i$. Such a monomial $g$ is determined by the condition $g \in J_i$. Hence

$$a_z^i(S/\text{in}(I)) = \sup\{r \mid (J_i)_r \neq (J_i)\} = c_i(I).$$

As a consequence of Lemma 3.1 we can use the invariants $c_i(I)$ to check when $x_n, \ldots, x_{n-t}$ is a regular resp. filter-regular sequence for $S/I$.

**Corollary 3.2.** $x_{n-i}$ is a non-zerodivisor in $S/(I, x_n, \ldots, x_{n-i+1})$ if and only if $c_i(I) = -\infty$.

**Proof.** By definition, $a_z^i(S/I) = -\infty$ if and only if $x_{n-i}$ is a non-zerodivisor in $S/(I, x_n, \ldots, x_{n-i+1})$. Hence the conclusion follows from Lemma 3.1.

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Corollary 3.3. Let $z = x_n, \ldots, x_{n-t}$. Then $z$ is a filter-regular sequence for $S/I$ if and only if $c_i(I) < \infty$ for $i = 0, \ldots, t$.

Proof. It is known that $z$ is a filter-regular sequence for $S/I$ if and only if $a^z_i(S/I) < \infty$ for $i = 0, \ldots, t$ [12]. Lemma 2.1]

Now we can characterize $\text{reg}_i(S/I)$ as follows.

Theorem 3.4. Assume that $c_i(I) < \infty$ for $i = 0, \ldots, t$. Then
\[ \text{reg}_i(S/I) = \max\{c_i(I) \mid i = 0, \ldots, t\}. \]

Proof. This follows from Theorem 2.4, Lemma 3.1, Corollary 3.3 and Lemma 3.5.

We can also give a characterization of $\text{reg}(S/I)$ which involves $r(I)$.

Lemma 3.5. Assume that $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$. Then
\[ r^z(S/I) = r(I). \]

Proof. By Corollary 3.3, $z = x_n, \ldots, x_{n-d+1}$ is a filter-regular sequence for $S/I$. By Lemma 2.3 and [12] Theorem 4.1, this implies that $z$ is a homogeneous system of parameters for $S/\text{in}(I)$ with
\[ r^z(S/I) = r(z(S/\text{in}(I))). \]

Note that $S/(x_{n-d+1}, \ldots, x_n) \cong S_d$ and that $J_d$ is the ideal obtained from $\text{in}(I)$ by the evaluation $x_{n-d+1} = \cdots = x_n = 0$. Then
\[
\begin{align*}
\text{reg}(S/\text{in}(I)) &= \max\{r\mid (S/(\text{in}(I), x_n, \ldots, x_{n-d+1}))_r \neq 0\} \\
&= \max\{r\mid (S_d/J_d)_r \neq 0\} \\
&= r(I).
\end{align*}
\]

Theorem 3.6. Assume that $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$. Then
\[ \text{reg}(S/I) = \max\{c_0(I), \ldots, c_{d-1}(I), r(I)\}. \]

Proof. This follows from Theorem 2.4, Lemma 3.1, Corollary 3.3 and Lemma 3.5.

4. Combinatorial Description

First, we want to show that the condition $c_i(I) < \infty$ can be easily checked in terms of the lattice vectors of the generators of $\text{in}(I)$. Let $B$ be the (finite) set of monomials which minimally generates $\text{in}(I)$. We set
\[ E_i := \{v \in \mathbb{N}^{n-i} \mid x^v \in B\}, \]
where $x^v = x_1^{e_1} \cdots x_s^{e_s}$ if $v = (e_1, \ldots, e_s)$. For $j = 1, \ldots, n-i$ we denote by $p_j$ the projection from $\mathbb{N}^{n-i}$ to $\mathbb{N}^{n-i-1}$ which deletes the $j$th coordinate. For two lattice vectors $a = (\alpha_1, \ldots, \alpha_n)$ and $b = (\beta_1, \ldots, \beta_n)$ of the same size we say $a \geq b$ if $\alpha_j \geq \beta_j$ for $j = 1, \ldots, s$.

Lemma 4.1. $c_i(I) < \infty$ if and only if for every element $a \in p_{n-i}(E_i) \setminus E_{i+1}$ there are elements $b_j \in E_{i+1}$ such that $p_j(a) \geq p_j(b_j)$, $j = 1, \ldots, n-i-1$. 

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For such a monomial \( x \), the set of monomials which minimally generates \( \langle x \rangle \) follows that \( x \in E_i \). From this it follows that \( \tilde{J}_i \) is generated by \( J_i \) and the monomials \( x^a \) with \( a \in p_{n-i}(E_i) \setminus E_{i+1} \).

Proposition 4.2. Assume that \( c_j(I) = \infty \). Then \( c_j(I) = -\infty \) if \( F_i = \emptyset \) and \( c_j(I) = \max_{a \in F_i} |a| \) if \( F_i \neq -\emptyset \).

Proof. Let \( a \) be an arbitrary element of \( F_i \). Then \( a = \max(v_1, \ldots, v_{n-i}) - 1, \ldots, 1) \) for some family \( v_1, \ldots, v_{n-i} \) of \( S_i \). Let \( v_j = (\varepsilon_{j1}, \ldots, \varepsilon_{jn-i}), j = 1, \ldots, n-i \). Then \( a = (\varepsilon_{11} - 1, \ldots, \varepsilon_{n-i} - 1) \). Since \( \varepsilon_{jj} > \varepsilon_{hj} \) for \( h \neq j \), we get \( a \geq (\varepsilon_{n-i1}, \ldots, \varepsilon_{n-in-i-1}, 0) \). Therefore, \( x^a \) is divided by the monomial obtained from \( x^{v_{n-i}} \) by setting \( x_{n-i} = 1 \). Note that \( J_i \) is generated by the monomials \( x^a \) with \( a \in E_i \). Since \( v_{n-i} \in E_i \), we have \( x^{v_{n-i}} \in J_i \), whence \( x^a \in \tilde{J}_i \). On the other hand, \( x^a \notin J_i \) because \( a \nless v \) for any element \( v \in E_i \). Since \( |a| = \deg x^a \), this implies \( (\tilde{J}_i/J_i)|a| \neq 0 \). Hence \( |a| \leq c_i(I) \). So we obtain \( c_i(I) = \max_{a \in F_i} |a| \) if \( F_i \neq \emptyset \).

To prove the converse inequality we assume that \( \tilde{J}_i/J_i \neq 0 \). Since \( c_i(I) < \infty \), there is a monomial \( x^b \in \tilde{J}_i \setminus J_i \) such that \( \deg x^b = c_i(I) \). Since \( x^b \notin J_i \), \( b \nless v \) for any element \( v \in E_i \). For \( j = 1, \ldots, n-i \), we have \( x_j x^b \in J_i \) because \( \deg x_j x^b = c_i(I) + 1 \). Therefore, \( x_j x^b \) is divided by some monomial \( x^{v_j} \) with \( v_j \in E_i \). Let \( b = (\beta_1, \ldots, \beta_{n-i}) \) and \( v_j = (\varepsilon_{j1}, \ldots, \varepsilon_{jn-i}) \). Then \( \beta_h \geq \varepsilon_{jh} \) for \( h \neq j \) and \( \beta_j + 1 \geq \varepsilon_{jj} \).
Since $b \not\geq v_j$, we must have $\beta_j < \varepsilon_{jj}$, hence $\beta_j = \varepsilon_{jj} - 1$. It follows that

$$
\varepsilon_{jj} = \beta_j + 1 > \varepsilon_{hh} \quad \text{for all } h \neq j.
$$

Thus, the family $v_1, \ldots, v_{n-i}$ belongs to $S_i$ and $b = \max(v_1, \ldots, v_{n-1}) - (1, \ldots, 1)$. So we have proved that $b \in F_i$. Hence

$$
c_i(I) = \deg x^b = |b| \leq \max_{a \in F_i} |a|.
$$

The above argument also shows that $F_i \neq \emptyset$ if $\tilde{J}_i \neq J_i$. So $c_i(I) = -\infty$ if $F_i = \emptyset$. 

By Corollary 3.3 if $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$, then $z = x_{n-d+1}$ is a filter-regular sequence for $S/I$. By Lemma 2.3 and Lemma 4.3 that implies $r(I) = r_d(S/I) < \infty$. In this case, we have the following description of $r(I)$.

**Proposition 4.3.** Assume that $r(I) < \infty$. Then $r(I) = \max_{a \in F_d} |a|$.

**Proof.** This can be proved similarly to the proof of Lemma 1.2.

Combining the above results with Theorem 3.4 and Theorem 3.6 we get a simple method to compute $\reg_r(S/I)$ and $\reg(S/I)$. We will illustrate the above method by an example at the end of the next section. Moreover, we get the following estimation for $\reg_r(S/I)$.

**Corollary 4.4.** Let $x_n, \ldots, x_{n-\ell}$ be a filter-regular sequence for $S/I$. Let $g_i$ denote the least common multiple of the minimal generators of $\ini(I)$ which are not divided by any of the variables $x_{n-\ell+1}, \ldots, x_n$. Then

$$
\reg_r(S/I) \leq \max\{\deg g_i - n + i \mid i = 0, \ldots, t\}.
$$

**Proof.** By Corollary 3.3 the assumption implies that $c_i(I) < \infty$ for $i = 0, \ldots, t$. Thus, combining Theorem 3.4 and Lemma 1.2 we get

$$
\reg_r(S/I) \leq \max\{|a| \mid a \in F_i, \ i = 0, \ldots, t\}.
$$

It is easily seen from the definition of $F_i$ that $\max_{a \in F_i} |a| \leq \deg g_i - n + i$, $i = 0, \ldots, t$, hence the conclusion.

**Remark.** Bruns and Herzog [BH, Theorem 3.1(a)], resp. Hoa and Trung [HT, Theorem 3.1], proved that for any monomial ideal $I$, $\reg(S/I) \leq \deg f - \deg I$, where $f$ is the least common multiple of the minimal generators of $I$. Note that the mentioned result of Bruns and Herzog is valid for multigraded modules.

### 5. The case of projective curves

Let $I_C \subset k[x_1, \ldots, x_n]$ be the defining saturated ideal of a (not necessarily reduced) projective curve $C \subset \mathbb{P}^{n-1}$, $n \geq 3$. We will assume that $k[x_{n-1}, x_n] \hookrightarrow S/I_C$ is a Noether normalization of $S/I_C$. In this case, Theorem 3.6 can be reformulated as follows.

**Proposition 5.1.** $\reg(S/I_C) = \max\{c_1(I_C), r(I_C)\}$.

**Proof.** By the above assumption $S/I_C$ is a generalized Cohen-Macaulay ring of positive depth and $x_n, x_{n-1}$ is a homogeneous system of parameters for $S/I_C$. Therefore, $x_n, x_{n-1}$ is a filter-regular sequence for $S/I_C$. In particular, $x_n$ is a non-zerodivisor in $S/I_C$. By Lemma 3.2 $c_0(I_C) = -\infty$. Hence the conclusion follows from Theorem 3.6.
Since $S/I_C$ has positive depth, the graded minimal free resolution of $S/I_C$ ends at most at the $(n - 1)$th place:
\[ 0 \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I_C \rightarrow 0. \]
From Theorem 3.4 we obtain the following information on the shifts of $F_{n-1}$. Note that $F_{n-1} = 0$ if $S/I_C$ is a Cohen-Macaulay ring or, in other words, if $C$ is an arithmetically Cohen-Macaulay curve.

**Proposition 5.2.** If $C$ is not an arithmetically Cohen-Macaulay curve, $c_1(I_C) + n - 1$ is the maximum degree of the generators of $F_{n-1}$.

**Proof.** Let $b_{n-1}$ be the maximum degree of the generators of $F_{n-1}$. As we have seen in the introduction, $b_{n-1} - n + 1 = (n - 1) - \text{reg}(S/I_C) = \text{reg}(S/I_C)$. By Theorem 3.4 \( \text{reg}(S/I_C) = \max\{c_0(I_C), c_1(I_C)\} = c_1(I_C) \) because $c_0(I_C) = -\infty$. So we obtain $b_{n-1} = c_1(I_C) + n - 1$.

Now we shall see that Proposition 5.1 contains all main results of Bermejo and Gimenez in [BG]. It should be noted that they did not use strong results such as Theorem 2.4. We follow the notations of [BG].

Let $E := \{a \in \mathbb{N}^{n-2} \mid x^a \in \text{in}(I_C)\}$ and denote by $H(E)$ the smallest integer $r$ such that $a \in E$ if $|a| = r$.

**Corollary 5.3 ([BG] Theorem 2.4).** Assume that $C$ is an arithmetically Cohen-Macaulay curve. Then \( \text{reg}(S/I_C) = H(E) - 1 \).

**Proof.** Since $x_n, x_{n-1}$ is a regular sequence in $S/I_C$, we have $c_1(I_C) = -\infty$ by Corollary 3.2. By Proposition 5.1 this implies \( \text{reg}(S/I_C) = r(I_C) \).

But \( r(I_C) = \sup\{r \mid (S_2/J_2)_r \neq 0\} = H(E) - 1 \) because $J_2$ is generated by the monomials $x^a$, $a \in E$.

Let $I_0$ be the ideal in $S$ generated by the polynomials obtained from $I_C$ by the evaluation $x_{n-1} = x_n = 0$. Then $S/I_0$ is a two-dimensional Cohen-Macaulay ring. Let $\tilde{I}$ denote the ideal in $S$ generated by the monomials obtained from $\text{in}(I_C)$ by the evaluation $x_{n-1} = x_n = 1$. Let
\[ F := \{a \in \mathbb{N}^{n-2} \mid x^a \in \tilde{I} \setminus \text{in}(I_0)\}. \]
For every vector $a \in F$ let
\[ E_a := \{(\mu, \nu) \in \mathbb{N}^2 \mid x^\mu x_{n-1}^\nu x_n^a \in \text{in}(I_C)\}. \]
Let $\mathcal{R} := \bigcup_{a \in F} \{a \times [\mathbb{N}^2 \setminus E_a]\}$ and denote by $H(\mathcal{R})$ the smallest integer $r$ such that the number of the elements $b \in \mathcal{R}$ with $|b| = s$ becomes a constant for $s \geq r$.

**Corollary 5.4 ([BG] Theorem 2.7).** \( \text{reg}(S/I_C) = \max\{\text{reg}(S/I_0), H(\mathcal{R})\} \).

**Proof.** As in the proof of Corollary 5.3 we have \( \text{reg}(S/I_0) = r(I_0) \). But $r(I_0) = r(I_C)$ because $\text{in}(I_0)$ is the ideal generated by the monomials obtained from $\text{in}(I_C)$ by the evaluation $x_{n-1} = x_n = 0$. Thus,
\[ \text{reg}(S/I_0) = r(I_C). \]
It has been observed in [BG] that the number of the elements $b \in \mathcal{R}$ with $|b| = s$ is the difference $H_{S/I_C}(s) - H_{S/I}(s) = H_{S/\text{in}(I_C)}(s) - H_{S/\text{in}(I_C)}(s)$, where $H_E(s)$ denotes the Hilbert function of a graded $S$-module $E$. Since $x_n$ is a non-zerodivisor in $S/\text{in}(I_C)$, $H(\mathcal{R})+1$ is the least integer $r$ such that $H_{\tilde{I}/x_{n}^a}(\text{in}(I_C), x_n)(s)$
\[ = 0 \text{ for } s \geq r. \] On the other hand, since \( \text{in}(I_C) \) is generated by monomials which do not contain \( x_0 \), and since \( J_1 \) is the ideal in \( k[x_1, \ldots, x_{n-1}] \) obtained from \( \text{in}(I_C) \) by the evaluation \( x_0 = 0 \), we have \( \text{in}(I_C) = J_1 S \) and \( I = J_1 S \), whence \( (I, x_0)/(\text{in}(I_C), x_0) \cong \tilde{J}_1/J_1 \). Note that \( c_1(I_C) = \max\{r \mid \tilde{J}_1/J_1, r \neq 0 \} \) with \( c_1(I_C) = -\infty \) if \( \tilde{J}_1 = J_1 \). Then
\[
H(\mathcal{R}) = \max\{0, c_1(I_C)\}.
\]
Thus, applying Proposition 5.1 we obtain \( \text{reg}(S/I_C) = \max\{\text{reg}(S/I_0), H(\mathcal{R})\} \).

**Example.** Let \( C \subseteq \mathbb{P}^3 \) be the monomial curve \( (t^\alpha s^\beta : t^\beta s^\alpha : s^{\alpha+\beta} : t^{\alpha+\beta}) \), \( \alpha > \beta > 0 \), g.c.d. \( (\alpha, \beta) = 1 \). It is known that the defining ideal \( I_C \subset k[x_1, x_2, x_3, x_4] \) is generated by the quadric \( x_1 x_2 - x_3 x_4 \) and the forms \( x_1^\beta + x_3^{\alpha - \beta - r} - x_3^{-r} x_1^r \), \( r = 0, \ldots, \alpha - \beta \), and that this is a Gröbner basis of \( I_C \) for the reverse lexicographic order with \( x_1 > x_2 > x_3 > x_4 \) [CM, Théorème 3.9]. Therefore,
\[
\text{in}(I_C) = (x_1 x_2, x_2^\alpha, x_1^\beta + 1, x_3^{\alpha - \beta - 1}, x_1^\beta + 1, x_3^{\alpha - \beta - 2}, \ldots, x_1^\alpha).
\]
Using the notations of Section 3 we have
\[
E_1 = \{(1, 1, 0), (0, \alpha, 0), (\beta + 1, 0, \alpha - \beta - 1), (\beta + 2, 0, \alpha - \beta - 2), \ldots, (\alpha, 0, 0)\},
\]
\[
E_2 = \{(1, 1), (0, \alpha), (\alpha, 0)\}.
\]
From this it follows that
\[
F_1 = \{((1, \beta + 1, 0, \alpha - \beta - 2), (\beta + 2, 0, \alpha - \beta - 3), \ldots, (\alpha - 1, 0, 0)\},
\]
\[
F_2 = \{(0, \alpha - 1), (\alpha - 1, 0)\}.
\]
By Proposition 4.3 \( c_1(I_C) = \alpha - 1 \) if \( \alpha - \beta \geq 2 \) \( (\alpha, 1) = -\infty \) if \( \alpha - \beta = 1 \) and \( r(I_C) = \alpha - 1 \) by Proposition 4.3. Applying Proposition 5.1 we obtain \( \text{reg}(S/I_C) = \alpha - 1 \).

The direct computation of the invariant \( H(\mathcal{R}) \) is more complicated than that of \( c_1(I_C) \). First, we should interpret \( F \) as the set of the elements of the form \( a \in \mathbb{N}^2 \) such that \( a \geq b \) for some elements \( b \in \text{p}(E_1) \) but \( a \not\geq c \) for any element \( c \in E_2 \). Then we get
\[
F = \{((\beta + 1, 0), (\beta + 2, 0), \ldots, (\alpha - 1, 0)\}.
\]
For all \( \varepsilon = \beta + 1, \ldots, \alpha - 1 \) we verify that \( E_{(\varepsilon, \alpha)} = (\alpha - \varepsilon, 0) + \mathbb{N}^2 \). It follows that
\[
\mathcal{R} = \{(\varepsilon, 0, \mu, \nu) \in \mathbb{N}^4 \mid \varepsilon = \beta + 1, \ldots, \alpha - 1; \mu \leq \alpha - \varepsilon - 1\}.
\]
If \( \alpha - \beta = 1 \), we have \( \mathcal{R} = \emptyset \), hence \( H(\mathcal{R}) = 0 \). If \( \alpha - \beta \geq 2 \), we can check that \( H(\mathcal{R}) = \alpha - 1 \).

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**References**


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