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EXISTENCE OF MULTIWAVELETS IN $\mathbb{R}^n$

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ABSTRACT. For a q-regular Multiresolution Analysis of multiplicity r with arbitrary dilation matrix A for a general lattice Γ in $\mathbb{R}^n$, we give necessary and sufficient conditions in terms of the mask and the symbol of the vector scaling function in order that an associated wavelet basis exists. We also show that if $2r(m-1) \geq n$ where m is the absolute value of the determinant of A, then these conditions are always met, and therefore an associated wavelet basis of q-regular functions always exists. This extends known results to the case of multiwavelets in several variables with an arbitrary dilation matrix A for a lattice Γ.

1. Introduction

The concept of Multiresolution Analysis (MRA) due to Mallat [Mal89] and Meyer [Mey92] provided the first systematic way to construct orthonormal wavelet bases of $\boldsymbol{\mathcal{L}}^2(\mathbb{R})$ (i.e., orthonormal bases generated by translations and dilations of a single function). They showed that for every MRA there exists an associated orthonormal wavelet basis. The rich structure of MRA is generated by another function (the scaling function) that satisfies a certain self-similarity condition. The problem of constructing orthonormal wavelets was then shifted to the problem of constructing MRAs. Using this structure, Daubechies [Dau88] was able to prove the existence of compactly supported orthonormal wavelets with arbitrary regularity on the line. After these results, the theory was generalized to different directions in the search for better wavelets bases with prescribed properties. In particular, the concept of MRA was extended in higher dimensions, with dilation matrix $2I_n$. In that case, 2^n-1 wavelets are required to generate the basis. The use of dilation matrices other than $2I_n$ allowed the construction of wavelet bases using fewer wavelets. A further generalization consists in considering MRAs generated by a finite number of scaling functions. The wavelets associated to these MRAs are known as multiwavelets. There is an extensive literature in multiwavelets; see for example [Alp93], [GLT93], [GHM94], [HC96], [HSS96], [CDP97], [Ald97], [JRZ99], [Cal99] and [CHM99]. Finally the lattice $\mathbb{Z}^n$ can be replaced by any general lattice in $\mathbb{R}^n$. For each of these generalizations it is important to know if and under which conditions an associated

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wavelet basis exists. This problem has been studied for particular cases of importance in [Gro87], [Mey92], [Che97], [Woj97], [AK97] and [DS97], but none of these provide a treatment valid for all generalizations. In this paper we are concerned with *q-regular* multiwavelets (see Definition 2.2). We give necessary and sufficient conditions in terms of the mask and the symbol of the scaling function for the existence of an associated wavelet basis. We prove these results for the very general case of orthonormal regular multiwavelets in higher dimensions for an arbitrary dilation matrix A for a lattice Γ. These wavelets are associated to an MRA of multiplicity r (i.e., r scaling functions). In particular we show that if a wavelet basis exists, then it is required to have $(det(A)-1)r$ wavelet functions. We further prove that if a regular MRA of multiplicity r in $\mathbb{R}^n$ is given and $2r(det(A)-1) \geq n$, then the necessary and sufficient conditions hold and a set of regular multiwavelets always exists. The regularity of the wavelets is at least of the same order as the regularity of the scaling functions.

2. Lattices, tiles and Multiresolution Analysis

Let Γ be an arbitrary lattice in $\mathbb{R}^n$ (i.e., $\Gamma = R(\mathbb{Z}^n)$ with R any invertible $n \times n$ matrix with real entries), and let $\widetilde{\Gamma}$ denote the dual lattice, that is $\widetilde{\Gamma} = (R^*)^{-1}(\mathbb{Z}^n)$. (Here $*$ denotes transpose conjugate.) Now let $\boldsymbol{\mathcal{K}}$ and $\widetilde{\boldsymbol{\mathcal{K}}}$ be fundamental domains for these lattices e.g., $\boldsymbol{\mathcal{K}} = R([0,1)^n)$ and $\widetilde{\boldsymbol{\mathcal{K}}} = (R^*)^{-1}([0,1)^n)$ and set $\kappa = |det(R)|$.

Let A be a *dilation matrix for* Γ, i.e., $A(\Gamma) \subset \Gamma$ and every eigenvalue λ of A satisfies $|\lambda| > 1$. Then A^* is a dilation for $\widetilde{\Gamma}$. The determinant of a dilation matrix for a lattice is always an integer and its absolute value is the number of cosets of the quotient group $\Gamma/A(\Gamma)$. A *digit set for* A *and* Γ is any set of representatives of this group.

Let us call $B = A^{-1}$ and $m = |det(A)|$. Given a digit set $D \doteq \{d_0, \dots, d_{m-1}\}$ for A and Γ, the set $\boldsymbol{\mathcal{Q}} \doteq \{\sum_{k=1}^{\infty} A^{-k}\xi_k \ : \xi_k \in D\}$ is compact and always satisfies (see [GM92], [Hut81]) that $\boldsymbol{\mathcal{Q}} + \Gamma = \mathbb{R}^n$ and the self-similarity condition

$$A(\boldsymbol{\mathcal{Q}}) = \bigcup_{s=0}^{m-1} \boldsymbol{\mathcal{Q}} + d_s.$$

The measure of $\boldsymbol{\mathcal{Q}}$ is equal to the measure of the fundamental domain $\boldsymbol{\mathcal{K}}$ of Γ, if and only if $\boldsymbol{\mathcal{Q}} \cap (\boldsymbol{\mathcal{Q}} + k)$ has measure zero for every $k \in \Gamma \setminus \{0\}$. In that case $\boldsymbol{\mathcal{Q}}$ is a *tile* in the sense that the Γ-translates $\{\boldsymbol{\mathcal{Q}} + k\}_{k\in\Gamma}$ cover $\mathbb{R}^n$ with overlaps of measure zero. For a given dilation matrix it is not always true that there exists a digit set with the property that $\boldsymbol{\mathcal{Q}}$ is a tile. (A counterexample was found in [Pot97].)

In this paper we will assume that a digit set D for A and Γ exists such that $\boldsymbol{\mathcal{Q}}$ is a tile, i.e. the measure of $\boldsymbol{\mathcal{Q}}$ is κ. Without loss of generality we will assume that $d_0 = 0$.

We will denote by Γ_h the coset $A\Gamma + d_h$, $h = 0, \dots, m-1$.

We will also assume that a digit set $\widetilde{D} \doteq \{\gamma_0, \dots, \gamma_{m-1}\}$ for the matrix A^* and the lattice $\widetilde{\Gamma}$ exists, such that the set $\widetilde{\boldsymbol{\mathcal{Q}}} \doteq \{\sum_{k=1}^{\infty} (A^*)^{-k}\xi_k \ : \xi_k \in \widetilde{D}\}$ has measure $1/\kappa$. Consequently, $\widetilde{\boldsymbol{\mathcal{Q}}}$ is compact, tiles the space by $\widetilde{\Gamma}$-translates and satisfies the self-similarity condition

$$A^*(\widetilde{\boldsymbol{\mathcal{Q}}}) = \bigcup_{s=0}^{m-1} \widetilde{\boldsymbol{\mathcal{Q}}} + \gamma_s.$$

The systems $\{\frac{1}{\sqrt{\kappa}}\, e^{-2\pi i\gamma\cdot\omega}\}_{\gamma\in\widetilde{\Gamma}}$ and $\{\sqrt{\kappa}\, e^{-2\pi i k\cdot\omega}\}_{k\in\Gamma}$ are orthonormal bases of $\boldsymbol{\mathcal{L}}^2(\boldsymbol{\mathcal{Q}})$ and $\boldsymbol{\mathcal{L}}^2(\widetilde{\boldsymbol{\mathcal{Q}}})$ respectively.

2.1. **A matrix of exponentials.** The following lemma, that we will need later, is a consequence of known orthogonality relations between characters of a group. We will give a direct short proof here.

Lemma 2.1. *Let A be a dilation matrix for a lattice Γ, with $|det(A)| = m$. If D and $\widetilde{D}$ are digit sets for A and A^*, respectively, then the $m\times m$ matrix Θ defined by*

$$\Theta \;=\; \left\{\frac{1}{\sqrt{m}}e^{-2\pi i(A^{-1}d\cdot\gamma)}\right\}_{(d,\gamma)\in D\times\widetilde{D}}$$

is unitary.

Proof. For functions $f, g : D \to \mathbb{C}$ we define the symbol $[f, g] \;=\; \sum_{d\in D} f(d)\,\overline{g(d)}$. We consider in D the group operation from $\Gamma/A(\Gamma)$. If now for each $\gamma \in \widetilde{D}$ we consider f_γ to be the column γ of Θ, i.e. $f_\gamma(d) = \frac{1}{\sqrt{m}}e^{-2\pi i(A^{-1}d\cdot\gamma)}$, $d \in D$, we will show that $[f_{\gamma_1}, f_{\gamma_2}] = \delta_{\gamma_1\gamma_2}$. Note that by the definition of f_γ, $f_\gamma(-d) = \overline{f_\gamma(d)}$ and also $f_{\gamma_1} = f_{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$.

Now fix any $d_0 \in D$ and consider

$$\begin{aligned} f_{\gamma_1}(d_0)[f_{\gamma_1}, f_{\gamma_2}] \;&=\; \sum_{d\in D} f_{\gamma_1}(d_0)\, f_{\gamma_1}(d)\,\overline{f_{\gamma_2}(d)} \;=\; \sum_{d\in D} f_{\gamma_1}(d_0 + d)\overline{f_{\gamma_2}(d)} \\ =\; \sum_{d\in D} f_{\gamma_1}(d)\,\overline{f_{\gamma_2}(d - d_0)} \;&=\; \sum_{d\in D} f_{\gamma_1}(d)\,\overline{f_{\gamma_2}(-d_0)}\;\overline{f_{\gamma_2}(d)} \;=\; f_{\gamma_2}(d_0)\,[f_{\gamma_1}, f_{\gamma_2}]. \end{aligned}$$

Then $(f_{\gamma_1} - f_{\gamma_2})(d)[f_{\gamma_1}, f_{\gamma_2}] = 0$ for each $d \in D$. This shows that if $\gamma_1 \neq \gamma_2$, then $[f_{\gamma_1}, f_{\gamma_2}] = 0$. Otherwise, if $\gamma_1 = \gamma_2$, then $[f_{\gamma_1}, f_{\gamma_2}] = \sum_{d\in D} |f_{\gamma_1}(d)|^2 = \#D\frac{1}{m} = 1$, which completes the proof. ◻

2.2. **Multiresolution Analysis.** A *Multiresolution Analysis* (MRA) *of multiplicity r* associated to a dilation matrix A and a lattice Γ is a sequence of closed subspaces $\{\boldsymbol{\mathcal{V}}_j\}_{j\in\mathbb{Z}}$ of $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ which satisfy:

P1 $\boldsymbol{\mathcal{V}}_j \subset \boldsymbol{\mathcal{V}}_{j+1}$ for each $j \in \mathbb{Z}$,
P2 $g(x) \in \boldsymbol{\mathcal{V}}_j \iff g(Ax) \in \boldsymbol{\mathcal{V}}_{j+1}$ for each $j \in \mathbb{Z}$,
P3 $\bigcap_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j = \{0\}$,
P4 $\bigcup_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j$ is dense in $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$, and
P5 there exist functions $\varphi_1, \dots, \varphi_r \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ such that the collection of lattice translates $\{\varphi_i(x - k)\}_{k\in\Gamma,\, i=1,\dots,r}$ forms an orthonormal basis for $\boldsymbol{\mathcal{V}}_0$.

If these conditions are satisfied, then the vector function $\varphi = (\varphi_1, \dots, \varphi_r)^{\mathrm{T}}$ is referred to as a *scaling vector* for the MRA.

Definition 2.2 ([Mey92])**.** Let q be a non-negative integer. A function f on $\mathbb{R}^n$ is *q-regular* if f is in the class C^q and $\left|\frac{\partial^\alpha}{\partial x^\alpha} f(x)\right| \le \frac{C_k}{(1+|x|)^k}$ for each $k \;=\; 0, 1, 2, \dots$ and each multi-index α with $|\alpha| \le q$. A *q-regular MRA* is an MRA where the scaling vector is q-regular.

Given an MRA we define, as usual, the subspaces $\boldsymbol{\mathcal{W}}_j \equiv \boldsymbol{\mathcal{V}}_{j+1} \ominus \boldsymbol{\mathcal{V}}_j,\ j \in \mathbb{Z}$, i.e., $\boldsymbol{\mathcal{W}}_j$ is the orthogonal complement of $\boldsymbol{\mathcal{V}}_j$ in $\boldsymbol{\mathcal{V}}_{j+1}$. We seek a set of functions in $\boldsymbol{\mathcal{V}}_1$ whose lattice translates form an orthonormal basis of $\boldsymbol{\mathcal{W}}_0$, i.e., a set of functions $f_1, \ldots, f_l \in \boldsymbol{\mathcal{V}}_1$ such that the system $\{f_i(x-k) : i = 1, \ldots, l,\ k \in \Gamma\}$ is complete and orthonormal in $\boldsymbol{\mathcal{W}}_0$. If such a set of functions exists, then the MRA structure will guarantee that the set $\{m^{j/2} f_i(A^j x - k) : i = 1, \ldots, l,\ j \in \mathbb{Z},\ k \in \Gamma\}$ is an orthonormal basis of $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$. We will show that it is necessary to have exactly $(m-1)r$ functions to generate the basis and that a set of functions with this property always exists provided that $2(m-1)r \geq n$. Note that in this case (since $\boldsymbol{\mathcal{V}}_0 \perp \boldsymbol{\mathcal{W}}_0$ and $\boldsymbol{\mathcal{V}}_1 = \boldsymbol{\mathcal{V}}_0 \oplus \boldsymbol{\mathcal{W}}_0$) the system $\{\varphi_i(x-k)\,,\, f_{si}(x-k) \colon\ i = 1, \ldots, r,\ \ s = 1, \ldots, m-1,\ \ k \in \Gamma\}$ forms an orthonormal basis of $\boldsymbol{\mathcal{V}}_1$.

Therefore, the problem reduces to completing the set of vectors $\{\varphi_i(x-k) : i = 1, \ldots, r,\ \ k \in \Gamma\}$ to an orthonormal basis of $\boldsymbol{\mathcal{V}}_1$.

The following proposition, that we will need later, is an immediate generalization of the one-dimensional case (see for example [AK97]).

Proposition 2.3. *The system* $\{\varphi_i(x-k), i = 1, \ldots, r,\ \ k \in \Gamma\}$ *is orthonormal if and only if* $\sum_{\gamma \in \widetilde{\Gamma}} \hat{\varphi}(w+\gamma)\hat{\varphi}^*(w+\gamma) = \kappa\, I_r$.

3. Characterization of the subspace $\mathcal{V}_1$

From now on we assume that an MRA of multiplicity r associated to a dilation matrix A and a lattice Γ in $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$, with scaling vector $\varphi = (\varphi_1, \ldots, \varphi_r)^T \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n, \mathbb{C}^r)$, is given. From the properties of the MRA we see that the system

$$\{m^{\frac{1}{2}} \varphi_i(Ax-k) : i = 1, \ldots, r, k \in \Gamma\}$$

form an orthonormal basis of $\boldsymbol{\mathcal{V}}_1$. This means that for each $f = (f_1, \ldots, f_r)^T$ with $f_i \in \boldsymbol{\mathcal{V}}_1$, $i = 1, \ldots, r$, there exists a unique sequence $\alpha = \{\alpha_k\}_{k \in \Gamma}$ of $r \times r$ matrices with complex entries such that

$$f(x) = \sum_{k \in \Gamma} \alpha_k \varphi(Ax-k). \tag{3.1}$$

The (s,t) entry of α_k is $\alpha_k^{st} = m\langle f_s(x), \varphi_t(Ax-k)\rangle$, and $\sum_{k\in\Gamma} |\alpha_k^{st}|^2 < +\infty$ for $s, t = 1, \ldots, r$.

Let $\hat{f}$ denote the Fourier Transform of the vector f (i.e., $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_r)^T$, where $\hat{f}_s(w) = \int_{\mathbb{R}^n} f_s(x) e^{-2\pi i x\cdot w} dx$). Then equation (3.1) can be written in the Fourier domain as $\hat{f}(\omega) = M_f(B^*\omega)\hat{\varphi}(B^*\omega)$, or $\hat{f}(A^*\omega) = M_f(\omega)\hat{\varphi}(\omega)$ with $M_f(\omega) = \frac{1}{m}\sum_{k\in\Gamma} \alpha_k e^{-2\pi i k\cdot w}$.

Note that $M_f \in \boldsymbol{\mathcal{L}}^2(\widetilde{\boldsymbol{\mathcal{K}}}, \mathbb{C}^{r\times r})$; that is, $M_f = (m_f^{st})_{s,t=1,\ldots,r}$ and m_f^{st} is a $\widetilde{\Gamma}$ periodic function for $s, t = 1, \ldots, r$. We will call M_f the *symbol* of f.

It is clear that the converse also holds, that is, if $M \in \boldsymbol{\mathcal{L}}^2(\widetilde{\boldsymbol{\mathcal{K}}}, C^{r\times r})$, then the function g defined by $\hat{g}(w) = M(B^*w)\hat{\phi}(B^*w)$ has components $g_s \in \boldsymbol{\mathcal{V}}_1$. It is straightforward to see that $\sum_t \|m_f^{st}\|^2_{\boldsymbol{\mathcal{L}}^2(\widetilde{\boldsymbol{\mathcal{K}}})} = \frac{1}{\kappa m}\|f_s\|^2_{\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)}$. The following proposition summarizes the results just obtained:

Proposition 3.1. *Assume* $f = (f_1, \ldots, f_r)^T \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n, \mathbb{C}^r)$. *Then* $f_s \in \boldsymbol{\mathcal{V}}_1$, $s = 1, \ldots, r$, *if and only if there exist* $\widetilde{\Gamma}$*-periodic and* $\widetilde{\boldsymbol{\mathcal{K}}}$*-square-integrable functions* m_f^{st}, $s, t = 1, \ldots, r$, *such that* $\hat{f}(A^*\omega) = M_f(\omega)\hat{\varphi}(\omega)$ *with* $M_f(\omega) = (m_f^{st})_{s,t=1,\ldots,r}$.

Moreover,

$$\sum_{s=1}^{r}\sum_{t=1}^{r}\|m_f^{st}\|^2_{\mathcal{L}^2(\widetilde{\mathcal{K}})} = \frac{1}{\kappa m}\sum_{s=1}^{r}\|f_s\|^2_{\mathcal{L}^2(\mathbb{R}^n)}.$$

Note that since the system $\{\sqrt{m}\,\varphi(Ax-k), k\in\Gamma\}$ is an o.n.b. of $\boldsymbol{\mathcal{V}}_1$, the matrix M_f is unique defined for each $f\in(\boldsymbol{\mathcal{V}}_1)^r$ up to a set of zero measure. Now we see that M_f can be decomposed into the different cosets of Γ, that is

$$M_f \;=\; M_{f0}+\cdots+M_{f(m-1)} \quad\text{with}\quad M_{fh}(\omega)=\frac{1}{m}\sum_{k\in\Gamma_h}\alpha_k e^{-2\pi ik\cdot\omega}.$$

If we define

$$u_{fh}(\omega)=\frac{1}{\sqrt{m}}\sum_{k\in\Gamma}\alpha_{Ak+d_h}e^{-2\pi ik\cdot\omega} \qquad\text{with}\quad h=0,\dots,m-1, \tag{3.2}$$

then we obtain

$$M_{fh}(\omega)=\frac{e^{-2\pi id_h\cdot\omega}}{\sqrt{m}}u_{fh}(A^*\omega) \quad\text{and}\quad M_f(w)=\sum_{h=0}^{m-1}\frac{e^{-2\pi id_h\cdot\omega}}{\sqrt{m}}u_{fh}(A^*\omega). \tag{3.3}$$

With this notation we have the following lemma:

Lemma 3.2. *If $f=(f_1,\dots,f_r)^T$ and $g=(g_1,\dots,g_r)^T$ with $f_s,g_t\in\boldsymbol{\mathcal{V}}_1$, then we have for k and k' in Γ,*

(3.4)

$$[\langle f_s(x-k),g_t(x-k')\rangle]_{s,t=1,\dots,r} \;=\; \kappa\int_{\widetilde{\mathcal{Q}}}\left(\sum_{h=0}^{m-1}u_{fh}(\omega)u_{gh}^*(\omega)\right)e^{-2\pi i(k-k')\cdot\omega}d\omega.$$

Proof. Let us call H the left-hand side of (3.4). Then by Plancherel's Theorem, changing variables and using (3.1) we obtain

$$H=m\int_{\mathbb{R}^n}M_f(\omega)\hat\varphi(\omega)\hat\varphi(\omega)^*M_g(\omega)^*e^{-2\pi iA(k-k')\cdot\omega}d\omega.$$

Since the symbols of f,g are $\widetilde\Gamma$-periodic and $\widetilde{\mathcal{Q}}$ is a $\widetilde\Gamma$-tile, we have

$$H=m\int_{\widetilde{\mathcal{Q}}}M_f(\omega)(\sum_{\gamma\in\widetilde\Gamma}\hat\varphi(\omega+\gamma)\hat\varphi^*(\omega+\gamma))M_g^*(\omega)e^{-2\pi iA(k-k')\cdot\omega}d\omega.$$

Using that φ has orthonormal Γ-translates and Proposition 2.3 we have

$$H=m\kappa\int_{\widetilde{\mathcal{Q}}}\left(\sum_{h=0}^{m-1}M_{fh}(\omega)\right)\left(\sum_{h'=0}^{m-1}M_{gh'}^*(\omega)\right)e^{-2\pi iA(k-k')\cdot\omega}d\omega.$$

Now we observe that if $h\neq h'$, then

$$\int_{\widetilde{\mathcal{Q}}}M_{fh}(\omega)M_{qh'}^*(\omega)e^{-2\pi iA(k-k')\cdot\omega}d\omega=0.$$

To see this, note that the entries of M_{fh} are trigonometric series in which the exponentials are of the type $e^{-2\pi i\gamma\cdot\omega}$ with $\gamma\in\Gamma_h$, and the entries of $M_{gh'}$ are exponentials of the type $e^{-2\pi i\gamma'\cdot\omega}$ with $\gamma'\in\Gamma_{h'}$. Consequently, the products in the integral will involve the exponentials

$$e^{-2\pi i(Ak+\gamma)\cdot\omega}e^{2\pi i(Ak'+\gamma')\cdot\omega}, \tag{3.5}$$

and since $h \neq h'$, then $\Gamma_h \cap \Gamma_{h'} = \emptyset$. Thus $Ak + \gamma \neq Ak' + \gamma'$, which implies that the exponentials (3.5) are orthogonal. Hence we have

$$H = m\kappa \int_{\widetilde{\mathcal{Q}}} \left(\sum_{h=0}^{m-1} M_{fh}(\omega) M_{gh}^*(\omega) \right) e^{-2\pi i (k-k')\cdot A^*\omega} d\omega,$$

and using (3.3) and changing variables again we obtain

$$H = \frac{\kappa}{m} \int_{A^*(\widetilde{\mathcal{Q}})} \left(\sum_{h=0}^{m-1} u_{fh}(\omega) u_{gh}^*(\omega) \right) e^{-2\pi i (k-k')\cdot\omega} d\omega.$$

Now since $A^*(\widetilde{\mathcal{Q}}) = \bigcup_{i=0}^{m-1} (\widetilde{\mathcal{Q}} + \gamma_i)$, we obtain that

$$H = \kappa \int_{\widetilde{\mathcal{Q}}} \left(\sum_{h=0}^{m-1} u_{fh}(\omega) u_{gh}^*(\omega) \right) e^{-2\pi i (k-k')\cdot\omega} d\omega.$$

□

4. Main results

Let us now consider m functions $g_0, \dots, g_{m-1} \in (\boldsymbol{\mathcal{V}}_1)^r$ and set

$$M_s = M_{g_s} \quad \text{and} \quad u_{sh} = u_{g_s h} \quad \text{for} \quad h = 0, \dots, m-1.$$

We associate to the functions $g_0, \dots, g_{m-1}$ the following two block matrices:

$$\boldsymbol{\mathcal{U}}(g_0, \dots, g_{m-1})(\omega) \; = \; \left[\; u_{jh}(\omega) \; \right]_{j,h=0,\dots,m-1} \tag{4.1}$$

and

$$\boldsymbol{\mathcal{M}}(g_0, \dots, g_{m-1})(\omega) \; = \; \left[\; M_j(\omega + B^*\gamma_h) \; \right]_{j,h=0,\dots,m-1},$$

where as before $B = A^{-1}$. Note that $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{M}}$ are matrix-valued $\widetilde{\Gamma}$-periodic functions (i.e., $\boldsymbol{\mathcal{U}}(\omega), \boldsymbol{\mathcal{M}}(\omega) \in (\mathbb{C}^{r\times r})^{m\times m}$ for each individual ω and $\boldsymbol{\mathcal{U}}(\omega + \widetilde{\gamma}) = \boldsymbol{\mathcal{U}}(\omega)$, $\boldsymbol{\mathcal{M}}(\omega + \widetilde{\gamma}) = \boldsymbol{\mathcal{M}}(\omega)$, for all $\widetilde{\gamma} \in \widetilde{\Gamma}$). It is easy to see that the matrix $\boldsymbol{\mathcal{M}}$ *is unitary a.e.* if and only if for each i, $j = 0, \dots, m-1$ we have

$$\sum_{k=0}^{m-1} M_i(\omega + B^*\gamma_k)\, M_j^*(\omega + B^*\gamma_k) \; = \; \delta_{i,j}\, I_{r\times r} \quad \text{for a.e. } \omega \in \mathbb{R}^n.$$

A similar observation applies to $\boldsymbol{\mathcal{U}}$. With this notation we have the following proposition. The proof is an adaptation of [Woj97], Prop. 5.9, for this more general context.

Proposition 4.1. *If* $S_l = \{g_{js}(x-k) : \quad j = 0, \dots, l-1, \quad s = 1, \dots, r, \quad k \in \Gamma\}$, *then we have:*

1. *The system* S_1 *is orthonormal if and only if the block vector* $(u_{0h})_{h=0,\dots,m-1}$ *satisfy* $\sum_{h=0}^{m-1} u_{0h} u_{0h}^* = I_r$ *a.e.*
2. *For each* $l \leq m$, *the system* S_l *is orthonormal if and only if the first* l *block-rows of the matrix* $\boldsymbol{\mathcal{U}}$ *are block orthonormal a.e. (i.e.,* $\sum_{h=0}^{m-1} u_{ih} u_{jh}^* = \delta_{i,j} I_r$ *a.e.,* $i, j = 0, \dots, l-1$*).*
3. *The system* S_m *is an orthonormal basis for* $\boldsymbol{\mathcal{V}}_1$ *if and only if the block matrix* $\boldsymbol{\mathcal{U}}(g_0, \dots, g_{m-1})$ *is unitary a.e.*

Proof. Define the $r \times r$ matrix

$$H_{pq} = \left[\langle g_{ps}(x-k), g_{qt}(x-k')\rangle\right]_{s,t=1,\dots,r}.$$

Using Lemma 3.2, we see that the entries of the matrix H_{pq} are the Fourier coefficients of order $k-k'$ of the $\widetilde{\Gamma}$-periodic function $\sqrt{\kappa}\sum_{h=0}^{m-1} u_{ph}(\omega)u_{qh}^*(\omega)$.

For the proof of part (1) let us consider $p=q=0$. If the function g_0 has orthonormal translates, then $H_{00} = \delta_{kk'}I_r$, and this is equivalent to $\sum_{h=0}^{m-1} u_{0h}(\omega)u_{0h}^*(\omega) = I_r$ for a.e. ω.

For part (2) we see that $H_{pq} = \delta_{pq}I_r$ if and only if $\sum_{h=0}^{m-1} u_{ph}(\omega)u_{qh}^*(\omega) = \delta_{pq}I_r$.

Let us now prove part (3). For this, from part (2) we see that if the functions $g_0, \dots, g_{m-1}$ have orthonormal translates, then the matrix $\boldsymbol{\mathcal{U}}$ has to be unitary. So we only need to see if the system S_m is complete. Assume not, and let $\tilde{g} \in \boldsymbol{\mathcal{V}}_1$ be a function such that

$$\langle \tilde{g}, g_{js}(x-k)\rangle = 0 \qquad \forall\, j = 0,\dots,m-1, \quad s = 1,\dots,r, \quad k \in \Gamma.$$

If $g = (\tilde{g}, 0, \dots, 0)^T$, then $\hat{g}(A^*\omega) = M_g(\omega)\hat{\varphi}(\omega)$ with $M_g \in (\boldsymbol{\mathcal{L}}^2(\widetilde{\boldsymbol{\mathcal{K}}}))^{r\times r}$. The matrix M_g defines a block vector u_g in the same way that the block vector u_s was defined by the matrix M_s. The block vector u_g is block orthogonal to each of the block vectors $(u_{i0}, \dots, u_{i(m-1)})$ for $s = 0, \dots, m-1$. That is, $u_g \equiv 0$, which implies $g = 0$. ∎

As a consequence of Proposition 4.1, we see that it is necessary to have mr functions to generate a basis of $\boldsymbol{\mathcal{V}}_1$ by orthogonal lattice translates. The MRA provides r of these functions (the scaling vector), and these generate the space $\boldsymbol{\mathcal{V}}_0$. The remaining $(m-1)r$ will generate the basis of $\boldsymbol{\mathcal{W}}_0$ and, consequently, the wavelet basis when scaling is allowed. Thus, a corollary of Proposition 4.1 is that a wavelet basis associated to our MRA requires $(m-1)r$ functions. In terms of the unitary matrix $\boldsymbol{\mathcal{U}}$, this translates to the fact that the MRA provides the first r rows and the wavelets the other $(m-1)r$.

4.1. **Necessary and sufficient conditions for the existence of a wavelet basis.** We will now see equivalent conditions for the existence of a wavelet basis.

If $\varphi = (\varphi_1, \dots, \varphi_r)^T \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n, \mathbb{C}^r)$ is the scaling vector for an MRA, then φ satisfies a refinement equation of the form

$$\varphi(x) \;=\; \sum_{k\in\Gamma} c_k\, \varphi(Ax-k),$$

for some matrices c_k in $\mathbb{C}^{r\times r}$. The symbol of this refinement equation is the $\widetilde{\Gamma}$-periodic matrix-valued function $M_0 \in \boldsymbol{\mathcal{L}}^2(\widetilde{\boldsymbol{\mathcal{K}}},\, \mathbb{C}^{r\times r})$ defined by

$$M_0(\omega) \;=\; \frac{1}{m}\sum_{k\in\Gamma} c_k\, e^{-2\pi i k\cdot\omega}, \qquad \omega \in \mathbb{R}^n.$$

The function φ is the unique function satisfying $\hat{\varphi}(A^*\omega) = M_0(\omega)\,\hat{\varphi}(\omega), \omega \in \mathbb{R}^n$.

Now suppose that $M_1, \dots, M_{m-1}$ are in $\boldsymbol{\mathcal{L}}^2(\widetilde{\boldsymbol{\mathcal{K}}},\, \mathbb{C}^{r\times r})$. Let us write these functions together with the function M_0 as

$$M_\ell(\omega) \;=\; \frac{1}{m}\sum_{k\in\Gamma} c_{\ell,k}\, e^{-2\pi i k\cdot\omega}, \qquad \ell = 0,\dots,m-1.$$

In particular set $c_{0,k} = c_k$. Let $\psi_1, \dots, \psi_{m-1}$ be the vector functions in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^r)$ whose Fourier transforms are defined by the formula

$$\hat{\psi}_\ell(A^*\omega) \;=\; M_\ell(\omega)\,\hat{\varphi}(\omega), \qquad \ell = 1, \dots, m-1.$$

We seek necessary and sufficient conditions on $M_1, \dots, M_{m-1}$ such that the lattice translates of $\{\psi_{\ell,i} : \ell = 1, \dots, m-1,\ i = 1, \dots, r\}$ will form an orthonormal basis for $\mathcal{W}_0$. These will be formulated in terms of the matrix $\mathcal{M}(\varphi, \psi_1, \dots, \psi_{m-1})$ known in the engineering literature as the *modulation matrix*, and also in terms of the matrix $\mathcal{U}(\varphi, \psi_1, \dots, \psi_{m-1})$, known as the *polyphase matrix*.

Let us recall here the definitions of $\mathcal{M}$ and $\mathcal{U}$ for these particular functions:

$$\mathcal{M}(\varphi, \psi_1, \dots, \psi_{m-1}) \;=\; \left[\; M_j(\omega + B^*\gamma_h) \;\right]_{j,h=0,\dots,m-1},$$

where γ_h are the digits in $\widetilde{D}$, and

$$\mathcal{U}(\varphi, \psi_1, \dots, \psi_{m-1}) \;=\; [u_{j,h}]_{j,h=0,\dots,m-1},$$

where $u_{jh}(\omega) = \frac{1}{\sqrt{m}} \sum_{k\in\Gamma} c_{j,Ak+d_h} e^{-2\pi i k\cdot\omega}$.

Theorem 4.2. *Let $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ be an MRA for $\mathcal{L}^2(\mathbb{R}^n)$ of multiplicity r. Then, using the notation above, the following statements are equivalent*:

(a) $\{\psi_{\ell,i}(x-k)\}_{k\in\Gamma,\, i=1,\dots,r,\, \ell=1,\dots,m-1}$ *forms an orthonormal basis for* $\mathcal{W}_0$.
(b) $\mathcal{U}$ *is unitary a.e.*
(c) $\mathcal{M}$ *is unitary a.e.*
(d) *The matrix coefficients* $\{c_{ik}\}_{i\,=\,0,\dots,m-1,\, k\in\,\Gamma}$ *satisfy*

$$\frac{1}{m}\sum_{k\in\Gamma} c_{i,k}\, c^*_{j,k-A\nu} \;=\; \delta_{0,\nu}\,\delta_{i,j}\, I_{r\times r} \text{ for every } \nu\in\Gamma \text{ and } i,j = 0,\dots,m-1. \tag{4.2}$$

Proof. The equivalence of (a) and (b) has been proved in Proposition 4.1.

To show the equivalence between (b) and (c) we will use Lemma 2.1. First, recall that by (3.3),

$$M_j(\omega) = \sum_{l=0}^{m-1} \frac{e^{-2\pi i d_l\cdot\omega}}{\sqrt{m}} u_{jl}(A^*\omega) \quad \text{with} \quad u_{jl}(\omega) = \frac{1}{\sqrt{m}} \sum_{s\in\Gamma} c_{jAs+d_l} e^{-2\pi i s\cdot\omega}.$$

Now let $0 \le h \le m-1$ and $\gamma \in \tilde{D}$. We can write

$$\begin{aligned} M_h(\omega + B^*\gamma) \;&=\; \sum_{l=0}^{m-1} \frac{e^{-2\pi i d_l\cdot(\omega+B^*\gamma)}}{\sqrt{m}} u_{hl}(A^*(\omega + B^*\gamma)) \\ &=\; \sum_{l=0}^{m-1} \frac{e^{-2\pi i (d_l\cdot B^*\gamma)}}{\sqrt{m}} \Big(e^{-2\pi i d_l\cdot\omega} u_{hl}(A^*\omega)\Big). \end{aligned} \tag{4.3}$$

For $0 \le t \le m-1$ consider the sum

$$\sum_{\gamma\in\tilde{D}} e^{2\pi i (d_t\cdot B^*\gamma)} M_h(\omega + B^*\gamma) = \sum_{l=0}^{m-1} \Big(\sum_{\gamma\in\tilde{D}} \frac{e^{-2\pi i ((d_l - d_t)\cdot B^*\gamma)}}{\sqrt{m}}\Big) e^{-2\pi i d_l\cdot\omega} u_{hl}(A^*\omega).$$

Now we use Lemma 2.1 to obtain

$$\frac{1}{\sqrt{m}} \sum_{\gamma\in\tilde{D}} e^{2\pi i (d_t\cdot B^*\gamma)} M_h(\omega + B^*\gamma) = e^{-2\pi i d_t\cdot\omega} u_{ht}(A^*\omega),$$

and therefore

$$u_{ht}(\omega) = \sum_{\gamma \in \tilde{D}} \frac{e^{2\pi i(d_t \cdot B^*\gamma)}}{\sqrt{m}} e^{2\pi i(d_t \cdot B^*\omega)} M_h(B^*\omega + B^*\gamma). \tag{4.4}$$

Equations (4.3) and (4.4) show that $\boldsymbol{\mathcal{U}}$ is unitary a.e. if and only if $\boldsymbol{\mathcal{M}}$ is unitary a.e., which shows the equivalence between (b) and (c).

Now we will prove that (c) is equivalent to (d). Let us first prove that if $\boldsymbol{\mathcal{M}}$ is unitary, then the coefficients have to satisfy equation (4.2). Since $\boldsymbol{\mathcal{M}}$ is unitary, we have

$$\sum_{\gamma \in \widetilde{D}} M_i(\omega + B^*\gamma)\, M_j^*(\omega + B^*\gamma) \;=\; \delta_{ij} I_{r\times r} \quad \text{for a.e. } \omega \in \mathbb{R}^n.$$

Writing out the matrix products, this means

$$\frac{1}{m^2} \sum_{\gamma \in \widetilde{D}} \sum_{t=1}^{r} \sum_{k,k' \in \Gamma} c_{ik}^{st}\, \overline{c_{jk'}^{ht}} e^{-2\pi i((k-k')\cdot(\omega + B^*\gamma))} \;=\; \delta_{ij}\delta_{sh} \quad \text{for a.e. } \omega \in \mathbb{R}^n.$$

Changing variables and the order of summation ($k - k' = \ell$) we obtain

$$\frac{1}{m^2} \sum_{t=1}^{r} \sum_{k,\ell \in \Gamma} c_{ik}^{st}\, \overline{c_{j(k-\ell)}^{ht}} \left[\sum_{\gamma \in \widetilde{D}} e^{-2\pi i(\ell \cdot B^*\gamma)} \right] e^{-2\pi i(\ell \cdot \omega)} \;=\; \delta_{ij}\delta_{sh} \quad \text{for a.e. } \omega \in \mathbb{R}^n.$$

Considering $\ell = Au + d$, $u \in \Gamma$, $d \in D$, interchanging the order of summation, and noting that $e^{-2\pi i(Au \cdot B^*\gamma)} = 1$, we have for a.e. $\omega \in \mathbb{R}^n$ that

(4.5)

$$\frac{1}{m^2} \sum_{k,u \in \Gamma} \sum_{d \in D} \sum_{t=1}^{r} c_{ik}^{st}\, \overline{c_{j(k-Au-d)}^{ht}} \left[\sum_{\gamma \in \widetilde{D}} e^{-2\pi i(d \cdot B^*\gamma)} \right] e^{-2\pi i(Au + d\cdot\omega)} \;=\; \delta_{ij}\delta_{sh}.$$

Now, note that $\sum_{\gamma \in \widetilde{D}} e^{-2\pi i(d \cdot B^*\gamma)} = \sum_{\gamma \in \widetilde{D}} e^{-2\pi i(d \cdot B^*\gamma)} \cdot \overline{e^{-2\pi i(0 \cdot B^*\gamma)}}$. Since we assume that $0 \in D$, this is just the dot product between rows d and 0 of the unitary matrix $\{\frac{1}{\sqrt{m}} e^{-2\pi i(d \cdot B^*\gamma)}\}_{d \in D, \gamma \in \widetilde{D}}$ (see Lemma 2.1) and therefore the expression in the square brackets is simply $m\, \delta_{0d}$. Therefore (4.5) becomes

$$\frac{1}{m} \sum_{u \in \Gamma} \left[\sum_{k \in \Gamma} \left(\sum_{t=1}^{r} c_{ik}^{st}\, \overline{c_{j(k-Au)}^{ht}} \right) \right] e^{-2\pi i(u \cdot A^*\omega)} \;=\; \delta_{ij}\delta_{sh} \quad \text{for a.e. } \omega \in \mathbb{R}^n.$$

Looking at the last equality, and interpreting it as the Fourier expansion of a periodic function, we have that

$$\frac{1}{m} \sum_{k \in \Gamma} \left(\sum_{t=1}^{r} c_{ik}^{st}\, \overline{c_{j(k-Au)}^{ht}} \right) \;=\; \delta_{0u}\delta_{ij}\delta_{sh}, \quad \text{or} \quad \frac{1}{m} \sum_{k \in \Gamma} c_{ik}\, c_{j(k-Au)}^{*} \;=\; \delta_{0u}\delta_{ij} I_{r\times r},$$

which proves our claim. For the converse, note that all steps are reversible. □

Thus, once an MRA has been found, we can construct a q-regular wavelet basis for $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ if we can construct a particular unitary matrix function $\boldsymbol{\mathcal{M}}(\omega)$. For each ω, the matrix $\boldsymbol{\mathcal{M}}(\omega)$ is of size $rm \times rm$, and the first r rows of this matrix are known. The q-regularity of φ implies that these first r rows are C^∞ periodic functions. If the remaining rows can be completed so that $\boldsymbol{\mathcal{M}}(\omega)$ is C^∞ and unitary a.e., then we can find the wavelets that generate the wavelet basis. The smoothness of $\boldsymbol{\mathcal{M}}$ will imply the q-regularity of the wavelet functions.

The question of whether this completion can always be accomplished is a very difficult open question. The single function multivariate case, with dilation $2I_n$ is solved by the fundamental Lemma of Gröchenig [Gro87].

We will see in the next section that if $(2m-2)r \geq n$, then $\boldsymbol{\mathcal{M}}(\omega)$ can always be completed so as to be smooth and unitary a.e. However, even in this case it is usually difficult to complete the matrix in such a way that the associated wavelets have some specific properties. For example, it is not known whether, given a compactly supported vector scaling function, the matrix can be completed so that the wavelet is compactly supported.

4.2. **The matrix completion.** In this section we will see that if $2(m-1)r \geq n$ and given r vectors $v_i \in C^\infty\big(\widetilde{\boldsymbol{\mathcal{K}}}, \mathbb{C}^{mr}\big)$, such that $\{v_i(\omega) : i = 1, \dots, r\}$ is an orthonormal set for all $\omega \in \mathbb{R}^n$, then there exist vectors $v_i \in C^\infty\big(\widetilde{\boldsymbol{\mathcal{K}}}, \mathbb{C}^{mr}\big), i = r+1, \dots, mr$, such that $\{v_1(\omega), \dots, v_{mr}(\omega)\}$ is an orthonormal basis of $\mathbb{C}^{mr}$ for all $\omega \in \widetilde{\boldsymbol{\mathcal{K}}}$, or, equivalently, the matrix with rows $v_i^T(\omega)$ is unitary for all $\omega \in \mathbb{R}^n$. This result is a consequence of a proposition due to Ashino and Kametani [AK97] that extends Gröchenig's Lemma to the case of multiplicity r.

Proposition 4.3 ([AK97]). *Let X be a real, compact, C^∞ manifold with dimX=n, and let s,n and d be positive integers satisfying $2(s-d) \geq n$.*

Then, for all C^∞-mappings $f_l : X \longrightarrow \mathbb{C}^s$, $\quad l = 1, \dots, d$, having the property

$$\big(f_k(x), f_l(x)\big) = \delta_{kl} \qquad for\ k, l \in \{1, \dots, d\}, x \in X,$$

there exist C^∞-mappings $f_l : X \longrightarrow \mathbb{C}^s$, $\quad l = d+1, \dots, s$, with the property

$$\big(f_k(x), f_l(x)\big) = \delta_{kl} \qquad for\ k, l \in \{1, \dots, s\}, x \in X.$$

Now we will apply this proposition to prove the existence of a wavelet set.

Theorem 4.4. *For each q-regular MRA of $\mathbb{R}^n$ of multiplicity r with general dilation A such that $2\big(|det(A)| - 1\big)r \geq n$, there exists a wavelet set containing $\big(|det(A)| - 1\big)r$ $q-$regular functions.*

Proof. Set $X = \widetilde{\boldsymbol{\mathcal{K}}}$, $d = r$ and $s = mr$ in Proposition 4.3 and define $v_j(\omega)$ to be the row j of the block vector $\big(u_{\varphi 0}(\omega), \dots, u_{\varphi(m-1)}(\omega)\big)$, where $u_{\varphi h}$ is the matrix-valued function defined in (3.2) and φ is the scaling vector. Since Proposition 4.1 (1) holds, the vectors v_j, $j = 1, \dots, r$, are orthonormal and we can apply Proposition 4.3 to obtain $(m-1)r$ vectors v_j, $j = r+1, \dots, mr$, to construct a unitary matrix with rows $v_j(\omega)$ for each $\omega \in \mathbb{R}^n$. This unitary matrix defines a block unitary matrix $\boldsymbol{\mathcal{U}}$ and consequently a block unitary matrix $\boldsymbol{\mathcal{M}}$. Therefore by Proposition 3.1 the vector functions $\psi_1, \dots, \psi_{m-1}$ constructed as in (4.1) form a wavelet set. ☐

Acknowledgments

After submitting this manuscript it was brought to our attention by the referee that there is some overlap between the present manuscript and the papers *General existence theorems for orthonormal wavelets, an abstract approach*, by L. Baggett, A. Carey, W. Moran and P. Ohring, Publ. Inst. Math. Sci. Kyoto Univ. **31** (1995), 95-111, and *Multiresolution analysis of abstract Hilbert spaces and wandering subspaces*, by D. Han, D. R. Larson, M. Papadakis and T. Stavropoulos, Cont. Math.,

247, (1999), 259-284. We remark that the present work was done completely independently from these papers, and our techniques of proof are substantially different.

The results in these papers are very general and apply to MRAs of abstract Hilbert spaces, in particular, MRAs in multidimensions are special cases of these more general MRAs. They prove the existence of MRA multiwavelets of abstract Hilbert spaces and characterize their corresponding high-pass filter. In particular the equivalence between (a) and (c) of Theorem 4.2 in our paper follows with some work from these more general MRAs.

In the present paper we are interested in the existence of regular multiwavelets, i.e. multiwavelet functions with certain smoothness and decay. This requires the completion of the modulation matrix to a C^∞ unitary matrix. This extra smoothness requirement on the entries of the modulation matrix is essential in order to obtain regular multiwavelets. This is accomplished in the present paper with certain restrictions on the dimensions (imposed by Adam's Theorem) using an extension of Gröchenig's Lemma.

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References

[Ald97] A. Aldroubi, *Oblique and hierarchical multiwavelet bases*, Appl. Comput. Harmon. Anal. **4** (1997), no. 3, 231-263. MR **98k:**42037

[Alp93] B. Alpert, *A class of bases in L^2 for the sparse representation of integral operators*, SIAM J. Math. Anal., **24**, (1993), no. 1, 246-262. MR **93k:**65104

[AK97] R. Ashino and M. Kametani, *A lemma on matrices and a construction of multiwavelets*, Math. Japan, **45** (1997), 267-287. MR **98c:**42025

[CHM99] C. Cabrelli, C. Heil and U. Molter, *Self-Similarity and Multiwavelets in Higher Dimensions*, preprint (1999).

[Cal99] A. Calogero, *Wavelets on general lattices associated with general expanding maps of* $\mathbf{R}^n$, Electron. Res. Announc. Amer. Math. Soc., **5**, (1999), 1-10 (electronic). MR **99i:**42042

[Che97] D.-R. Chen, *On the existence and construction of orthonormal wavelets on* $L^2(\mathbb{R}^s)$, Proc. Amer. Math. Soc., **125**, (1997), 2883-2889. MR **98h:**42030

[CDP97] A. Cohen, I. Daubechies and G. Plonka, *Regularity of refinable function vectors*, J. Fourier Anal. Appl., **3**, (1997), no. 3, 295-324. MR **98e:**42031

[Dau88] I. Daubechies *Orthonormal bases of compactly supported wavelets*, Comm. Pure and Appl. Math., **41**, (1988), 909-996. MR **90m:**42039

[DS97] L. De Michele and P.-M. Soardi, *On multiresolution analysis of multiplicity d*, Mh. Math. **124**, (1997), 255-272. MR **98k:**42039

[GHM94] J. Geronimo, D. Hardin, and P. Massopust, *Fractal functions and wavelet expansions based on several scaling functions*, J. Approx. Theory, **78**, (1994), no. 3, 373-401. MR **95h:**42033

[GLT93] T.N.T. Goodman, S.L. Lee, W.S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc., **338**, (1993), no. 2, 639-654. MR **93j:**42017

[Gro87] K. Gröchenig, *Analyse multiéchelles et bases d'ondelettes*, C. R. Acad. Sci. Paris Sér. I Math., **305**, (1987), 13-15. MR **88j:**47036

[GM92] K. Gröchenig and W. Madych, *Multiresolution analysis, Haar bases and self-similar tilings*, IEEE Trans. Inform. Theory, **38**, (1992), 556-568. MR **93i:**42001

[HC96] C. Heil and D. Colella, *Matrix refinement equations: existence and uniqueness*, J. Fourier Anal. Appl., **2**, (1996), no. 4, 363-377. MR **97k:**39021

[HSS96] C. Heil, G. Strang and V. Strela, *Approximation by translates of refinable functions*, Numer. Math., **73**, (1996), no. 1, 75-94. MR **97c:**65033

[Hut81] J. Hutchinson, *Fractals and Self-similarity*, Indiana Univ. Math. J., **3**, (1981), 713-747. MR **82h:**49026

[JRZ99] R.-Q. Jia, S. Riemenschneider and D.-X. Zhou, *Smoothness of multiple refinable functions and multiple wavelets*, SIAM J. Matrix Anal. Appl., **21**, (1999), no. 1, 1-28 (electronic). MR **2000k:**42050

[Mal89] S. Mallat *Multiresolution approximations and wavelet orthonormal basis of* $L^2(R)$, Trans. Amer. Math. Soc., **315**, (1989), 69-87. MR **90e:**42046

[Mey92] Y. Meyer, *Wavelets and Operators*, Cambridge University Press, Cambridge, 1992. MR **94f:**42001

[Pot97] A. Potiopa, *A problem of Lagarias and Wang*, Master's thesis, Siedlce University, Siedlce, Poland June (1997), (Polish).

[Woj97] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge University Press, 1997. MR **98j:**42025

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