PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 5, Pages 1425–1435
S 0002-9939(01)06228-1
Article electronically published on December 27, 2001

ULTRASTABILITY OF IDEALS OF HOMOGENEOUS POLYNOMIALS AND MULTILINEAR MAPPINGS ON BANACH SPACES

KLAUS FLORET AND STEPHAN HUNFELD

(Communicated by Dale Alspach)

ABSTRACT. Using the theory of full and symmetric tensor norms on normed spaces, a theorem of Kürsten and Heinrich on ultrastability and maximality of normed operator ideals is extended to ideals of n-homogeneous polynomials and n-linear mappings—scalar-valued and vector-valued. The motivation for these results is the following important special case: the "uniterated" Aron-Berner extension $\overline{q}^{\mathfrak{U}}$: $E'' \longrightarrow F''$ of an n-homogeneous polynomial $q : E \longrightarrow F$ to the bidual remains in certain ideals under preservation of the norm. Moreover, Lotz's characterization of maximal normed ideals of linear mappings through appropriate tensor norms is proved for ideals of n-homogeneous scalar-valued polynomials and ideals of n-linear mappings.

1. PRELIMINARIES

1.1. For vector spaces $E_1, \dots, E_n, E, F$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ the "full" n-fold *tensor product* of $(E_1, \dots, E_n)$ will be denoted by $E_1 \otimes \cdots \otimes E_n = \otimes(E_1, \dots, E_n) = \otimes_{j=1}^n E_j$. The universal definition of the n-fold tensor product identifies the n-linear mappings $E_1 \times \cdots \times E_n \longrightarrow F$ with the linear mappings $\otimes_{j=1}^n E_j \longrightarrow F$:

$$L(E_1, \dots, E_n; F) = L(\otimes_{j=1}^n E_j; F), \quad \varphi \rightsquigarrow \varphi^L .$$

Symmetric linear mappings $E^n \longrightarrow F$ are linearized by the *n-th symmetric tensor product* $\otimes^{n,s} E$:

$$L_s({}^n E; F) = L(\otimes^{n,s} E; F), \quad \varphi \rightsquigarrow \varphi^{L,s}$$

(see e.g. [F1] for more details). A mapping $q : E \longrightarrow F$ is, by definition, an n-homogeneous polynomial (notation: $q \in P^n(E; F)$) if there is a $\varphi \in L({}^n E; F)$ with $q(x) = \varphi(x, \dots, x)$ for all $x \in E$. Actually there is a unique $\check{q} \in L_s({}^n E; F)$ with this property:

$$P^n(E; F) = L_s({}^n E; F) = L(\otimes^{n,s} E; F), \quad q \rightsquigarrow \check{q} \rightsquigarrow q^L .$$

If the spaces are normed, then the continuous n-linear mappings and n-homogeneous polynomials are denoted by $\mathcal{L}(E_1, \dots, E_n; F)$, $\mathcal{L}_s({}^n E; F)$ or $\mathcal{P}^n(E; F)$, respectively.

Received by the editors February 9, 1999 and, in revised form, November 20, 2000.
2000 *Mathematics Subject Classification.* Primary 46B08; Secondary 46B28, 46G25.
Key words and phrases. Tensor products, symmetric tensor products, ideals of polynomials, ideals of n-linear mappings, ultraproducts.

1.2. Again for fixed $n \in \mathbb{N}$ the *projective norm* π is uniquely defined by the property

$$\big[\otimes_\pi(E_1,\dots,E_n)\big]' \overset{1}{=} \mathcal{L}(E_1,\dots,E_n) :\overset{1}{=} \mathcal{L}(E_1,\dots,E_n;\mathbb{K})$$

if $E_1,\dots,E_n$ are normed spaces ($\overset{1}{=}$ means isometrically equal).

The *injective norm* ε satisfies, by definition ($\overset{1}{\hookrightarrow}$ stands for a metric injection),

$$\otimes_\varepsilon(E_1,\dots,E_n) \overset{1}{\hookrightarrow} \big(\otimes_\pi(E_1',\dots,E_n')\big)';$$

this shows that ε is somehow dual to π. A *tensor norm β of order n* assigns to each n-tuple $(E_1,\dots,E_n)$ a norm $\beta(\cdot;E_1,\dots,E_n)$ on $\otimes(E_1,\dots,E_n)$ (notation: $\otimes_\beta(E_1,\dots,E_n)$ or $\otimes_{\beta,j=1}^n E_j$) such that

(a) $\varepsilon \le \beta \le \pi$,
(b) $\|\otimes_{j=1}^n T_j : \otimes_{\beta,j=1}^n E_j \longrightarrow \otimes_{\beta,j=1}^n F_j\| \le \prod_{j=1}^n \|T_j : E_j \longrightarrow F_j\|$ for all operators $T_j \in \mathcal{L}(E_j;F_j)$ (the metric mapping property).

Note that (a) and (b) imply that in (b) there is actually equality. β is called *finitely generated* if for all E_j and $z \in \otimes_{j=1}^n E_j$

$$\beta(z;E_1,\dots,E_n) = \inf\big\{\beta(z;M_1,\dots,M_n) \;\big|\; M_j \in \mathrm{FIN}(E_j), z \in \otimes_{j=1}^n M_j\big\}$$

(where $\mathrm{FIN}(E)$ denotes the set of finite-dimensional subspaces of E). ε and π are finitely generated tensor norms of order n. There is no general reference for the theory of tensor norms of order $n > 2$; many results, however, are straightforward generalizations of the case $n = 2$ which, e.g., is treated in [DF].

1.3. In the same spirit the natural *projective* and *injective* norms π_s and ε_s on the n-th symmetric tensor product satisfy

$$\big(\otimes_{\pi_s}^{n,s} E\big)' \overset{1}{=} \mathcal{P}^n(E;\mathbb{K}) =: \mathcal{P}^n(E),$$

$$\otimes_{\varepsilon_s}^{n,s} E \overset{1}{\hookrightarrow} \mathcal{P}^n(E')\,.$$

An *s-tensor norm α of order n* (or shortly s-tensor norm, if $n \in \mathbb{N}$ is clear) assigns to each normed space E a norm $\alpha(\cdot;\otimes^{n,s}E)$ on $\otimes^{n,s}E$ (notation: $\otimes_\alpha^{n,s}E$) such that

(a) $\varepsilon_s \le \alpha \le \pi_s$,
(b) the metric mapping property $\|\otimes^{n,s}T : \otimes_\alpha^{n,s}E \longrightarrow \otimes_\alpha^{n,s}F\| \le \|T : E \longrightarrow F\|^n$ for all $T \in \mathcal{L}(E;F)$.

α is called *finitely generated* if for all E and $z \in \otimes^{n,s}E$,

$$\alpha(z;\otimes^{n,s}E) = \inf\big\{\alpha(z;\otimes^{n,s}M) \;\big|\; M \in \mathrm{FIN}(E),\ z \in \otimes^{n,s}M\big\}\,.$$

A detailed study of ε_s and π_s can be found in [F1]; the theory of s-tensor norms (in the spirit of Grothendieck's theory of tensor norms of order 2, see [DF]) will be developed in a forthcoming paper [F2]. We do not need anything from the general theory in this paper. Note that for convenience the definitions allow n to be 1: in this case $\otimes_\alpha^1 E \overset{1}{=} E \overset{1}{=} \otimes_\alpha^{1,s}E$.

1.4. Let $\mathfrak{U}$ be an ultrafilter on a set I; the ultraproduct (along $\mathfrak{U}$) of a family $(E_\iota)_{\iota\in I}$ of Banach spaces E_ι will be denoted by $(E_\iota)_\mathfrak{U}$ (see e.g. [DF, 18.4]). Now take $\varphi_\iota \in \mathcal{L}(E_{1,\iota},\dots,E_{n,\iota};F_\iota)$ for all $\iota \in I$ such that $\sup_{i\in I}\|\varphi_\iota\| < \infty$. Then

$$\overline{(\varphi_\iota)}_\mathfrak{U}\big((x_\iota^1)_\mathfrak{U},\dots,(x_\iota^n)_\mathfrak{U}\big) := \big(\varphi_\iota(x_\iota^1,\dots,x_\iota^n)\big)_\mathfrak{U}$$

defines an n-linear map $(E_{1,\iota})_{\mathfrak{U}} \times \cdots \times (E_{n,\iota})_{\mathfrak{U}} \longrightarrow (F_\iota)_{\mathfrak{U}}$ between the ultraproducts; it easily follows that $\|\overline{(\varphi_\iota)}_{\mathfrak{U}}\| = \lim_{\mathfrak{U}} \|\varphi_\iota\|$. If $F_\iota = \mathbb{K}^m$, then

$$\overline{(\varphi_\iota)}_{\mathfrak{U}}\big((x_\iota^n)_{\mathfrak{U}}, \dots, (x_\iota^n)_{\mathfrak{U}}\big) = \lim_{\mathfrak{U}} \varphi_\iota(x_\iota^1, \dots, x_\iota^n)$$

since $(\mathbb{K}^m)_{\mathfrak{U}} = \mathbb{K}^m$; in this case we write $\lim_{\mathfrak{U}} \varphi_\iota$ for $\overline{(\varphi_\iota)}_{\mathfrak{U}}$. If $E_{1,\iota} = \cdots = E_{n,\iota} = E_\iota$, it is clear that $(\varphi_\iota)_{\mathfrak{U}}$ is symmetric if all φ_ι are; in particular, for polynomials $q_\iota \in \mathcal{P}^n(E_\iota; F_\iota)$ with $\sup_\iota \|q_\iota\| < \infty$,

$$\overline{(q_\iota)}_{\mathfrak{U}}\big((x_\iota)_{\mathfrak{U}}\big) := \big(q_\iota(x_\iota)\big)_{\mathfrak{U}} = \overline{(\check{q}_\iota)}_{\mathfrak{U}}\big((x_\iota)_{\mathfrak{U}}, \dots, (x_\iota)_{\mathfrak{U}}\big)$$

is an n-homogeneous polynomial $(E_\iota)_{\mathfrak{U}} \longrightarrow (F_\iota)_{\mathfrak{U}}$. Again it is rather immediate to see that

$$\big(\mathcal{P}^n(E_\iota; F_\iota)\big)_{\mathfrak{U}} \longrightarrow \mathcal{P}^n\big((E_\iota)_{\mathfrak{U}}; (F_\iota)_{\mathfrak{U}}\big), \quad (q_\iota)_{\mathfrak{U}} \longrightarrow \overline{(q_\iota)}_{\mathfrak{U}}$$

is an isometry; for $F_\iota = \mathbb{K}^m$ we write $\lim_{\mathfrak{U}} q_\iota := \overline{(q_\iota)}_{\mathfrak{U}}$. The purpose of this paper is to study under which circumstances $\overline{(q_\iota)}_{\mathfrak{U}}$ (resp. $\overline{(\varphi_\iota)}_{\mathfrak{U}}$) is in a certain class of polynomials (resp. n-linear mappings) if all q_ι (resp. φ_ι) are.

1.5. For this, two special properties of ultraproducts will be needed: local determination and local duality.

Proposition (local determination of ultraproducts)**.** *Let E_ι be normed spaces for $\iota \in I$ and $\{0\} \neq M \in \mathrm{FIN}\big((E_\iota)_{\mathfrak{U}}\big)$. Then there are, for all $\iota \in I$, operators $R_\iota \in \mathcal{L}(M; E_\iota)$ such that*

- (a) *$x = (R_\iota x)_{\mathfrak{U}}$ for all $x \in M$;*
- (b) *$\|R_\iota\| \le 1$ for all $\iota \in I$ and there is an $\mathcal{U} \in \mathfrak{U}$ with $\|R_\iota\| = 1$ for all $\iota \in \mathcal{U}$;*
- (c) *for all $\varepsilon > 0$ there is an $\mathcal{U}_\varepsilon \in \mathfrak{U}$ such that the inverse $R_\iota^{-1} : R_\iota(M) \longrightarrow M$ exists and $\|R_\iota^{-1}\| \le 1 + \varepsilon$ for all $\iota \in \mathcal{U}_\varepsilon$.*

This is due to Kürsten [K, Satz 4.1] and Heinrich [H, Prop. 6.1]. Here we shall only need (a) and the first part of (b).

1.6. For the local duality, take normed spaces E_ι and denote by

$$J : (E'_\iota)_{\mathfrak{U}} \longrightarrow (E_\iota)'_{\mathfrak{U}}, \quad J(x'_\iota)_{\mathfrak{U}} = \lim_{\mathfrak{U}} x'_\iota$$

the natural map from 1.4., i.e., $\langle J(x'_\iota)_{\mathfrak{U}}, (x_\iota)_{\mathfrak{U}} \rangle = \lim_{\mathfrak{U}} \langle x'_\iota, x_\iota \rangle$, and by

$$K : (E_\iota)_{\mathfrak{U}} \hookrightarrow (E_\iota)''_{\mathfrak{U}} \xrightarrow{J'} (E'_\iota)'_{\mathfrak{U}}\,.$$

Proposition (local duality of ultraproducts)**.** *Let E_ι be normed spaces for all $\iota \in I$, $N \in \mathrm{FIN}\big((E_\iota)'_{\mathfrak{U}}\big)$ and $L \in \mathrm{FIN}\big((E_\iota)_{\mathfrak{U}}\big)$. Then for every $\varepsilon > 0$ there is an operator $T \in \mathcal{L}(N; (E'_\iota)_{\mathfrak{U}})$ such that*

- (a) *$\|T\| = 1$ and $\|T^{-1} : TN \longrightarrow N\| \le 1 + \varepsilon$,*
- (b) *$JTx' = x'$ for all $x' \in N \cap \operatorname{im} J$,*
- (c) *$\langle JTx', x \rangle = \langle Kx, Tx' \rangle = \langle x', x \rangle$ for all $x' \in N$ and $x \in L$.*

This result is due to Kürsten [K], Stern (see [H] for references) and Heinrich [H]; the present formulation is taken from the proof of [H, Theorem 7.3]. We shall only need $\|T\| = 1$ and (c).

2. The main theorem on ultrastability

2.1. Every $z' \in \left[\otimes_{k=1}^{m}\left(\otimes^{n_k,s}E_k\right)\right]^*$ defines a $\left(\sum_{k=1}^{m} n_k\right)$-linear functional φ on $\prod_{k=1}^{m}(E_k)^{n_k}$ which is symmetric in the n_k variables in E_k for each $k = 1,\dots,m$, and vice-versa. In the spirit of the notations of 1.1 we may define $\varphi^L := z'$. For ultraproducts and $z'_\iota = \varphi_\iota^L$ the notation

$$\lim_{\mathfrak{U}} z'_\iota := \left[\lim_{\mathfrak{U}} \varphi_\iota\right]^L$$

will be used in the following:

Theorem. *Let $m, n_1, \dots, n_m \in \mathbb{N}$, $\mathfrak{U}$ an ultrafilter on I, normed spaces $E_{k,\iota}$, a finitely generated tensor norm β of order m and finitely generated s-tensor norms α_k of order n_k be given. If $z'_\iota \in \left[\otimes_{\beta,k=1}^{m}\left(\otimes_{\alpha_k}^{n_k,s}E_{k,\iota}\right)\right]' =: H_\iota$ for all $\iota \in I$ such that $\sup_{\iota\in I}\|z'_\iota\|_{H_\iota} < \infty$, then*

$$\lim_{\mathfrak{U}} z'_\iota \in \left[\otimes_{\beta,k=1}^{m}\left(\otimes_{\alpha_k}^{n_k,s}\left(E_{k,\iota}\right)_{\mathfrak{U}}\right)\right]' =: H$$

and $\|\lim_{\mathfrak{U}} z'_\iota\|_H \le \lim_{\mathfrak{U}}\|z'_\iota\|_{H_\iota}$.

Proof. Since β and all α_k are finitely generated, it is enough to show that for all $M_k \in \mathrm{FIN}\big((E_{k,\iota})_{\mathfrak{U}}\big)$ and all $z \in \otimes_{k=1}^{m}[\otimes^{n_k,s}M_k] =: M$,

$$|\langle \lim_{\mathfrak{U}} z'_\iota, z\rangle| \le \lim_{\mathfrak{U}}\|z'_\iota\|_{H_\iota}\cdot\beta\big(z;\otimes_{\alpha_1}^{n_1,s}M_1,\dots,\otimes_{\alpha_m}^{n_m,s}M_m\big)$$

holds. Given these M_k the local determination of ultraproducts (see 1.5) gives operators $R_{k,\iota}: M_k \longrightarrow E_{k,\iota}$ with $\|R_{k,\iota}\| \le 1$ and $(R_{k,\iota}x_k)_{\mathfrak{U}} = x_k$ for all $x_k \in M_k$; it follows that

$$\begin{aligned}
&\Big\langle \lim_{\mathfrak{U}} z'_\iota, \underbrace{\left[\otimes^{n_1}x_1\right]\otimes\cdots\otimes\left[\otimes^{n_m}x_m\right]}_{=:x}\Big\rangle\\
&\qquad= \Big\langle \lim_{\mathfrak{U}} z'_\iota, \left[\otimes^{n_1}(R_{1,\iota}x_1)_{\mathfrak{U}}\right]\otimes\cdots\otimes\left[\otimes^{n_m}(R_{m,\iota}x_m)_{\mathfrak{U}}\right]\Big\rangle\\
&\qquad= \lim_{\mathfrak{U}}\Big\langle z'_\iota, \underbrace{\left[\otimes^{n_1,s}R_{1,\iota}\right]\otimes\cdots\otimes\left[\otimes^{n_m,s}R_{m,\iota}\right]}_{=:R_\iota}x\Big\rangle
\end{aligned}$$

and hence for all $z \in M$,

$$\begin{aligned}
|\langle \lim_{\mathfrak{U}} z'_\iota, z\rangle| &= |\lim_{\mathfrak{U}}\langle z'_\iota, R_\iota z\rangle|\\
&\le |\lim_{\mathfrak{U}}\|z'_\iota\|_{H_\iota}\beta\big(R_\iota z;\otimes_{\alpha_1}^{n_1,s}E_{1,\iota},\dots,\otimes_{\alpha_m}^{n_m,s}E_{m,\iota}\big)|\\
&\le |\lim_{\mathfrak{U}}\|z'_\iota\|_{H_\iota}\|R_{1,\iota}\|^{n_1}\cdots\|R_{m,\iota}\|^{n_m}\beta\big(z;\otimes_{\alpha_1}^{n_1}M_1,\dots,\otimes_{\alpha_m}^{n_m}M_m\big)|
\end{aligned}$$

by the metric mapping properties of β and $\alpha_1,\dots,\alpha_m$. This is the desired inequality. □

In other words, the natural map

$$\left(\left[\otimes_{\beta,k=1}^{m}\left(\otimes_{\alpha_k}^{n_k,s}E_{k,\iota}\right)\right]'\right)_{\mathfrak{U}} \longrightarrow \left[\otimes_{\beta,k=1}^{m}\left(\otimes_{\alpha_k}^{n_k,s}\left(E_{k,\iota}\right)_{\mathfrak{U}}\right)\right]'$$

has norm ≤ 1. It is rather immediate that the norm is 1 if all $E_\iota \ne \{0\}$ and that for $\beta = \pi$ and $\alpha_k = \pi_s$ this is even an isometry (as in the special cases of 1.4, one has to take $x_{k,\iota} \in B_{E_{k,\iota}}$ with $|\langle x'_\iota, \dots\rangle| \ge \|x'_\iota\|(1-\varepsilon)\big)$.

2.2. Note the special cases of $m = 1$ (polynomials) and all $n_k = 1$ (no symmetry):

(a) If α is a finitely generated s-tensor norm of order n, then for all normed spaces E_ι the natural map

$$\Big(\big(\otimes_\alpha^{n,s} E_\iota\big)'\Big)_{\mathfrak{U}} \longrightarrow \Big(\otimes_\alpha^{n,s} (E_\iota)_{\mathfrak{U}}\Big)'$$

has norm ≤ 1.

(b) If β is a finitely generated tensor norm of order m, then for all normed spaces $E_{k,\iota}$ the natural map

$$\Big(\big(\otimes_\beta(E_{1,\iota}, \dots , E_{m,\iota})\big)'\Big)_{\mathfrak{U}} \longrightarrow \Big(\otimes_\beta\big((E_{1,\iota})_{\mathfrak{U}}, \dots , (E_{m,\iota})_{\mathfrak{U}}\big)\Big)'$$

has norm ≤ 1.

2.3. It is clear that the same reasoning gives that the natural map

$$\Big(\Big[\otimes_\alpha^{m,s}\big(\otimes_{\beta,k=1}^{n} E_{k,\iota}\big)\Big]'\Big)_{\mathfrak{U}} \longrightarrow \Big[\otimes_\alpha^{m,s}\Big(\otimes_{\beta,k=1}^{n}\big(E_{k,\iota}\big)_{\mathfrak{U}}\Big)\Big]'$$

has norm ≤ 1.

3. Scalar-valued ideals of polynomials and multilinear mappings

3.1. A subclass $\mathcal{Q} \subset \mathcal{P}^n$ of n-homogeneous continuous scalar-valued polynomials on Banach spaces is called an *ideal*, if

(a) $\mathcal{Q}(E) := \mathcal{P}^n(E) \cap \mathcal{Q}$ *is a linear subspace of* $\mathcal{P}^n(E)$ *for all Banach spaces* E,
(b) *if* $T \in \mathcal{L}(E; F)$ *and* $q \in \mathcal{Q}(F)$, *then* $q \circ T \in \mathcal{Q}$,
(c) $[\mathbb{K} \ni z \rightsquigarrow z^n \in \mathbb{K}] \in \mathcal{Q}$.

If $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \longrightarrow [0, \infty]$ satisfies

(a′) $\|\cdot\|_{\mathcal{Q}}\big|_{\mathcal{Q}(E)}$ *is a norm for all Banach spaces* E,
(b′) $\|q \circ T\|_{\mathcal{Q}} \le \|T\|^n \, \|q\|_{\mathcal{Q}}$ *in the situation of* (b),
(c′) $\|\mathbb{K} \ni z \rightsquigarrow z^n \in \mathbb{K}\|_{\mathcal{Q}} = 1$,

then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a *normed ideal of n-homogeneous polynomials.* It can easily be seen that always $\|q\| \le \|q\|_{\mathcal{Q}}$ and that $\mathcal{Q}(M) = \mathcal{P}^n(M) = \otimes^{n,s} M'$ for all finite-dimensional M. It would also be possible to define ideals of polynomials on normed spaces (not only Banach spaces) – but this would not make much of a difference.

It is rather immediate to see that for each s-tensor norm α of order n

$$\mathcal{Q}(E) := (\otimes_\alpha^{n,s} E)'$$

defines a normed ideal of n-homogeneous polynomials. For $\alpha = \varepsilon_s$ one obtains the integral polynomials (see e.g. [F1, chap. 3] for their properties). It is not difficult to see that all *extendible* n-homogeneous polynomials (i.e., those $q \in \mathcal{P}^n(E)$ such that for all super spaces $G \supset E$ there is an extension $\widetilde{q} \in \mathcal{P}^n(G)$ of G; see Kirwan and Ryan [KR]) are also of this form; in this case the s-tensor norm α of order n is the "injective associate" of the projective s-norm π_s, i.e., satisfies

$$\otimes_\alpha^{n,s} E \overset{1}{\hookrightarrow} \otimes_{\pi_s}^{n,s} \ell_\infty(B_{E'}) \, ;$$

we omit the details.

3.2. For $(\mathcal{Q}, \| \ \|_{\mathcal{Q}})$ and $q \in \mathcal{P}^n(E)$ define

$$\|q\|_{\mathcal{Q}_{\max}} := \sup\bigl\{\|q\big|_M\|_{\mathcal{Q}} \bigm| M \in \mathrm{FIN}(E)\bigr\} \in [0, \infty]\,.$$

$(\mathcal{Q}, \| \ \|_{\mathcal{Q}})$ is called *maximal* if every $q \in \mathcal{P}^n(E)$ with $\|q\|_{\mathcal{Q}_{\max}} < \infty$ is in $\mathcal{Q}$ and $\|q\|_{\mathcal{Q}} = \|q\|_{\mathcal{Q}_{\max}}$. $(\mathcal{Q}, \| \ \|_{\mathcal{Q}})$ is called *ultrastable* if for $q_\iota \in \mathcal{Q}(E_\iota)$ with $\sup_{i \in I} \|q_\iota\|_{\mathcal{Q}} < \infty$ one has $\lim_{\mathfrak{U}} q_\iota \in \mathcal{Q}\bigl((E_\iota)_{\mathfrak{U}}\bigr)$ and $\| \lim_{\mathfrak{U}} q_\iota\|_{\mathcal{Q}} \le \sup \|q_\iota\|_{\mathcal{Q}}$ (and hence $\le \lim_{\mathfrak{U}} \|q_\iota\|_{\mathcal{Q}}$).

Theorem. *For each normed ideal $(\mathcal{Q}, \| \ \|_{\mathcal{Q}})$ of n-homogeneous (scalar-valued) polynomials the following statements are equivalent:*

(1) $(\mathcal{Q}, \| \ \|_{\mathcal{Q}})$ *is maximal.*
(2) $(\mathcal{Q}, \| \ \|_{\mathcal{Q}})$ *is ultrastable.*
(3) *There is a finitely generated s-tensor norm α of order n such that*

$$\mathcal{Q}(E) \overset{1}{=} (\otimes_\alpha^{n,s} E)'$$

for all Banach spaces E.

It is clear that this result can be conjectured when knowing the Kürsten-Heinrich characterization of maximal Banach operator ideals [H, Theorem 8.1] and Lotz' representation with tensor norms of order 2 (see [DF, 17.5]. It follows from (3) that $\mathcal{Q}(E)$ is a Banach space if $\mathcal{Q}$ is maximal.

Proof. (1) $\curvearrowright$ (3). For finite-dimensional Banach spaces M define α by

$$\otimes_\alpha^{n,s} M := \mathcal{Q}(M)'$$

and for arbitrary normed spaces E by

$$\alpha(z; \otimes^{n,s} E) := \inf\bigl\{\alpha(z; \otimes^{n,s} M) \bigm| M \in \mathrm{FIN}(E),\, z \in \otimes^{n,s} M\bigr\}\,.$$

It is straightforward to see that α is a finitely generated s-tensor norm of order n with $(*)$ $\mathcal{Q}(M) \overset{1}{=} (\otimes_\alpha^{n,s} M)'$ if M is finite-dimensional. Now

$$\begin{aligned} \|q^L\|_{(\otimes_\alpha^{n,s} E)'} &= \sup\bigl\{|\langle q^L, z\rangle| \bigm| \alpha(z; \otimes^{n,s} E) < 1\bigr\} \\ &= \sup\bigl\{|\langle (q\big|_M)^L, z\rangle| \bigm| M \in \mathrm{FIN}(E),\, z \in \otimes^{n,s} M,\, \alpha(z; \otimes^{n,s} M) < 1\bigr\} \\ &\overset{(*)}{=} \sup\bigl\{\|q\big|_M\|_{\mathcal{Q}} \bigm| M \in \mathrm{FIN}(E)\bigr\} = \|q\|_{\mathcal{Q}} \end{aligned}$$

by the maximality of $\mathcal{Q}$.

(3) $\curvearrowright$ (2). This is the special case 2.2(a) of the main theorem.

(2) $\curvearrowright$ (1). Let $q \in \mathcal{P}^n(E)$ with $\|q\|_{\mathcal{Q}_{\max}} < \infty$ and let $\mathfrak{U}$ be an ultrafilter finer than the order filter in $\mathrm{FIN}(E)$. For $q\big|_M$ on $M \in \mathrm{FIN}(E)$ one obtains $\|q\big|_M\|_{\mathcal{Q}} \le \|q\|_{\mathcal{Q}_{\max}}$ and $\lim_{\mathfrak{U}} q\big|_M : (M)_{\mathfrak{U}} \longrightarrow \mathbb{K}$ extends q (via the natural isometric embedding $E \hookrightarrow (M)_{\mathfrak{U}}$). If follows from (2) that $q \in \mathcal{Q}$ and

$$\|q\|_{\mathcal{Q}} \le \| \lim_{\mathfrak{U}} q\big|_M\|_{\mathcal{Q}} \le \sup \|q\big|_M\|_{\mathcal{Q}} = \|q\|_{\mathcal{Q}_{\max}}$$

and hence $\|q\|_{\mathcal{Q}} = \sup \|q\big|_M\|_{\mathcal{Q}}$ since always $\| \cdot \|_{\mathcal{Q}_{\max}} \le \| \cdot \|_{\mathcal{Q}}$. $\square$

3.3. Every $q \in \mathcal{P}^n(E)$ has an extension to $\mathcal{P}^n(E'')$: the Aron-Berner extension (see [AB]) can easily be seen as an iterated limit along local ultrafilters of E (see [LR] or [F1, 6.9]). This motivated Dineen and Timoney [DT] and Lindström and Ryan [LR] independently to define an "uniterated" Aron-Berner extension (as it is called in [F1]) as follows: for $\iota := (M, N, \varepsilon) \in I := \mathrm{FIN}(E'') \times \mathrm{FIN}(E') \times]0,1]$ choose with the strong principle of local reflexity an operator $T_\iota \in \mathcal{L}(M; E)$ with $T_\iota x = x$ for all $x \in M \cap E$ such that $\|T_\iota\| \le 1 + \varepsilon$ and $\langle T_\iota x'', x' \rangle$ for all $(x'', x') \in M \times N$; for $x'' \in E''$ define $f_\iota(x'') := T_\iota x''$ if $x'' \in M$ and $f_\iota(x'') := 0$ otherwise. Take a local ultrafilter $\mathfrak{U}$ on I, i.e., an ultrafilter which is finer than the order filter on I, then the natural mappings

$$\begin{array}{llllll} J_E : E'' & \longrightarrow & (E)_{\mathfrak{U}}; & x'' & \rightsquigarrow & (f_\iota(x''))_{\mathfrak{U}}, \\ Q_E : (E)_{\mathfrak{U}} & \longrightarrow & E''; & (x_\iota)_{\mathfrak{U}} & \rightsquigarrow & \lim_{\iota,\mathfrak{U}} x_\iota \end{array}$$

($\sigma(E'', E')$-limit) have the following properties: J_E is an isometry which extends the natural embedding $E \hookrightarrow (E)_{\mathfrak{U}}$, the mapping Q_E has norm 1 (if $E \ne \{0\}$) and $J_E Q_E : (E)_{\mathfrak{U}} \longrightarrow \operatorname{im} J_E \overset{1}{=} E''$ is a norm-1-projection. For $q \in \mathcal{P}^n(E)$ define

$$\overline{q}^{\mathfrak{U}} := \big[\lim_{\iota,\mathfrak{U}} q\big] \circ J_E \in \mathcal{P}^n(E'')$$

hence $\overline{q}^{\mathfrak{U}}(x'') = \lim_{\iota,\mathfrak{U}} q(f_\iota(x''))$; it is clear that $\overline{q}^{\mathfrak{U}}$ extends q. The foregoing theorem implies the following:

Theorem. *Let $\mathcal{Q}$ be a maximal normed ideal of n-homogeneous polynomials, $q \in \mathcal{P}^n(E)$ and $\mathfrak{U}$ a local ultrafilter of E. Then $q \in \mathcal{Q}(E)$ if and only if $\overline{q}^{\mathfrak{U}} \in \mathcal{Q}(E'')$; in this case $\|\overline{q}^{\mathfrak{U}}\|_{\mathcal{Q}} = \|q\|_{\mathcal{Q}}$.*

In particular, this applies to the class of integral polynomials. Note that it took considerable effort to prove this result for the iterated Aron-Berner extension in the cases $\mathcal{Q} := \mathcal{P}^n$ (Davie-Gamelin [DG]) and $\mathcal{Q} := \{\text{integral polynomials}\}$ (due to [CZ]; in [F1, 6.8] there is an alternative proof).

3.4. It is clear that a theorem like 3.1 holds also for normed ideals of n-linear functionals originally defined by Pietsch [P] in 1983: a subclass $\mathcal{A}$ of all n-linear continuous functionals on Banach spaces is an *ideal* if (for all Banach spaces E_j and F_j)

(a) $\mathcal{A}(E_1, \dots, E_n) := \mathcal{A} \cap \mathcal{L}(E_1, \dots, E_n)$ *is a linear subspace of* $\mathcal{L}(E_1, \dots, E_n)$,
(b) *if* $T_j \in \mathcal{L}(E_j; F_j)$ *and* $\varphi \in \mathcal{A}(E_1, \dots, E_n)$, *then* $\varphi \circ (T_1, \dots, T_n) \in \mathcal{A}$,
(c) $[\mathbb{K}^n \ni (x_1, \dots, x_n) \rightsquigarrow x_1 \cdots x_n \in \mathbb{K}] \in \mathcal{A}$.

If $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \longrightarrow [0, \infty]$ satisfies

(a′) $\|\cdot\|_{\mathcal{A}}\big|_{\mathcal{A}(E_1, \dots, E_n)}$ *is a norm on* $\mathcal{A}(E_1, \dots, E_n)$,
(b′) $\|\varphi \circ (T_1, \dots, T_n)\|_{\mathcal{A}} \le \|T\| \cdots \|T_n\| \, \|\varphi\|_{\mathcal{A}}$ *in the situation of* (b),
(c′) $\|\mathbb{K}^n \ni (x_1, \dots, x_n) \rightsquigarrow x_1 \cdots x_n \in \mathbb{K}\|_{\mathcal{A}} = 1$,

then $(\mathcal{A}, \|\ \|_{\mathcal{A}})$ is called a *normed ideal of n-linear functionals*. It is obvious from 3.2 how to define that $(\mathcal{A}, \|\ \|_{\mathcal{A}})$ is maximal or ultrastable. The same kind of ideas as in the proof of Theorem 3.2 gives the following:

Theorem. *For every normed ideal $(\mathcal{A}, \|\ \|_{\mathcal{A}})$ of n-linear functionals the following statements are equivalent:*

(1) $(\mathcal{A}, \|\ \|_{\mathcal{A}})$ *is maximal.*
(2) $(\mathcal{A}, \|\ \|_{\mathcal{A}})$ *is ultrastable.*

(3) *There exists a finitely generated tensor norm β of order n with*

$$\big(\otimes_\beta(E_1,\dots,E_n)\big)' \overset{1}{=} \mathcal{A}(E_1,\dots,E_n)$$

for all Banach spaces $E_1,\dots,E_n$.

4. Vector-valued ideals of polynomials and multilinear mappings

4.1. *A normed ideal of n-homogeneous continuous vector-valued polynomials on Banach spaces* is, by definition, a pair $(\mathcal{Q},\|\ \|_{\mathcal{Q}})$ such that:

(a) $\mathcal{Q}(E;F):=\mathcal{Q}\cap\mathcal{P}^n(E;F)$ *is a linear subspace of* $\mathcal{P}^n(E;F)$ *and* $\|\ \|_{\mathcal{Q}}\big|_{\mathcal{Q}(E;F)}$ *is a norm on it.*

(b) *If* $T\in\mathcal{L}(E_1;E_2)$, $q\in\mathcal{Q}(E_2;E_3)$ *and* $S\in\mathcal{L}(E_3;E_4)$, *then* $S\circ q\circ T\in\mathcal{Q}$ *and* $\|S\circ q\circ T\|_{\mathcal{Q}}\le\|T\|^n\,\|q\|_{\mathcal{Q}}\,\|S\|$.

(c) $[\mathbb{K}\ni z\rightsquigarrow z^n\in\mathbb{K}]$ *is in* $\mathcal{Q}$ *and has norm* 1.

4.2. We are only interested in ideals coming from tensor norms. For this the following notation will be used:

$$\mathcal{L}(E;G')=(E\otimes_\pi G)',\quad T\rightsquigarrow\beta_T,$$

$$\mathcal{P}^n(E;G')=\mathcal{L}\big(\otimes_{\pi_s}^{n,s}E;G'\big)=\big((\otimes_{\pi_s}^{n,s}E)\otimes_\pi G\big)',\quad q\rightsquigarrow\beta_q:=\beta_{q^L}.$$

Let α be a finitely generated s-tensor norm of order n and β a finitely generated tensor norm of order 2. We define $q\in\mathcal{P}^n_{(\alpha,\beta)}(E;F)$ if $\beta_{(\varkappa_F\circ q)}\in\big((\otimes_\alpha^{n,s}E)\otimes_\beta F'\big)'$ and in this case $\|q\|_{(\alpha,\beta)}:=\|\beta_{\varkappa_F\circ q}\|_{(\dots)}$ (here $\varkappa_F:F\hookrightarrow F''$ is the natural injection). It is easy to see that $(\mathcal{P}^n_{(\alpha,\beta)},\|\ \|_{(\alpha,\beta)})$ is a normed ideal in the sense of 4.1. If $\mathcal{C}$ is the maximal Banach operator ideal associated with the dual tensor norm β' of β, then from the representation theorem [DF, 17.5] we get

$$\mathcal{P}^n_{(\alpha,\beta)}(E;F)\overset{1}{=}\mathcal{C}\big(\widetilde{\otimes}_\alpha^{n,s}E;F\big).$$

A consequence of this is that $\mathcal{P}^n_{(\alpha,\beta)}$ is *regular*, i.e., $q\in\mathcal{P}^n_{(\alpha,\beta)}(E;F)$ if (and only if) $\varkappa_F\circ q\in\mathcal{P}^n_{(\alpha,\beta)}(E;F'')$ and $\|q\|_{(\alpha,\beta)}=\|\varkappa_F\circ q\|_{(\alpha,\beta)}$. It is also rather routine to show that $\mathcal{P}^n_{(\alpha,\beta)}$ is *maximal* in the following sense: $q\in\mathcal{P}^n(E;F)$ is in $\mathcal{P}^n_{(\alpha,\beta)}$ if

$$\sup\big\{\|Q_L^F\circ q\big|_M\|_{(\alpha,\beta)}\ \big|\ L\in\mathrm{COFIN}(E),\ M\in\mathrm{FIN}(E)\big\}<\infty$$

and then this number is $\|q\|_{(\alpha,\beta)}$ (where $\mathrm{COFIN}(E):=\{L\subset E/L$ finite-codimensional, closed subspace of $E)=\{N^0\mid N\in\mathrm{FIN}(E')\}$ and $Q_L^F:F\longrightarrow F/L$ the canonical quotient map). However, it seems to be unlikely that every maximal and regular $(\mathcal{Q},\|\ \|_{\mathcal{Q}})$ is of the form $\mathcal{P}^n_{(\alpha,\beta)}$, contrary to the scalar case.

Just one example: Since $[\otimes_{\varepsilon_s}^{n,s}E]\otimes_\varepsilon F\overset{1}{\hookrightarrow}\mathcal{P}^n(E';F)$, one can deduce from [JM, Lemma 2.1.] (see also [A2] for the reflexive case) that $\mathcal{P}^n_{\varepsilon_s,\varepsilon}(E;F')$ is the space of integral n-homogeneous polynomials $E\longrightarrow F'$ in the sense of Alencar [A1].

Theorem. *The ideal* $\mathcal{P}^n_{(\alpha,\beta)}$ *is ultrastable, i.e., if* $\mathfrak{U}$ *is an ultrafilter on* I *and* $q_\iota\in\mathcal{P}^n_{(\alpha,\beta)}(E_\iota;F_\iota)$ *such that* $\sup\|q_\iota\|_{(\alpha,\beta)}<\infty$, *then* $\overline{(q_\iota)_{\mathfrak{U}}}\in\mathcal{P}^n_{(\alpha,\beta)}\big((E_\iota)_{\mathfrak{U}};(F_\iota)_{\mathfrak{U}}\big)$ *and* $\|\overline{(q_\iota)_{\mathfrak{U}}}\|_{(\alpha,\beta)}\le\lim_{\mathfrak{U}}\|q_\iota\|_{(\alpha,\beta)}$.

See 1.4 for the notation.

Proof. Let $V \in \mathcal{L}\big(\otimes_{\pi_s}^{n,s}(E_\iota)_{\mathfrak{U}};(F_\iota)_{\mathfrak{U}}\big)$ be the operator associated with $\overline{(q_\iota)}_{\mathfrak{U}}$. One has to show that $\widetilde{V} \in \mathcal{C}\big(\widetilde{\otimes}_\alpha^{n,s}(E_\iota)_{\mathfrak{U}};(F_\iota)_{\mathfrak{U}}\big)$ where $(\mathcal{C},\mathbf{C})$ is the Banach operator ideal associated with β' and $\widetilde{V}$ is the extension of V to the completion $\widetilde{\otimes}_\alpha^{n,s}(E_\iota)_{\mathfrak{U}}$. From the main theorem in Section 2 we get

$$z' := \lim_{\mathfrak{U}} \beta_{\varkappa_{F_\iota} \circ q_\iota} \in \Big[\big(\otimes_\alpha^{n,s}(E_\iota)_{\mathfrak{U}}\big) \otimes_\beta (F_\iota')_{\mathfrak{U}}\Big]' =: H$$

and $\|z'\|_H \le \lim_{\mathfrak{U}} \|q_\iota\|_{(\alpha,\beta)}$. If $z' = \beta_U$ with $U \in \mathcal{L}\big(\widetilde{\otimes}_\alpha^{n,s}(E_\iota)_{\mathfrak{U}};(F_\iota')'_{\mathfrak{U}}\big)$ (hence $\mathbf{C}(U) = \|z'\|_H$), then $U = K \circ \widetilde{V}$ where $K : (F_\iota)_{\mathfrak{U}} =: F \longrightarrow (F_\iota')'_{\mathfrak{U}} =: G'$ is the natural mapping from 1.6 (check on $[\otimes^n(x_\iota)_{\mathfrak{U}}] \otimes (x_\iota')_{\mathfrak{U}}$). Using maximality it is enough to show that

$$\|M \hookrightarrow \widetilde{\otimes}_\alpha^{n,s}(E_\iota)_{\mathfrak{U}} \overset{\widetilde{V}}{\longrightarrow} F \overset{Q_{N^0}^F}{\longrightarrow} F/N^0 = N'\| \le \|z'\|_H$$

for all $M \in \mathrm{FIN}\big(\widetilde{\otimes}_\alpha^{n,s}(E_\iota)_{\mathfrak{U}}\big)$ and $N \in \mathrm{FIN}(F')$. With $L := \widetilde{V}(M) \subset F$ the local duality 1.6 of ultraproducts gives an operator $T \in \mathcal{L}(N;G)$ with $\|T\| = 1$ and

$$\langle K\widetilde{V}(u), Tx'\rangle_{G',G} = \langle x', \widetilde{V}(u)\rangle_{F',F} = \langle Q_{N^0}^F \widetilde{V}(u), x'\rangle_{N',N}$$

for all $u \in M$ and $x' \in N$. This means $Q_{N^0}^F \circ \widetilde{V}\big|_M = T' \circ K \circ \widetilde{V}\big|_M$. It follows that $\mathbf{C}(Q_{N^0}^F \circ \widetilde{V}\big|_M) \le \|T'\|\mathbf{C}(K \circ \widetilde{V}\big|_M) = \mathbf{C}(U\big|_M) \le \|z'\|_H$ and therefore $\|\overline{(q_\iota)}_{\mathfrak{U}}\|_{(\alpha,\beta)} = \mathbf{C}(\widetilde{V}) \le \lim_{\mathfrak{U}} \|q_\iota\|_{(\alpha,\beta)}$. ☐

4.3. Note the special case $\beta = \pi$ and $q_\iota^0 : E_\iota \longrightarrow \otimes_\alpha^{n,s} E_\iota$ being the "canonical" polynomial $q_\iota^0(x) := \otimes^n x$, i.e., $(q_\iota^0)^L = \mathrm{id}_{\otimes_\alpha^{n,s} E_\iota}$ and $\|q_\iota^0\|_{(\alpha,\pi)} = 1$. Then $\overline{(q_\iota^0)}_{\mathfrak{U}}$ is in $\mathcal{P}_{\alpha,\pi}\big((E_\iota)_{\mathfrak{U}};\big(\widetilde{\otimes}_\alpha^{n,s} E_\iota\big)_{\mathfrak{U}}\big)$ which means that the natural map

$$\otimes_\alpha^{n,s}(E_\iota)_{\mathfrak{U}} \longrightarrow \big(\otimes_\alpha^{n,s} E_\iota\big)_{\mathfrak{U}}$$

defined by $\otimes^n(x_\iota)_{\mathfrak{U}} \rightsquigarrow (\otimes^n x_\iota)_{\mathfrak{U}}$ has norm ≤ 1, if α is a finitely generated s-tensor norm.

4.4. For any ultrafilter $\mathfrak{U}$ and normed space F the map $Q_F : (F)_{\mathfrak{U}} \longrightarrow F''$ from 3.3 is well-defined, extends the natural embedding $F \hookrightarrow (F)_{\mathfrak{U}}$ and satisfies

$$\langle Q_F((y_\iota)_{\mathfrak{U}}), y'\rangle_{F'',F'} = \lim_{\iota,\mathfrak{U}} \langle y_\iota, y'\rangle_{F,F'}\,.$$

If $\mathfrak{U}$ is a local ultrafilter of another space E and $q \in \mathcal{P}^n(E;F)$, then (see 3.3 for the notation)

$$\overline{q}^{\mathfrak{U}} := Q_F \circ \overline{(q)}_{\mathfrak{U}} \circ J_E : E'' \longrightarrow F''$$

is an n-homogeneous polynomial which extends q and can be calculated as follows:

$$\langle \overline{q}^{\mathfrak{U}}(x''), y'\rangle_{F'',f'} = \lim_{\iota,\mathfrak{U}} \langle q(f_\iota(x'')), y'\rangle\,.$$

Theorem 4.2 and the regularity of $\mathcal{P}_{(\alpha,\beta)}^n$ imply the following:

Corollary. *Take $q \in \mathcal{P}^n(E;F)$ and a local ultrafilter on E. Then the extension $\overline{q}^{\mathfrak{U}}$ is in $\mathcal{P}_{(\alpha,\beta)}^n(E'';F'')$ if and only if q is; in this case $\|q\|_{(\alpha,\beta)} = \|\overline{q}^{\mathfrak{U}}\|_{(\alpha,\beta)}$.*

4.5. For n-linear operators $E_1 \times \cdots \times E_n \longrightarrow F$ the same ideas apply: a *normed ideal of n-linear continuous operators between Banach spaces* is a pair $(\mathcal{A}, \| \ \|_{\mathcal{A}})$ such that

(a) $\mathcal{A}(E_1, \ldots, E_n; F) = \mathcal{A} \cap \mathcal{L}(E_1, \ldots, E_n; F)$ *is linear and* $\| \ \|_{\mathcal{A}}\big|_{\mathcal{A}(E_1,\ldots,E_n;F)}$ *is a norm,*

(b) *if* $T_j \in \mathcal{L}(G_j; E_j)$, $\varphi \in \mathcal{A}(E_1, \ldots, E_n; F)$ *and* $S \in \mathcal{L}(F; G)$, *then the composition* $S \circ \varphi \circ (T_1, \ldots, T_n)$ *is in* $\mathcal{A}$ *and*

$$\|S \circ \varphi \circ (T_1, \ldots, T_n)\|_{\mathcal{A}} \leq \|S\| \, \|\varphi\|_{\mathcal{A}} \, \|T_1\| \cdots \|T_n\| \, ,$$

(c) $[\mathbb{K}^n \ni (x_1, \ldots, x_n) \rightsquigarrow x_1 \cdots x_n \in \mathbb{K}]$ *is in* $\mathcal{A}$ *and* $\| \cdots \|_{\mathcal{A}} = 1$.

Every tensor norm β of order $n+1$ defines an ideal $\mathcal{A}_\beta$ as follows: an n-linear map φ is in $\mathcal{A}_\beta(E_1, \ldots, E_n; F)$ if and only if the $(n+1)$-linear form associated with $\varkappa_F \circ \varphi$ is in $[\otimes_\beta(E_1, \ldots, E_n, F')]'$.

An ideal $(\mathcal{A}, \| \ \|_{\mathcal{A}})$ is called *maximal* if

$$\|\varphi\|_{\mathcal{A}^{\max}} := \sup\bigl\{ \|Q_L^F \circ \varphi\big|_{M_1 \times \cdots \times M_n}\|_{\mathcal{A}} \bigm| M_j \in \mathrm{FIN}(E_j),\, L \in \mathrm{COFIN}(F) \bigr\} < \infty$$

implies $\varphi \in \mathcal{A}$ and $\|\varphi\|_{\mathcal{A}} = \|\varphi\|_{\mathcal{A}^{\max}}$ holds. The ideal is called *regular* if $\varkappa_F \circ \varphi \in \mathcal{A}(E_1, \ldots, E_n; F'')$ implies $\varphi \in \mathcal{A}$ and $\|\varphi\|_{\mathcal{A}} = \|\varkappa_F \circ \varphi\|_{\mathcal{A}}$; it is easy to see that the ideals $\mathcal{A}_\beta$ are regular. $\mathcal{A}$ is *ultrastable* if for $\varphi_\iota \in \mathcal{A}(E_{1,\iota}, \ldots, E_{n,\iota}; F_\iota)$ with $\|\varphi\|_{\mathcal{A}} \leq c$ the operator $\overline{(\varphi_\iota)}_{\mathfrak{U}}$ is in $\mathcal{A}\bigl((E_{1,\iota})_{\mathfrak{U}}, \ldots, (E_{n,\iota})_{\mathfrak{U}}; (F_\iota)_{\mathfrak{U}}\bigr)$ and $\|\overline{(\varphi_\iota)}_{\mathfrak{U}}\|_{\mathcal{A}} \leq \sup \|\varphi_\iota\|_{\mathcal{A}}$.

Theorem. *Let $(\mathcal{A}, \| \ \|)_{\mathcal{A}}$ be a normed ideal of n-linear continuous mappings between Banach spaces. Then the following statements are equivalent:*

(1) $(\mathcal{A}, \| \ \|_{\mathcal{A}})$ *is maximal.*

(2) $(\mathcal{A}, \| \ \|_{\mathcal{A}})$ *is ultrastable and regular.*

(3) *There is a finitely generated tensor norm β of order $n+1$ such that*

$$\mathcal{A}(E_1, \ldots, E_n; F') \stackrel{1}{=} \bigl(\otimes_\beta(E_1, \ldots, E_n, F)\bigr)',$$

$$\mathcal{A}(E_1, \ldots, E_n; F) \stackrel{1}{=} \bigl(\otimes_\beta(E_1, \ldots, E_n, F')\bigr)' \cap \mathcal{L}(E_1, \ldots, E_n; F) \, .$$

Proof. (1) $\curvearrowright$ (3) runs exactly as in the case $n = 1$ (see [DF, 17.5.], the extension lemma holds also for finitely generated tensor norms of arbitrary order) with a construction of β as in the proof of Theorem 3.2.

(3) $\curvearrowright$ (2). Ultrastability follows from the main theorem as in the proof of Theorem 4.2; the regularity follows immediately when looking at the $(n+1)$-linear functionals appearing in the two formulae in (3).

(2) $\curvearrowright$ (1). Take $\varphi \in \mathcal{L}(E_1, \ldots, E_n; F)$ with $\|\varphi\|_{\mathcal{A}^{\max}} < \infty$. Following Heinrich's proof for the case $n = 1$ (see [H]) consider $I := \mathrm{FIN}(E_1) \times \cdots \times \mathrm{FIN}(E_n) \times \mathrm{FIN}(F')$ and let $\mathfrak{U}$ be an ultrafilter finer than the order filter. For $\iota = (M_1, \ldots, M_n, N)$ define $E_{k,\iota} := M_k$ and $F_\iota := F/N^0$ and metric embeddings

$$J_k : E_k \longrightarrow (E_{k,\iota})_{\mathfrak{U}} \quad x_k \rightsquigarrow (x_{k,\iota})_{\mathfrak{U}}$$

where $x_{k,\iota} = x_k$ if $x_k \in M_k$ and $= 0$ otherwise. Moreover, define a mapping $Q : (F_\iota)_{\mathfrak{U}} \longrightarrow F''$ by

$$\langle Q(y_\iota)_{\mathfrak{U}}, y' \rangle := \lim_{\mathfrak{U}} \langle y_\iota, y'_\iota \rangle$$

where $y'_\iota = y'$ if $y' \in N$ and $= 0$ otherwise; Q has norm ≤ 1.

For $\varphi_\iota := Q_{N^0}^F \circ \varphi\big|_{M_1 \times \cdots \times M_n}$ one obtains

$$\varkappa_F \circ \varphi = Q \circ \overline{(\varphi_\iota)}_{\mathfrak{U}} \circ (J_1, \ldots, J_n) \in \mathcal{A}$$

with $\|\varkappa_F \circ \varphi\|_{\mathcal{A}} \leq \lim_\iota \|\varphi\|_{\mathcal{A}} \leq \|\varphi\|_{\mathcal{A}^{\max}}$; the regularity gives $\varphi \in \mathcal{A}$, and $\|\varphi\|_{\mathcal{A}} = \|\varkappa_F \circ \varphi\|_{\mathcal{A}} \leq \|\varphi\|_{\mathcal{A}^{\max}}$. □

For $n = 1$ the equivalence (1) $\curvearrowleft$ (2) is the Heinrich-Kürsten result for Banach operator ideals and (2) $\curvearrowleft$ (3) Lotz' respresentation theorem. Note that the result implies that all $\mathcal{A}(E_1, \dots, E_n; F)$ are Banach spaces if $(\mathcal{A}, \| \ \|_{\mathcal{A}})$ is maximal.

4.6. As in 4.3 one obtains that the natural map

$$\otimes^n_{\gamma,k=1}(E_{k,\iota})_{\mathfrak{U}} \longrightarrow \Big(\otimes^n_{\gamma,k=1} E_{k,\iota}\Big)_{\mathfrak{U}}$$

defined by $(x'_\iota)_{\mathfrak{U}} \otimes \cdots \otimes (x^n_\iota)_{\mathfrak{U}} \rightsquigarrow (x^1_\iota \otimes \cdots \otimes x^n_\iota)_{\mathfrak{U}}$ has norm ≤ 1 if γ is a finitely generated tensor norm or order n. For a proof define the tensor norm β of order $n+1$ by

$$\otimes_\beta(E_1, \dots, E_n, E_{n+1}) := [\otimes_\gamma(E_1, \dots, E_n)] \otimes_\pi E_{n+1}\,,$$

and apply the theorem to $\otimes : E_{1,\iota} \times \cdots \times E_{n,\iota} \longrightarrow \widetilde{\otimes}_\gamma(E_{1,\iota}, \dots, E_{n,\iota}) =: F_\iota$.

References

[A1] R. Alencar, *On reflexivity and basis of* $P(^mE)$, Proc. Roy. Irish Acad. **85** (1985), 131–138. MR **87i:**46101

[A2] R. Alencar, *An application of Singer's theorem to homogeneous polynomials*, Contemp. Math. **144** (1993), 1–8. MR **94c:**53087

[AB] R. Aron, P. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France **106** (1978), 3–24. MR **80e:**46029

[CD] D. Carando, V. Dimant, *Duality in spaces of nuclear and integral polynomials*, J. Math. Anal. Appl. **241** (2000), 107–121. MR **2001c:**46089

[CZ] D. Carando, I. Zalduendo, *A Hahn-Banach theorem for integral polynomials*, Proc. Amer. Math. Soc. **127** (1999), 241–250. MR **99b:**46067

[DF] A. Defant, K. Floret, *Tensor Norms and Operator Ideals*, North Holland Math. Studies 176, 1993. MR **94e:**46130

[DG] A. Davie, T. Gamelin, *A theorem on polynomial-star approximation*, Proc. Amer. Math. Soc. **106** (1989), 351–356. MR **89k:**46023

[DT] S. Dineen, R.M. Timoney, *Complex geodesics on convex domains*, in: Progress Funct. Anal. (eds.: Bierstedt, Bonet, Horvath, Maestre) North Holland Math. Studies **170** (1992), 333–365. MR **92m:**46066

[F1] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note di Matematica (Trier-conference 1997) **17** (1997), 153–188. MR **2001g:**46038

[F2] K. Floret, *The metric theory of symmetric tensor products of normed spaces*, in preparation.

[H] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. **313** (1980), 77–104. MR **82b:**46013

[JM] J.A. Jaramillo, L.A. de Moraes, *Duality and reflexivity in spaces of polynomials*, Arch. Math. (Basel) **74** (2000), 282–293. MR **2000k:**46063

[K] K.D. Kürsten, *s-Zahlen und Ultraprodukte von Operatoren in Banachräumen*, Doctoral Thesis, Leipzig, 1976.

[KR] P. Kirwan, R. Ryan, *Extendibility of homogeneous polynomials on Banach spaces*, Proc. AMS **126** (1998), 1023–1029. MR **98f:**46042

[LR] M. Lindström, R. Ryan, *Applications of ultraproducts to infinite dimensional holomorphy*, Math. Scand. **71** (1992), 229–242. MR **94c:**46090

[P] A. Pietsch, *Ideals of multilinear functionals*, Proc. 2nd Int. Conf. Operator Alg., etc. (Teubner Texte Math. 62), Leipzig (1984), 185–199. MR **85g:**00027

Department of Mathematics, University of Oldenburg, D-26111 Oldenburg, Germany
E-mail address: floret@mathematik.uni-oldenburg.de

Werstener Dorfstrasse 209, D-40591 Düsseldorf, Germany