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FREDHOLMNESS AND INVERTIBILITY OF TOEPLITZ OPERATORS WITH MATRIX ALMOST PERIODIC SYMBOLS

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Abstract. We consider Toeplitz operators with symbols that are almost periodic matrix functions of several variables. It is shown that under certain conditions on the group generated by the Fourier support of the symbol, a Toeplitz operator is Fredholm if and only if it is invertible.

1. Preliminaries and main result

Let (AP^k) be the algebra of complex valued almost periodic functions of k real variables, i.e., the closed subalgebra of $L^\infty(\mathbb{R}^k)$ (with respect to the standard Lebesgue measure) generated by all the functions $e_\lambda(t) = e^{i\langle\lambda,t\rangle}$, where $\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k$. Here the variable $t = (t_1, \cdots, t_k) \in \mathbb{R}^k$, and

$$\langle\lambda, t\rangle = \sum_{j=1}^{k} \lambda_j t_j$$

is the standard inner product of λ and t. For every $f(t) \in (AP^k)$ its *Fourier series* is defined by the formal sum

$$\sum_\lambda f_\lambda e^{i\langle\lambda,t\rangle}, \tag{1.1}$$

where

$$f_\lambda = \lim_{T\to\infty} \frac{1}{(2T)^k} \int_{[-T,T]^k} e^{-i\langle\lambda,t\rangle} f(t)dt,\ \lambda \in \mathbb{R}^k, \tag{1.2}$$

and the sum in (1.1) is taken over the set $\sigma(f) = \{\lambda \in \mathbb{R}^k :\ f_\lambda \neq 0\}$, called the *spectrum* of f. The spectrum of every $f \in (AP^k)$ is at most a countable set. Denote by $\Lambda(f)$ the smallest additive subgroup of $\mathbb{R}^k$ which contains $\sigma(f)$. The *mean* $M\{f\}$ of $f \in (AP^k)$ is defined by $M\{f\} = f_0 = \lim_{T\to\infty} \frac{1}{(2T)^k} \int_{[-T,T]^k} f(t)dt$. For the general theory of almost periodic functions of one and several variables we refer the reader to [4, 8, 9] and to Chapter 1 in [10].

Let Δ be a non-empty subset of $\mathbb{R}^k$. Denote

$$(AP^k)_\Delta = \{f \in (AP^k) :\ \sigma(f) \subseteq \Delta\}. \tag{1.3}$$

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If Δ is an additive subgroup of $\mathbb{R}^k$, then $(AP^k)_\Delta$ is a unital subalgebra of (AP^k). We denote by $(AP^k)_\Delta^{n\times n}$ the set (algebra if Δ is a subgroup of $\mathbb{R}^k$) of $n\times n$ matrices with entries in $(AP^k)_\Delta$.

Introduce an inner product on (AP^k) by the formula

$$\langle f,g\rangle = M\{fg^*\},\ f,g\in(AP^k). \tag{1.4}$$

The completion of (AP^k) with respect to this inner product is called the *Besikovitch space* and is denoted by (B^k). Thus (B^k) is a Hilbert space. The elementary exponentials $e_\lambda(t)$, $\lambda\in\mathbb{R}^k$, form an orthonormal basis in (B^k). For a nonempty set $\Lambda\subseteq\mathbb{R}^k$, define the projection

$$\Pi_\Lambda\left(\sum_{\lambda\in\sigma(f)} f_\lambda e_\lambda(t)\right) = \sum_{\lambda\in\sigma(f)\cap\Lambda} f_\lambda e_\lambda(t),$$

where $f\in(AP^k)$. The projection Π_Λ extends by continuity to the orthogonal projection (also denoted Π_Λ) on (B^k). We denote by $(B^k)_\Lambda$ the range of Π_Λ, or, equivalently, the completion of $(AP^k)_\Lambda$ with respect to the inner product (1.4). The vector valued Besikovitch space $(B^k)^{n\times 1}$ consists of $n\times 1$ columns with components in (B^k), with the standard Hilbert space structure. Similarly, $(B^k)_\Lambda^{n\times 1}$ is the Hilbert space of $n\times 1$ columns with components in $(B^k)_\Lambda$.

A subset S of $\mathbb{R}^k$ is called a *halfspace* if it is closed under multiplication by nonnegative real numbers and addition, and has the properties that $\mathbb{R}^k = S\cup(-S)$ and $S\cap(-S)=\{0\}$. A standard example of a halfspace is given by

$$E_k=\{(x_1,\cdots,x_k)^T\in\mathbb{R}^k\setminus\{0\} : x_1=x_2=\cdots=x_{j-1}=0,\ x_j\neq 0\Rightarrow x_j>0\}\cup\{0\}.$$

The vectors in $\mathbb{R}^k$ are understood as column vectors; the superscript ^{T} denotes the transpose. Clearly, when $k=1$ the only halfspaces are $[0,\infty)(=E_1)$ and $(-\infty,0]$. It turns out (see, for example, [12] for a proof) that a set $S\subset\mathbb{R}^k$ is a halfspace if and only if there exists a real invertible $k\times k$ matrix A such that

$$S=\{Ax\ :\ x\in E_k\}. \tag{1.5}$$

A halfspace S naturally induces a total order $>_S$ on $\mathbb{R}^k$ by the rule: $\lambda>_S\mu$ if and only if $\lambda-\mu\in S$ and $\lambda\neq\mu$.

Given $f\in(AP^k)^{n\times n}$ and a nonempty subset $\Omega\subseteq\mathbb{R}^k$, the *Toeplitz operator* $T_f(\Omega)$ is defined on $(B^k)_\Omega^{n\times 1}$ as follows:

$$T_f(\Omega)(g)=\Pi_\Omega(fg),\quad g\in(AP^k)_\Omega^{n\times 1};$$

the definition is extended by continuity to $g\in(B^k)_\Omega^{n\times 1}$. We will be interested in the case when $\Omega=\Lambda'\cap S$, where Λ' is an additive subgroup of $\mathbb{R}^k$ which contains $\Lambda(f)$, and S is a halfspace. Note that the cases $\Lambda'=\Lambda$ and $\Lambda'=\mathbb{R}^k$ are not excluded.

We state the main result of this paper.

Theorem 1.1. *Let $f\in(AP^k)^{n\times n}$. Assume that the additive subgroup $\Lambda'\supseteq\Lambda(f)$ of $\mathbb{R}^k$ and the halfspace S are such that the following condition is satisfied:*

(A) *For every $\mu>_S 0$, $\mu\in\Lambda'$, there exists a countable set $\lambda_j\in\Lambda'$, $j=1,2,\cdots$, such that $\mu>_S\lambda_j>_S 0$ for all j, and $\lambda_j-\lambda_k\notin\Lambda(f)$ for all $j\neq k$.*

Then the operator $T_f(\Lambda'\cap S)$ is Fredholm if and only if it is invertible.

Results in the spirit of Theorem 1.1 are known in the literature. It is proved in [3] that Toeplitz operators on $L_2(\mathbb{R}^n)$ with (suitably interpreted) almost periodic symbols are invertible if and only if they are Fredholm. A particular case of Theorem 1.1 (where $k = 1$ and $\Lambda' = \mathbb{R}$) was proved in [5] (see also [6]); another proof of this particular case using the theory of limit operators ([11], [1]) is given in [7] (in the equivalent language of Wiener-Hopf operators) and [2]. The approach of [3] is based on a far-from-trivial verification of the fact that the C^*-algebra generated by Toeplitz operators under consideration does not contain non-zero compact operators, and on the following known and simple fact: *A C^*-algebra $\mathcal{A}$ of linear bounded operators on a Hilbert space that contains I has only zero intersection with the ideal $\mathcal{K}$ of compact operators if and only if every Fredholm operator that belongs to $\mathcal{A}$ is invertible.* For the reader's convenience, here is a short proof of this fact: Assume $\mathcal{A} \cap \mathcal{K} = \{0\}$, and let $X \in \mathcal{A}$ be Fredholm. The limit $\lim_{\varepsilon \to 0} \varepsilon(X^*X + \varepsilon I)^{-1}$ exists in $\mathcal{A}$ and coincides with the orthogonal projection on the kernel of X. By assumption, this orthogonal projection, being a finite rank operator, must be zero. Applying this argument to X^*, we conclude also that $\operatorname{Range} X$ is the whole Hilbert space, i.e., X is invertible. Conversely, assume that every Fredholm operator in $\mathcal{A}$ is invertible, and let $K \in \mathcal{A} \cap \mathcal{K}$. Then for every real c the operator $I + cKK^* \in \mathcal{A}$ is Fredholm, hence invertible. It follows that $\sigma(KK^*) = \{0\}$, therefore $K = 0$.

Note that if condition (A) is satisfied, then the factor-group $\Lambda'/\Lambda(f)$ is infinite. However, the condition of the factor-group $\Lambda'/\Lambda(f)$ being infinite does not imply condition (A), as the following example shows: Let $\Lambda' = \mathbb{Z}^2 \subset \mathbb{R}^2$ (here $\mathbb{Z}$ stands for the set of integers),

$$\Lambda(f) = \{(0, m) \in \mathbb{Z}^2 \, : \, m \in \mathbb{Z}\}, \quad S = \{(x_1, x_2) \in \mathbb{R}^2 \, : \, x_1 > 0 \text{ or } x_1 = 0, x_2 \geq 0\}.$$

In this example, for every element $(0, m) \in \Lambda'$ such that $(0, m) >_S 0$ there exist only finitely many elements $\lambda \in \Lambda'$ for which $(0, m) >_S \lambda >_S 0$. Here, e.g., the Toeplitz operator with (scalar valued) symbol $f(t_1, t_2) = e^{it_2}$, considered as an operator on $(B^2)^{1\times 1}_{\mathbb{Z}^2 \cap S}$, is Fredholm but not invertible.

The following two corollaries are noteworthy:

Corollary 1.2. *If $f \in (AP^k)^{n\times n}$ and $S \subset \mathbb{R}^k$ is a halfspace, then the Toeplitz operator $T_f(S)$ is Fredholm if and only if $T_f(S)$ is invertible.*

For the proof observe that $\Lambda(f)$ is at most countable, whereas for every $\mu >_S 0$ the set

$$\{\lambda \in \mathbb{R}^k \, : \, \mu >_S \lambda >_S 0\}$$

is a continuum (as easily follows from (1.5)). It remains to apply Theorem 1.1 with $\Lambda' = \mathbb{R}^k$.

A subgroup $\Lambda \subseteq \mathbb{R}^k$ is called *divisible* if

$$\lambda \in \Lambda \quad \Longrightarrow \quad \frac{1}{n}\lambda \in \Lambda \quad \text{for every positive integer } n,$$

and it is called *discrete* if there exists an open neighborhood U of 0 in $\mathbb{R}^k$ such that $U \cap \Lambda = \{0\}$.

Corollary 1.3. *Let $f \in (AP^k)^{n\times n}$ and let $S \subset \mathbb{R}^k$ be a halfspace. If $\Lambda(f)$ is discrete and $\Lambda' \supset \Lambda(f)$ is divisible, then the operator $T_f(\Lambda' \cap S)$ is Fredholm if and only if it is invertible.*

In particular, $\Lambda(f)$ is discrete if f is a periodic matrix function.

2. Proof of the main result

We start with a lemma.

Lemma 2.1. *Let T be a Fredholm linear bounded operator on the orthogonal sum of Hilbert spaces $\mathcal{H} = \bigoplus_{j\in J} \mathcal{H}_j$, where the index set J is infinite. Assume that each $\mathcal{H}_j$ is T-invariant. Then there exist a finite subset $J_0 \subset J$ and $\varepsilon > 0$ such that for every $j \in J \setminus J_0$ the restriction $T|_{\mathcal{H}_j}$ of T to $\mathcal{H}_j$ is invertible and the inequality*

$$\|T|_{\mathcal{H}_j} x\| \geq \varepsilon \|x\|, \qquad x \in \mathcal{H}_j, \tag{2.1}$$

holds.

Proof. Denote $T_j = T|_{\mathcal{H}_j}$, $j \in J$, and let J_1 be the set of such $j \in J$ that $\dim \operatorname{Ker} T_j > 0$. Since $\operatorname{Ker} T = \bigoplus_{j\in J} \operatorname{Ker} T_j$, this set must be finite. A similar reasoning applied to adjoint operators shows that $J_2 = \{j \in J : \dim \operatorname{Ker} T_j^* > 0\}$ is finite as well. Let $J_0 = J_1 \cup J_2$, and represent T as the orthogonal sum of the operators $\hat{T} = \bigoplus_{j\in J_0} T_j$ and $\tilde{T} = \bigoplus_{j\in J\setminus J_0} T_j$. Observe that $\operatorname{Ker} \tilde{T} = \operatorname{Ker} \tilde{T}^* = \{0\}$. The Fredholmness of T implies the Fredholmness, and thus invertibility, of $\tilde{T}$. But then

$$\|Tx\| = \|\tilde{T}x\| \geq \|\tilde{T}^{-1}\|^{-1} \, \|x\|$$

for all $x \in \bigoplus_{j\in J\setminus J_0} \mathcal{H}_j$. In particular,

$$\|T|_{\mathcal{H}_j} x\| \geq \|\tilde{T}^{-1}\|^{-1} \|x\|$$

for $x \in \mathcal{H}_j$. □

It is easy to derive a necessary and sufficient condition for T to be Fredholm. It consists of (2.1) combined with Fredholmness of T_j for all $j \in J_0$. However, we do not need this statement in what follows.

Proof of Theorem 1.1. Assume the hypotheses and notation of Theorem 1.1, and let $T_f(\Lambda' \cap S)$ be a Fredholm operator. We show that the kernel of $T_f(\Lambda' \cap S)$ is trivial. Let $\{\Omega_j\}_{j\in J}$ be the collection of all distinct cosets of Λ' by $\Lambda(f)$, indexed with an index set J. It is easy to see that $(B^k)^{n\times 1}_{\Omega_j \cap S}$ is $T_f(\Lambda' \cap S)$-invariant, for every $j \in J$. Indeed, fix $j \in J$. For an elementary exponential $e_\lambda x$, where $\lambda \in \Omega_j \cap S$ and x a non-zero vector in $\mathbb{C}^n$, we have

$$\sigma(f e_\lambda x) \subseteq \sigma(f) + \lambda \subseteq \lambda + \Lambda(f) = \Omega_j.$$

So $\sigma\left(\Pi_{\Lambda'\cap S}(f e_\lambda x)\right) \subseteq \Omega_j$. But it is also clear that

$$\sigma\left(\Pi_{\Lambda'\cap S}(f e_\lambda x)\right) \subseteq S,$$

therefore

$$\sigma\left(\Pi_{\Lambda'\cap S}(f e_\lambda x)\right) \subseteq S \cap \Omega_j.$$

Since the elementary exponentials $e_\lambda x$ (where λ is arbitrary in $\Omega_j \cap S$ and x is an arbitrary element of a fixed orthonormal basis in $\mathbb{C}^n$) form an orthonormal basis in $(B^k)^{n\times 1}_{\Omega_j\cap S}$, it follows that the subspace $(B^k)^{n\times 1}_{\Omega_j\cap S}$ is $T_f(\Lambda' \cap S)$-invariant. We also have the orthogonal decomposition

$$(B^k)^{n\times 1}_{\Lambda'\cap S} = \bigoplus_{j\in J} (B^k)^{n\times 1}_{\Omega_j\cap S}.$$

Therefore, by Lemma 2.1, there exist a finite set $J_0 \subset J$ and $\varepsilon > 0$ such that the restriction of $T_f(\Lambda' \cap S)$ to $(B^k)^{n\times 1}_{\Omega_j \cap S}$, call it T_j, is invertible for $j \in J \setminus J_0$, and moreover

$$\|T_j y\| \geq \varepsilon \|y\| \quad \text{for all } \ y \in (B^k)^{n\times 1}_{\Omega_j \cap S} \quad (j \in J \setminus J_0). \tag{2.2}$$

Now assume that

$$\Pi_{\Lambda' \cap S}(fg_0) = 0$$

for some $g_0 \in (B^k)^{n\times 1}_{\Omega_{j_0} \cap S}$, $\|g_0\| = 1$. Let

$$h = \Pi_{\Lambda' \setminus S}(fg_0).$$

In the Fourier series for h, $\sum_{\nu \in \Lambda' \setminus S} h_\nu e_\nu$, we have

$$\sum_{\nu \in \Lambda' \setminus S} \|h_\nu\|^2 < \infty,$$

and therefore there exists a finite set $Q \subseteq \Lambda' \setminus S$ such that

$$\sum_{\nu \in \Lambda' \setminus S, \ \nu \notin Q} \|h_\nu\|^2 < \varepsilon^2.$$

Let $\mu \in Q$ be the largest element in Q (with respect to the total order induced by S); thus $\mu \geq_S q$ for every $q \in Q$. By condition (A), there is $\mu_0 \in \Lambda' \setminus S$ such that $\mu_0 >_S \mu$. We then have

$$\sum_{\nu \in \Lambda' \setminus S, \ -\mu_0 + \nu \in S} \|h_\nu\|^2 < \varepsilon^2. \tag{2.3}$$

In addition, by condition (A), we may select countably many elements $\lambda_j \in \Lambda' \setminus S$, $j = 1, 2, \ldots$, such that $\lambda_j >_S \mu_0$ and $\lambda_j - \lambda_k \notin \Lambda(f)$ for all $j \neq k$. Let $g_j = e_{\lambda_j} g_0$. Since $\Pi_{\Lambda' \cap S}(fg_0) = 0$ we have

$$\begin{aligned} T_f(\Lambda' \cap S)(g_j) &= \Pi_{\Lambda' \cap S}(fg_j) = \Pi_{\Lambda' \cap S}(e_{\lambda_j} fg_0) \\ &= \Pi_{\Lambda' \cap S}(e_{\lambda_j}(\Pi_{\Lambda' \setminus S}(fg_0))) = \Pi_{\Lambda' \cap S}\left(e_{\lambda_j}\left(\sum_{\nu \in \Lambda' \setminus S, \ \nu \geq_S \lambda_j} h_\nu e_\nu\right)\right). \end{aligned}$$

But in view of (2.3)

$$\| \sum_{\nu \in \Lambda' \setminus S, \ \nu \geq_S \lambda_j} h_\nu e_\nu \| < \varepsilon,$$

thus

$$\|T_f(\Lambda' \cap S)(g_j)\| < \varepsilon = \varepsilon \|g_j\|. \tag{2.4}$$

Since the elements λ_j belong to different cosets of Λ' by $\Lambda(f)$, we obtain a contradiction with (2.2), according to which the inequality (2.4) is possible only for $j \in J_0$.

We have proved that $\operatorname{Ker} T_f(\Lambda' \cap S) = \{0\}$. Denote by $\hat{f} \in (AP^k)^{n\times n}$ the matrix function such that $\hat{f}(t)$ is the conjugate transpose of the matrix $f(t)$, for every $t \in \mathbb{R}^k$. It is easy to see that $\Lambda(\hat{f}) = \Lambda(f)$. Also, $(T_f(\Lambda' \cap S))^* = T_{\hat{f}}(\Lambda' \cap S)$. Applying the already proved part of Theorem 1.1 to $\hat{f}$, we see that

$$\operatorname{Range} T_f(\Lambda' \cap S) = (B^k)^{n\times 1}_{\Lambda' \cap S}.$$

The invertibility of $T_f(\Lambda' \cap S)$ follows. ∎

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