PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 5, Pages 1391–1399
S 0002-9939(01)06282-7
Article electronically published on October 12, 2001

ON SEMIPROJECTIVITY OF C^*-ALGEBRAS OF DIRECTED GRAPHS

WOJCIECH SZYMAŃSKI

(Communicated by David R. Larson)

ABSTRACT. It is shown that if E is a countable, transitive directed graph with finitely many vertices, then $C^*(E)$ is semiprojective.

0.

Semiprojectivity of C^*-algebras was originally investigated by Effros and Kaminker [7] and soon afterwards by Blackadar [2] in the context of shape theory for C^*-algebras. In this article we follow Blackadar's approach. Semiprojectivity of C^*-algebras is a noncommutative analogue of the topological concept of absolute neighbourhood retract. It is closely related to the problem of stability of relations defining a C^*-algebra and has been investigated recently by a number of authors; e.g. see [8].

It is not easy for a simple C^*-algebra to be semiprojective. However, Blackadar has conjectured [4] that Kirchberg algebras (purely infinite, simple, separable, unital, and satisfying the Universal Coefficient Theorem) with finitely generated K-groups are semiprojective. He proved this for Cuntz-Krieger algebras corresponding to finite matrices and more recently for $\mathcal{O}_\infty$ [2, 4]. The latter result is highly non-trivial as no finite presentation of $\mathcal{O}_\infty$ is known.

It is the main purpose of the present article to show that semiprojectivity holds for a class of generalized Cuntz-Krieger algebras corresponding to transitive directed graphs on finitely many vertices, but with possibly infinitely many edges; cf. Theorem 5 below. With help of Proposition 2, below, it is possible to show that all Kirchberg algebras with arbitrary finitely generated K_0 group and finitely generated free abelian K_1 group whose rank is not greater than the rank of the K_0 group may be realized in this way (see also [13, Lemma 2]). Cuntz-Krieger algebras of infinite directed graphs were introduced in [11] and then studied by many authors; e.g. see [10, 1, 12]. Among many advantages of working with graph algebras is the fact that both their defining relations and their main properties can be easily read from the underlying graphs.

Our argument from Theorem 5 is based on Blackadar's proof of semiprojectivity of $\mathcal{O}_\infty$ [4]. In order to carry it out we must know the K-groups of C^*-algebras of infinite graphs on finitely many vertices. These we calculate in Proposition 2, which

Received by the editors June 1, 2000 and, in revised form, November 9, 2000.
2000 *Mathematics Subject Classification.* Primary 46L05, 46L80.

should be of independent interest. The proof of Proposition 2 depends heavily on the results of [12].

1.

We recall the definition of the C^*-algebra corresponding to a directed graph [9]. Let $E = (E^0, E^1, r, s)$ be a directed graph with (at most) countably many vertices E^0 and edges E^1, and range and source functions $r, s : E^1 \to E^0$, respectively. $C^*(E)$ is by definition the universal C^*-algebra generated by families of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and partial isometries $\{s_e \mid e \in E^1\}$ with mutually orthogonal ranges, subject to the following three relations:

(G1) $s_e^* s_e = p_{r(e)}$,

(G2) $s_e s_e^* \leq p_{s(e)}$,

(G3) $p_v = \sum_{e \in E^1 \colon s(e)=v} s_e s_e^*$, provided $s^{-1}(v)$ is finite and non-empty.

A directed graph E is row-finite if every vertex emits at most finitely many edges. For row-finite graphs the K-theory of $C^*(E)$ can be calculated as follows [12, Theorem 3.2]. Let V_E be the collection of all those vertices which emit at least one edge, and let $\mathbf{Z}V_E$ and $\mathbf{Z}E^0$ be free abelian groups on free generators V_E and E^0, respectively. We have

$$K_0(C^*(E)) \cong \operatorname{coker}(\Delta_E), \tag{1}$$

$$K_1(C^*(E)) \cong \ker(\Delta_E), \tag{2}$$

where $\Delta_E : \mathbf{Z}V_E \to \mathbf{Z}E^0$ is the map defined on generators as

$$\Delta_E(v) = \left(\sum_{e \in E^1 \colon s(e)=v} r(e) \right) - v. \tag{3}$$

However, this result is not sufficient for our purposes, since we need a description of the K-groups of the C^*-algebras corresponding to arbitrary countable graphs with finitely many vertices. This is done in the following proposition.

Proposition 2. *If E is a countable directed graph with finitely many vertices, then the K-groups of $C^*(E)$ are still described by formulae* (1) *and* (2) *above, provided V_E is defined as the collection of all those vertices which emit at least one but only finitely many edges.*

Proof. Let $E^0 = V_E \cup W_0 \cup W_\infty$, with W_0 sinks and W_∞ vertices emitting infinitely many edges. For $w \in W_\infty$ we denote the edges with source w by $e_1(w), e_2(w), \ldots$. Then we denote by B_n the C^*-subalgebra of $C^*(E)$ generated by $\{p_v \mid v \in E^0\}$, $\{s_e \mid s(e) \in V_E\}$ and $\{s_{e_k(w)} \mid w \in W_\infty,\ k = 1, \ldots, n\}$. We will use continuity of the K-theory to calculate the K-groups of $C^*(E)$ as $K_*(C^*(E)) \cong \lim_{n\to\infty} K_*(B_n)$.

We claim that B_n is naturally isomorphic to $C^*(E_n)$, with the graph E_n defined as follows. The vertex set $E_n^0 = E^0 \cup \overline{W}_\infty$, where $\overline{W}_\infty = \{\overline{w} \mid w \in W_\infty\}$. Let $F_n = (s^{-1}(V_E) \cup \{e_k(w) \mid w \in W_\infty,\ k = 1, \ldots, n\}) \cap r^{-1}(W_\infty)$. Then $E_n^1 = s^{-1}(V_E) \cup \{e_k(w) \mid w \in W_\infty,\ k = 1, \ldots, n\} \cup \overline{F}_n$, where $\overline{F}_n = \{\overline{e} \mid e \in F_n\}$ and $s(\overline{e}) = s(e)$, $r(\overline{e}) = \overline{r(e)}$. Indeed, let $\{P_a \mid a \in E_n^0\}$ and $\{S_b \mid b \in E_n^1\}$ be the canonical generators of $C^*(E_n)$. There exists a C^*-algebra homomorphism

$\phi_n : C^*(E_n) \to B_n$ such that

$$\begin{aligned}
\phi_n(P_w) &= \begin{cases} p_w & \text{if } w \in V_E \cup W_0, \\ \sum_{k=1}^{n} s_{e_k(w)} s^*_{e_k(w)} & \text{if } w \in W_\infty, \end{cases} \\
\phi_n(P_{\overline{w}}) &= p_w - \sum_{k=1}^{n} s_{e_k(w)} s^*_{e_k(w)}, \\
\phi_n(S_e) &= \begin{cases} s_e & \text{if } r(e) \notin W_\infty, \\ s_e \sum_{k=1}^{n} s_{e_k(w)} s^*_{e_k(w)} & \text{if } r(e) \in W_\infty, \end{cases} \\
\phi_n(S_{\overline{e}}) &= s_e - s_e \sum_{k=1}^{n} s_{e_k(w)} s^*_{e_k(w)},
\end{aligned}$$

since the target elements satisfy relations (G1)–(G3) for the graph E_n. Clearly, the generators of B_n belong to the range of ϕ_n and hence ϕ_n is surjective. Since $\phi_n(P_a) \neq 0$ for all $a \in E_n^0$, ϕ_n is injective by the gauge-invariant uniqueness theorem [1, Theorem 2.1], and the claim follows.

[12, Theorem 3.2] implies that for any n we have a commuting diagram with exact rows

$$\begin{array}{ccccccc}
0 \to K_1(B_n) \to & \mathbf{Z}V_{E_n} & \xrightarrow{\Delta_{E_n}} & \mathbf{Z}E_n^0 & \to K_0(B_n) \to 0 \\
\downarrow i_* & || & & || & \downarrow i_* \\
0 \to K_1(B_{n+1}) \to & \mathbf{Z}V_{E_{n+1}} & \xrightarrow{\Delta_{E_{n+1}}} & \mathbf{Z}E_{n+1}^0 & \to K_0(B_{n+1}) \to 0.
\end{array}$$

Consequently, we have $K_0(C^*(E)) \cong \lim_{n\to\infty} \operatorname{coker}(\Delta_{E_n})$ and $K_1(C^*(E)) \cong \lim_{n\to\infty} \ker(\Delta_{E_n})$. In fact, we are going to show that (i) $\ker(\Delta_{E_n}) = \ker(\Delta_E)$ and (ii) $\operatorname{coker}(\Delta_{E_n})$ is naturally isomorphic to $\operatorname{coker}(\Delta_E)$ for all n, and this will prove our proposition.

Proof of (i). Viewed as maps from $\mathbf{Z}V_E \oplus \mathbf{Z}W_\infty$ to $\mathbf{Z}V_E \oplus \mathbf{Z}W_0 \oplus \mathbf{Z}W_\infty \oplus \mathbf{Z}\overline{W}_\infty$ and from $\mathbf{Z}V_E$ to $\mathbf{Z}V_E \oplus \mathbf{Z}W_0 \oplus \mathbf{Z}W_\infty$, respectively, Δ_{E_n} and Δ_E have the following matrices:

$$\Delta_{E_n} = \begin{bmatrix} D_1 - 1 & D_4 \\ D_2 & D_5 \\ D_3 & D_6 - 1 \\ D_3 & D_6 \end{bmatrix}, \quad \Delta_E = \begin{bmatrix} D_1 - 1 \\ D_2 \\ D_3 \end{bmatrix},$$

where D_i, $i = 1, \dots, 6$, are some integer entry blocks depending on E and n. Thus, if $\Delta_{E_n}(x, y) = 0$, then $D_3 x + D_6 y - y = 0 = D_3 x + D_6 y$ and hence $y = 0$. Consequently,

$$\ker(\Delta_{E_n}) = \ker([D_1 - 1,\ D_2,\ D_3,\ D_3]^t) = \ker([D_1 - 1,\ D_2,\ D_3]^t) = \ker(\Delta_E).$$

Proof of (ii). Since $(0, 0, y, 0) = (D_4 y, D_5 y, D_6 y, D_6 y) - \Delta_{E_n}(0, y)$, $\mathbf{Z}V_E \oplus \mathbf{Z}W_0 \oplus \mathbf{Z}W_\infty \oplus \mathbf{Z}\overline{W}_\infty$ is equal to the sum of $X = (\mathbf{Z}V_E \oplus \mathbf{Z}W_0 \oplus 0 \oplus 0) + \operatorname{span}\{(0, 0, w, \overline{w}) \mid w \in W_\infty\}$ and the image $\operatorname{Im}(\Delta_{E_n})$ of Δ_{E_n}. But if $\Delta_{E_n}(x, y) \in X$, then $D_3 x + D_6 y - y = D_3 x + D_6 y$ and hence $y = 0$. Consequently,

$$\operatorname{coker}(\Delta_{E_n}) \cong X / \operatorname{Im}([D_1 - 1,\ D_2,\ D_3,\ D_3]^t) \cong \operatorname{coker}(\Delta_E),$$

as required. ☐

Lemma 3. *If E is a countable, transitive directed graph with finitely many vertices, but E is not a single loop, then one may find a finite subset $\Omega \subseteq E^1$ such that the following holds. For any vertex $v \in E^0$ emitting infinitely many edges $\{e_i(v) \mid i = 1, 2, \dots\}$, any positive integer d and any unitary $t \in C^*(E)$ with $(I - f)t = I - f$, where $f = \sum_{j=d}^{m} s_{e_j(v)} s^*_{e_j(v)}$ for some $m \geq d$, there exists a unitary $u \in C^*(\{s_e \mid e \in \Omega\} \cup \{s_{e_d(v)}\})$ such that $(q + I - p_v)u = q + I - p_v$ and $(p_v - q)u$ is homotopic in $(p_v - q)C^*(E)(p_v - q)$ to $(p_v - q)t$, where $q = \sum_{j=1}^{d-1} s_{e_j(v)} s^*_{e_j(v)}$.*

Proof. We fix a vertex $w \in E^0$. Since $p_w C^*(E) p_w$ is a full corner of $C^*(E)$ it is purely infinite and simple, just as $C^*(E)$, and $K_1(p_w C^*(E) p_w) \cong K_1(C^*(E))$ is finitely generated by virtue of Proposition 2. Thus, there exists a finite subset $\Omega \subseteq E^1$ for which $p_w C^*(\{s_e \mid e \in \Omega\}) p_w$ contains representatives of all homotopy classes of unitaries in $p_w C^*(E) p_w$ [5]. Adding a finite number of edges to Ω, if necessary, we may further assume that for any two vertices $w_1, w_2 \in E^0$ there is a path in Ω from w_1 to w_2.

Now let $v \in E^0$ and t be as in the hypothesis of the lemma. Then there exist in Ω paths α and β from w to v and from $r(s_d)$ to w, respectively. Let $\tilde{u}$ be a unitary in $p_w C^*(\{s_e \mid e \in \Omega\}) p_w$ in the same homotopy class as $s_\alpha t s^*_\alpha + p_w - s_\alpha s^*_\alpha$ in $p_w C^*(E) p_w$. We set

$$u = s_{e_d(v)\beta} \tilde{u} s^*_{e_d(v)\beta} + I - s_{e_d(v)\beta} s^*_{e_d(v)\beta}.$$

Clearly $(q + I - p_v)u = q + I - p_v$, and $(p_v - q)u$ is homotopic to $(p_v - q)t$ since these two unitaries have the same class in $K_1((p_v - q)C^*(E)(p_v - q))$ [5]. □

Lemma 4. *If E is a finite directed graph, then $C^*(E)$ is semiprojective.*

Proof. This follows from [2, Propositions 2.18 and 2.23]. □

Blackadar showed in [2, Corollary 2.24] that Cuntz-Krieger algebras (corresponding to finite 0-1 matrices) are semiprojective. Such algebras may be realized as graph algebras corresponding to finite graphs. However, the algebras of graphs with sinks cannot, strictly speaking, be described as Cuntz-Krieger algebras. Indeed, in order to allow for graphs with sinks one would have to consider matrices with zero rows, and this was not done in the original work of Cuntz and Krieger [6]. Therefore, for the sake of completeness, we have stated Lemma 4, above.

Theorem 5. *If E is a countable, transitive directed graph with finitely many vertices, then $C^*(E)$ is semiprojective.*

Proof. By virtue of Lemma 4 it suffices to consider the case of a directed graph E with finitely many vertices, say $E^0 = \{v_1, \dots, v_N\}$, and infinitely many edges E^1. As in the proof of Proposition 2 if a vertex v emits infinitely many edges, then we denote them by $\{e_j(v) \mid j = 1, 2, \dots\}$.

Let B be a unital C^*-algebra and let J_k, $k = 1, 2, \dots$, be an increasing sequence of closed, two-sided ideals of B such that $B/\overline{\bigcup_{k=1}^{\infty} J_k}$ is isomorphic to (and identified with) $C^*(E)$. We must show that for some n there exists a unital C^*-algebra homomorphism $\lambda : C^*(E) \to B/J_n$ such that $\pi_n \circ \lambda = \mathrm{id}$, where $\pi_n : B/J_n \to C^*(E)$ denotes the natural quotient map. To this end we first find, with the help of Lemma 3, a finite subset Ω of E^1 so large that the following hold:

(i) The conditions of Lemma 3 are satisfied.
(ii) If v emits finitely many edges, then $\{e \in E^1 \mid s(e) = v\} \subseteq \Omega$.
(iii) If v emits infinitely many edges, then the following hold:
  (a) Both $e_1(v)$ and $e_2(v)$ belong to Ω.
  (b) For each k there exists an $e \in \Omega$ such that $s(e) = v$ and $r(e) = r(e_k(v))$.
  (c) If $e_k(v) \in \Omega$, then $e_i(v) \in \Omega$ for all $i = 1, \dots, k-1$.

We note that the above conditions imply, in particular, that for any two vertices $v, w \in E^0$ there exists a path $(f_1, \dots, f_r)$ from v to w such that $f_i \in \Omega$ for all $i = 1, \dots, r$.

We denote $A_0 = C^*(\{s_e \mid e \in \Omega\})$ and

$$A_m = C^*\left(\{s_e \mid e \in \Omega\} \cup \bigcup_{i=1}^{m} \{s_e \mid e \in E^1,\ s(e) = v_i\}\right)$$

for $m = 1, \dots, N$. According to [12, Lemma 1.2] A_0 is canonically isomorphic to the C^*-algebra $C^*(E_\Omega)$, corresponding to a suitable finite graph E_Ω. The graph E_Ω is described in [12, Definition 1.1], but here we do not need to know it. Thus, Lemma 4 implies that there exists an n and a unital C^*-algebra homomorphism $\lambda^0 : A_0 \to B/J_n$ such that $\pi_n \circ \lambda^0 = \mathrm{id}$. We are going to show that this n is big enough to construct the required partial lifting $\lambda : C^*(E) \to B/J_n$. To this end we construct a sequence of unital C^*-algebra homomorphisms

$$\lambda^m : A_m \to B/J_n,$$

$m = 1, \dots, N$, such that $\pi_n \circ \lambda^m = \mathrm{id}$. Then $\lambda = \lambda^N$ will give the required partial lifting.

So suppose that the homomorphisms $\lambda^1, \dots, \lambda^{m-1}$ have already been constructed, and denote $v = v_m$. If v emits only finitely many edges, then we set $\lambda^m = \lambda^{m-1}$. If v emits infinitely many edges, then we let $M = \max\{j \mid e_j(v) \in \Omega\}$ and define λ^m to be the elementwise limit of the sequence of unital C^*-algebra homomorphisms

$$\lambda_k^m : C^*(A_{m-1} \cup \{s_{e_i(v)} \mid i = M+1, \dots, k\}) \to B/J_n,$$

$k = M, M+1, \dots$, defined inductively below, such that $\pi_n \circ \lambda_k^m = \mathrm{id}$ and λ_{k+2}^m coincides with λ_{k+1}^m on $A_{m-1} \cup \{s_{e_i(v)} \mid i = M+1, \dots, k\}$. This last condition ensures that the limit of $\lambda_k^m(x)$ exists for x in the dense $*$-subalgebra of A_m generated by A_{m-1} and $\{s_{e_i(v)} \mid i = M+1, M+2, \dots\}$, and hence the required elementwise limit $\lambda^m = \lim_{k\to\infty} \lambda_k^m$ exists.

To begin our construction we set $\lambda_M^m = \lambda^{m-1}$. For the inductive step we assume that λ_k^m has been already constructed for some $k \geq M$. Let

$$\Omega' = \Omega \cup \left(\bigcup_{i=1}^{m-1} s^{-1}(v_i)\right) \cup \{e_j(v) \mid j = M+1, \dots, k+1\}.$$

We will find, below, two elements b_k and $\tilde{b}_{k+1}$ of B/J_n so that the assignment

$$s_e \mapsto \begin{cases} b_k & \text{if } e = e_k(v), \\ \tilde{b}_{k+1} & \text{if } e = e_{k+1}(v), \\ \lambda_k^m(s_e) & \text{if } e \in \Omega' \setminus \{e_k(v), e_{k+1}(v)\}, \end{cases} \tag{4}$$

extends to the desired homomorphism λ_{k+1}^m. We use the tilde in $\tilde{b}_{k+1}$ since in the next inductive step we might have $b_{k+1} \neq \tilde{b}_{k+1}$.

We set $q = \sum_{j=1}^{k-1} s_{e_j(v)} s_{e_j(v)}^*$ and $Q = \sum_{j=1}^{k-1} \lambda_k^m(s_{e_j(v)} s_{e_j(v)}^*)$. Then q and Q are projections such that $\pi_n(Q) = q$. Let α be a path in Ω with source $r(e_1(v))$ and range $r(e_{k+1}(v))$, and let β be a path in Ω with source $r(e_k(v))$ and range v. Such paths can be found by transitivity of E and conditions (ii) and (iii) from the definition of Ω. Set

$$\begin{aligned} t_1 &= s_{e_k(v)} s^*_{e_k(v)\beta e_k(v)}, \\ t_2 &= s_{e_{k+1}(v)} s^*_{e_k(v)\beta e_1(v)\alpha}. \end{aligned}$$

Then t_1 and t_2 are partial isometries such that $t_1^* t_1$ is orthogonal to $t_2^* t_2$, $t_1 t_1^*$ is orthogonal to $t_2 t_2^*$, and all four are subprojections of $f = \sum_{i=k}^{k+2} s_{e_i(v)} s^*_{e_i(v)}$. Let t_3 be a partial isometry in $C^*(E)$ with domain $f - t_1^* t_1 - t_2^* t_2$ and range $f - t_1 t_1^* - t_2 t_2^*$. Such partial isometry exists since the two projections are non-zero and have the same K_0 class [5]. Then

$$t = t_1 + t_2 + t_3 + I - f$$

is a unitary in $C^*(E)$ such that $(I - f)t = (I - f)$. By virtue of condition (i) and Lemma 3 there exists a unitary u in $C^*(\{s_e \mid e \in \Omega\} \cup \{s_{e_k(v)}\})$ such that

$$(q + I - p_v)u = q + I - p_v$$

and $(p_v - q)u$ is homotopic in $(p_v - q)C^*(E)(p_v - q)$ to $(p_v - q)t$. Thus $(p_v - q)u^* t$ is homotopic to the identity in $(p_v - q)C^*(E)(p_v - q)$. Hence, since $\pi_n(\lambda_k^m(p_v) - Q) = p_v - q$, there exists a unitary Z_0 in $(\lambda_k^m(p_v) - Q)(B/J_n)(\lambda_k^m(p_v) - Q)$ such that $\pi_n(Z_0) = (p_v - q)u^* t$ [3, Proposition 3.4.5]. Therefore, $Z = Z_0 + Q + I - \lambda_k^m(p_v)$ is a unitary in B/J_n such that $(Q + I - \lambda_k^m(p_v))Z = (Q + I - \lambda_k^m(p_v))$ and $\pi_n(Z) = u^* t$ [4]. We set

$$\begin{aligned} b_k &= \lambda_k^m(u) Z \lambda_k^m(s_{e_k(v)\beta e_k(v)}), \\ \tilde{b}_{k+1} &= \lambda_k^m(u) Z \lambda_k^m(s_{e_k(v)\beta e_1(v)\alpha}). \end{aligned}$$

We claim that the assignment (4) gives rise to the desired homomorphism λ_{k+1}^m. To this end we first show that the algebra $C^*(A_{m-1} \cup \{s_{e_i(v)} \mid i = M+1, \dots, k+1\})$, which is obviously also generated by $\{s_e \mid e \in \Omega'\}$, is isomorphic to the graph algebra $C^*(E_{\Omega'})$ corresponding to a suitable graph $E_{\Omega'}$. Indeed, let W be the set of all those vertices $w \in E^0$ for which there exists an edge $e \in E^1 \setminus \Omega'$ with $s(e) = w$. We define $E_{\Omega'}$ to be the graph with the vertex set $E^0_{\Omega'} = E^0 \cup \overline{W}$, where $\overline{W} = \{\overline{w} \mid w \in W\}$, and the edge set $E^1_{\Omega'} = \Omega' \cup \overline{F}$, where $\overline{F} = \{\overline{f} \mid f \in \Omega',\ r(f) \in W\}$ with the source and range maps $s(\overline{f}) = s(f)$ and $r(\overline{f}) = \overline{r(f)}$, respectively. We note that for each $w \in W$ there are only finitely many edges $e \in \Omega'$ with $s(e) = w$. Let $\{P_v \mid v \in E^0\} \cup \{P_{\overline{w}} \mid w \in W\}$ and $\{S_e \mid e \in \Omega'\} \cup \{S_{\overline{f}} \mid f \in F\}$ be the canonical generators of $C^*(E_{\Omega'})$. There exists a C^*-algebra homomorphism

$$\psi : C^*(E_{\Omega'}) \to C^*(A_{m-1} \cup \{s_{e_i(v)} \mid i = M+1, \dots, k+1\})$$

such that

$$\begin{aligned}
\psi(P_v) &= \begin{cases} p_v & \text{if } v \notin W, \\ \displaystyle\sum_{g \in \Omega',\, s(g)=v} s_g s_g^* & \text{if } v \in W, \end{cases} \\
\psi(P_{\overline{w}}) &= p_w - \sum_{g \in \Omega',\, s(g)=w} s_g s_g^*, \\
\psi(S_e) &= \begin{cases} s_e & \text{if } r(e) \notin W, \\ s_e \displaystyle\sum_{g \in \Omega',\, s(g)=r(e)} s_g s_g^* & \text{if } r(e) \in W, \end{cases} \\
\psi(S_{\overline{f}}) &= s_f \left(I - \sum_{g \in \Omega',\, s(g)=r(f)} s_g s_g^* \right),
\end{aligned}$$

since the target elements satisfy relations (G1)–(G3) for the graph $E_{\Omega'}$. Clearly, the generators of $C^*(A_{m-1} \cup \{s_{e_i(v)} \mid i = M+1, \dots, k+1\})$ belong to the range of ψ and hence ψ is surjective. For injectiveness of ψ we first observe that in the graph $E_{\Omega'}$ all loops have exits. Indeed, if $(g_1, \dots, g_r)$ is a loop in $E_{\Omega'}$, then $g_i \in \Omega'$ for all $i = 1, \dots, r$, since the ranges of all elements of $\overline{F}$ are sinks. Thus $(g_1, \dots, g_r)$ is also a loop in E and hence has an exit in E, since the graph E is transitive and has infinitely many edges. Since $\Omega \subseteq \Omega'$ conditions (ii) and (iii-a) now imply that the loop $(g_1, \dots, g_r)$ has an exit in $E_{\Omega'}$, as claimed. Since $\psi(P_a) \neq 0$ for all $a \in E_n^0$, ψ is injective by the Cuntz-Krieger uniqueness theorem [12, Theorem 1.5].

Next we use the universal property of $C^*(E_{\Omega'})$ to define a C^*-algebra homomorphism $\tilde{\lambda}_{k+1}^m : C^*(E_{\Omega'}) \to B/J_n$, as follows. If $r(e_k(v)) \notin W$, then we set

$$\tilde{\lambda}_{k+1}^m(S_{e_k(v)}) = b_k.$$

Otherwise we put

$$\begin{aligned}
\tilde{\lambda}_{k+1}^m(S_{e_k(v)}) &= b_k \lambda_k^m \left(\sum_{g \in \Omega',\, s(g)=r(e_k(v))} s_g s_g^* \right), \\
\tilde{\lambda}_{k+1}^m(S_{\overline{e_k(v)}}) &= b_k \lambda_k^m \left(I - \sum_{g \in \Omega',\, s(g)=r(e_k(v))} s_g s_g^* \right).
\end{aligned}$$

Likewise, if $r(e_{k+1}(v)) \notin W$, then we set

$$\tilde{\lambda}_{k+1}^m(S_{e_{k+1}(v)}) = \tilde{b}_{k+1}.$$

Otherwise we put

$$\begin{aligned}
\tilde{\lambda}_{k+1}^m(S_{e_{k+1}(v)}) &= \tilde{b}_{k+1} \lambda_k^m \left(\sum_{g \in \Omega',\, s(g)=r(e_{k+1}(v))} s_g s_g^* \right), \\
\tilde{\lambda}_{k+1}^m(S_{\overline{e_{k+1}(v)}}) &= \tilde{b}_{k+1} \lambda_k^m \left(I - \sum_{g \in \Omega',\, s(g)=r(e_{k+1}(v))} s_g s_g^* \right).
\end{aligned}$$

For $v \in E^0$, $w \in W$, $e \in \Omega' \setminus \{e_k(v), e_{k+1}(v)\}$, and $f \in F \setminus \{e_k(v), e_{k+1}(v)\}$ we set

$$\tilde{\lambda}_{k+1}^m(P_v) = \begin{cases} \lambda_k^m(p_v) & \text{if } v \notin W, \\ \lambda_k^m\left(\sum_{g \in \Omega',\, s(g)=v} s_g s_g^*\right) & \text{if } v \in W, \end{cases}$$

$$\tilde{\lambda}_{k+1}^m(P_{\overline{w}}) = \lambda_k^m\left(p_w - \sum_{g \in \Omega',\, s(g)=w} s_g s_g^*\right),$$

$$\tilde{\lambda}_{k+1}^m(S_e) = \begin{cases} \lambda_k^m(s_e) & \text{if } r(e) \notin W, \\ \lambda_k^m\left(s_e \sum_{g \in \Omega',\, s(g)=r(e)} s_g s_g^*\right) & \text{if } r(e) \in W, \end{cases}$$

$$\tilde{\lambda}_{k+1}^m(S_{\overline{f}}) = \lambda_k^m\left(s_f\left(I - \sum_{g \in \Omega',\, s(g)=r(f)} s_g s_g^*\right)\right).$$

Such a homomorphism $\tilde{\lambda}_{k+1}^m$ exists since the target elements satisfy relations (G1)–(G3) for the graph $E_{\Omega'}$.

Finally, we define $\lambda_{k+1}^m = \tilde{\lambda}_{k+1}^m \circ \psi^{-1}$. It is clear from our construction that $\lambda_{k+1}^m : C^*(A_{m-1} \cup \{s_{e_i(v)} \mid i = M+1, \dots, k+1\}) \to B/J_n$ is a C^*-algebra homomorphism such that $\pi_n \circ \lambda_{k+1}^m = \text{id}$, condition (4) holds, and λ_{k+1}^m coincides with λ_{k-1}^m on $A_{m-1} \cup \{s_{e_i(v)} \mid i = M+1, \dots, k-1\}$. Thus the homomorphism λ_{k+1}^m has all the required properties.

Continuing inductively in this manner we construct homomorphisms λ_k^m and then $\lambda^m = \lim_{k\to\infty} \lambda_k^m$. □

References

[1] T. Bates, D. Pask, I. Raeburn and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324. CMP 2001:04

[2] B. Blackadar, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), 249–275. MR **87b:**46074

[3] B. Blackadar, *K-Theory for Operator Algebras*, 2nd ed., MSRI Publ., vol. **5**, Cambridge Univ. Press, 1998. MR **99g:**46104

[4] B. Blackadar, *Semiprojectivity in simple C^*-algebras*, lecture at the US-Japan Seminar, Fukuoka, 1999.

[5] J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **113** (1981), 181–197. MR **84c:**46058

[6] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268. MR **82f:**46073a

[7] E. G. Effros and J. Kaminker, *Homotopy continuity and shape theory for C^*-algebras*, in *Geometric methods in operator algebras (Kyoto, 1983)*, pp. 152–180, Pitman Res. Notes Math. **123**, Longman Sci. Tech., Harlow, 1986. MR **88a:**46082

[8] S. Eilers, T. A. Loring and G. K. Pedersen, *Stability of anticommutation relations: An application of noncommutative CW complexes*, J. Reine Angew. Math. **499** (1998), 101–143. MR **99e:**46067

[9] N. J. Fowler, M. Laca and I. Raeburn, *The C^*-algebras of infinite graphs*, Proc. Amer. Math. Soc. **128** (2000), 2319–2327. MR **2000k:**46079

[10] A. Kumjian, D. Pask and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174. MR **99i:**46049

[11] A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541. MR **98g:**46083

[12] I. Raeburn and W. Szymański, *Cuntz-Krieger algebras of infinite graphs and matrices*, preprint, 1999.

[13] W. Szymański, *The range of K-invariants for C^*-algebras of infinite graphs*, Indiana Univ. Math. J., to appear.

Department of Mathematics, The University of Newcastle, Callaghan, New South Wales 2308, Australia

E-mail address: `wojciech@frey.newcastle.edu.au`