THE EXTENSION OF POSITIVE DEFINITE OPERATOR-VALUED FUNCTIONS DEFINED ON A SYMMETRIC INTERVAL OF AN ORDERED GROUP

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Abstract. Let $G_1$ be an ordered abelian group and $a \in G_1$. Let $G_2$ be an abelian group and $f$ an operator-valued positive definite function on $(-a,a) \times G_2$. We prove that $f$ admits a positive definite extension to $G_1 \times G_2$, generalizing in this way existing results for the case when $G_1 = \mathbb{R}$ and $f$ is continuous.

1. Introduction

Let $G$ be an abelian group and let $\Lambda$ be a finite subset of $G$. A function $k : S = \Lambda - \Lambda \to \mathcal{L}(\mathcal{H})$ (the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$) is called positive (semi)definite with respect to $\Lambda$ if, for every finite subset $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$, the operator matrix \( \{ k(\lambda_i - \lambda_j) \}_{i,j=1}^n \) is positive (semi)definite. Without loss of generality, we assume in this paper that positive definite functions $k$ have the property that $k(0) = I_{\mathcal{H}}$. Let $G_1$ be an ordered abelian group, $a \in G_1$, and $G_2$ be an abelian group. A function $k : (-a,a) \times G_2 \to \mathcal{L}(\mathcal{H})$ is referred to as positive definite if it is positive definite with respect to $[0,a) \times G_2$, unless otherwise specified. M. G. Krein proved \[10\] that every positive definite continuous scalar function on a real interval $(-a,a)$ admits a continuous positive definite extension to $\mathbb{R}$. A. P. Artjomenko \[2\] (see also Theorem 4.2.3 in \[15\]) provided a new proof for Krein’s Extension Theorem without the continuity requirement. Y. M. Berezansky and I. M. Gali \[6\], see also Theorem 5.4.4.2 in \[5\] proved the following extension of Krein’s Theorem: “Given a Hilbert space $\mathcal{H}$, and a positive definite function $k$ on a layer in $\mathcal{H}$ that is $J$ continuous at 0, then $k$ can be extended to a positive definite function on $\mathcal{H}$ with the same property of continuity.” By a similar proof it follows that every continuous positive definite function on $(-a,a) \times G$, where $G$ is a topological group, can be extended to a continuous positive definite function on $\mathbb{R} \times G$.

J. Friedrich and L. Klotz \[9\] proved that, given $0 < a < \infty$ and a topological group $G$, any strongly continuous positive definite function $k : (-a,a) \times G \to \mathcal{L}(\mathcal{H})$ admits a positive definite extension to $\mathbb{R} \times G$. The aim of this paper is to generalize the above result by omitting the continuity requirement and by substituting $\mathbb{R}$ with an ordered abelian group. Finally, several corollaries of our main result are presented. One of them is a result recently proved in \[7\], and the others are
generalizations of extension results for positive definite functions in [10], [13], and [4].

For notation and results in group theory and Fourier Analysis on groups we refer to [14] and [15]. If $G$ is a locally compact abelian group, then its character group is denoted by $\Gamma$.

Suppose $P$ is a semigroup of the abelian group $G$ and that $P$ has the properties
\[ P \cap (-P) = \{0\}, \quad P \cup (-P) = G. \]
Under these conditions, $P$ induces an order in $G$. If we define $x \geq y$ to mean $x - y \in P$, then the axioms of linear order are satisfied. The choice of a semigroup $P$ with the above properties makes $G$ into an ordered group.

Let $K$ be a compact Hausdorff space and let $B(K)$ denote the class of Borel measurable sets in $K$ and $C(K)$ the set of all continuous complex functions on $K$. A function $F : B(K) \to \mathcal{L}(\mathcal{H})$ such that $F(X) = I_{\mathcal{H}}$ is called a semispectral measure if for every $h \in \mathcal{H}$, $\mu(\sigma) = (F(\sigma)h, h)$ is a positive Borel measure on $K$. If $F$ is a semispectral measure, define the Borel measures $\mu_{h,k}(\sigma) = (F(\sigma)h, k)$ for $\sigma \in B(\mathcal{H})$ and $h, k \in \mathcal{H}$; $\mu_{h,k}$ is called the semispectral family associated with $F$. The following results are well-known facts (see, e.g., Theorem 7.1 and Proposition 9.2 in [17]).

**Theorem 1.1.** Let $X$ be a compact Hausdorff space and let $L : C(K) \to \mathcal{L}(\mathcal{H})$ be a linear operator such that $L(1) = I_{\mathcal{H}}$. Then $L$ is positive in the sense that $L(q) \geq 0$ for every $q \in C(K)$ if and only if there exists a semispectral family $\mu_{h,k}$ on $K$ such that
\[ (L(q)h, k) = \int_K q(x)d\mu_{h,k}(x) \]
for every $q \in C(K)$ and $h, k \in \mathcal{H}$.

**Theorem 1.2.** Let $G$ be an abelian group. A function $f : G \to \mathcal{L}(\mathcal{H})$ with $f(0) = I_{\mathcal{H}}$ is positive definite if and only if there exists a semispectral family $\mu_{h,k}$ on $\Gamma$ such that
\[ (f(x)h, k) = \int_{\Gamma} \gamma(x)d\mu_{h,k}(\gamma) \]
for every $x \in G$ and $h, k \in \mathcal{H}$. ($G$ is considered with the discrete topology, thus $\Gamma$ is compact.)

2. Main results

The following is the main result of the paper. Its proof is modeled after Artjomenko’s proof of the Krein Extension Theorem (2, also presented in 15).

**Theorem 2.1.** Let $G_1$ be an ordered abelian group and $a \in G_1$. If $G_2$ is an abelian group, then every positive definite function $f : (-a, a) \times G_2 \to \mathcal{L}(\mathcal{H})$ admits a positive definite extension to $G_1 \times G_2$.

**Proof.** Let $G = G_1 \times G_2$ and consider on $G$ the discrete topology, and let $\Gamma$ be the character group of $G$. Denote $V = (-a, a) \times G_2$ and let $\mathcal{P}(V)$ be the set of all functions $g : V \to \mathbb{C}$ with finite support. For $g \in \mathcal{P}(V)$ define
\[ \phi(x) = \begin{cases} g(x)k(x), & x \in V, \\ 0, & x \not\in V. \end{cases} \]
We prove that $\phi$ is positive definite on $G$ for every positive definite $g \in \mathcal{P}(V)$. Consider $x_1, x_2, \ldots, x_n \in G$. We have to prove that the operator matrix $(\phi(x_i - x_j))_{i,j=1}^n$ is positive semidefinite. Without loss of generality, suppose that $x_i = (\lambda_1, \sigma_i)$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Consider the undirected graph $H = (V, E)$ with vertex set $V = \{1, 2, \cdots, n\}$ and edge set $E = \{(i, j) \mid i \neq j \text{ and } x_j - x_i \in V\}$. Then $H$ is a so-called proper interval graph [8]. Define the partial operator matrix

\begin{equation}
(A_{ij})_{i,j=1}^n = \begin{cases}
  k(x_i - x_j) & \text{for } (i, j) \in E, \\
  \text{unspecified} & \text{for } (i, j) \notin E.
\end{cases}
\end{equation}

All fully specified principal submatrices of $A$ are positive semidefinite and $H$ is the graph associated with the pattern of $A$. By Corollary 3.2 in [11], $A$ admits a positive semidefinite extension $B = (B_{ij})_{i,j=1}^n$. The matrix $(\phi(x_i - x_j))_{i,j=1}^n$ is the Schur product of $B$ and $(g(x_i - x_j))_{i,j=1}^n$ (all unspecified entries of $A$ correspond to zeros in $(g(x_i - x_j))_{i,j=1}^n$). By a generalized version of Schur’s Theorem [12], proof of Theorem 4.3), it follows that $(\phi(x_i - x_j))_{i,j=1}^n$ is positive semidefinite.

For $h \in \mathcal{H}$ define $\hat{\phi}_h(x) = \langle \phi(x)h, h \rangle$; $\hat{\phi}_h$ is a positive definite function on $G$ for every positive definite $g \in \mathcal{P}(V)$. Then

$$\hat{\phi}_h(1) = \int_G \phi_h(x)dx = \sum_{x \in G} \phi_h(x) \geq 0,$$

where the last inequality is a consequence of the fact that $\hat{\phi}_h$ is positive (see Theorem 1.9.8 in [13]). Consequently, $\sum_{x \in G} g(x)k(x) \geq 0$, for every positive definite $g \in \mathcal{P}(V)$.

Define $T_V = \{g \mid g \in \mathcal{P}(V)\}$ and let $T_V^+ = \{p \in T_V \mid p \geq 0\}$. Every element of $T_V^+$ is the Fourier transform of a positive definite function in $P(V)$. Define $l : T_V \to \mathcal{L}(\mathcal{H})$ by $l(q) = \sum_{x \in G} q(x)k(x)$. Then $l$ is a positive operator on $T_V$, which is an operator system in $C(\Gamma)$. We will prove that $l$ is completely positive.

Let $m > 1$ and let $\mathcal{M}_m$ denote the set of all complex $m \times m$ matrices. Let $Z : G \to \mathcal{M}_m$, $Z(x) = (z_{ij}(x))_{i,j=1}^m$, $z_{ij} \in \mathcal{P}(V)$, be a positive definite function. For $i, j = 1, \cdots, m$, define

$$\Phi_{ij}(x) = \begin{cases}
  z_{ij}(x)k(x), & x \in V; \\
  0, & x \notin V.
\end{cases}$$

We first prove that $\Phi(x) = (\Phi_{ij}(x))_{i,j=1}^m$ is a positive definite matrix function on $G$. Let $x_1, x_2, \cdots, x_n \in G$. Without loss of generality, suppose that $x_i = (\lambda_1, \sigma_i)$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

\begin{equation}
(\Phi(x_k - x_l))_{k,l=1}^n = [Z(x_k - x_l)]_{k,l=1}^n \odot [B \otimes J_m],
\end{equation}

where $\odot$ denotes the Schur product, $B$ is a positive semidefinite extension of the partial matrix $A$ defined by [8], and $J_m$ is the $m \times m$ matrix with all entries equal to 1. The matrix $[Z(x_k - x_l)]_{k,l=1}^n$ is positive semidefinite since $Z(x)$ is a positive definite function. Then [14] and the generalized version of Schur’s Theorem [12] imply that $(\Phi(x_k - x_l))_{k,l=1}^n$ is positive semidefinite, thus $\Phi(x)$ is a positive definite function.

Let $Q(x) = (q_{ij}(x))_{i,j=1}^m$ be a matrix-valued function such that $q_{ij} \in T_V$ for every $i, j = 1, \cdots, m$, and $Q(x) \geq 0$ for every $x \in G$. Then $Q(x) = (\hat{q}_{ij}(x))_{i,j=1}^m$ is a positive definite matrix function on $G$, which implies that the function $(\hat{q}_{ij}(x)k(x))_{i,j=1}^n$
is also positive definite. Let $h \in \mathcal{H}_n = \bigoplus_{i=1}^{n} \mathcal{H}$. Define $\Phi_h(x) = (\Phi(x) h, h)$; $\Phi_h$ is a positive definite (scalar) function on $G$. Then

\begin{equation}
\hat{\Phi}_h(1) = \int_{G} \Phi_h(x) dx = \sum_{x \in G} \Phi_h(x) \geq 0,
\end{equation}

the last inequality being a consequence of Theorem 1.9.8 in [15].

Relation (5) implies that the function $l : TV \rightarrow L(\mathcal{H}) \otimes M_m$, defined by

\[ l(q) = \left( \sum_{x \in G} \hat{q}_{ij}(x) k(x) \right)_{i,j=1}^{m}, \]

is positive definite for every $m \geq 1$, which means that $l$ is completely positive.

Since $TV$ is an operator system in $C(\Gamma)$, by Arveson’s Theorem ([3], see also Theorem 6.5 in [11]), $l$ admits a (completely) positive extension $L : C(\Gamma) \rightarrow L(\mathcal{H})$.

By Theorem 1.12 there exists a semispectral family $\{\mu_{h,k}\}_{h,k \in \mathcal{H}}$ on $\Gamma$ such that

\[(L(q)h, k) = \int_{\Gamma} q(\gamma) d\mu_{h,k}(\gamma) \]

for every $q \in C(\Gamma)$ and $h, k \in \mathcal{H}$. Define $K : G \rightarrow L(\mathcal{H})$ by

\[(K(x)h, k) = \int_{\Gamma} \overline{\gamma(x)} d\mu_{h,k}(\gamma),\]

for every $x \in G$ and $h, k \in \mathcal{H}$. Since $\{\mu_{h,k}\}_{h,k \in \mathcal{H}}$ is a semispectral family, Theorem 1.12 implies that $K$ is positive definite on $G$.

Let $x_0 \in V$ and consider the function $\chi_{\{x_0\}} \in \mathcal{P}(V)$. For $h \in \mathcal{H}$, we have that

\[(l(\chi_{\{x_0\}})h, h) = \sum_{x \in G} (\chi_{\{x_0\}} k(x) h, h) = (k(x_0) h, h).\]

Also,

\[\check{\chi}_{\{x_0\}}(\gamma) = \int_{G} \overline{\gamma(x)} \chi_{\{x_0\}} dx = \overline{\gamma(x_0)}.\]

Thus,

\[(L(\check{\chi}_{\{x_0\}})h, h) = \int_{\Gamma} \check{\chi}_{\{x_0\}}(\gamma) d\mu_h(\gamma) = \int_{\Gamma} \overline{\gamma(x_0)} d\mu_h(\gamma) = (K(x_0) h, h).\]

This implies that $(K(x_0) h, h) = (k(x_0) h, h)$ for every $h \in \mathcal{H}$, thus $K_{\{(-a,a) \times G_2\} = k}$ and this completes the proof.

**Corollary 2.2.** Let $a$ be a positive real number and let $f : (-a, a) \rightarrow L(\mathcal{H})$ be a positive definite function with a countable support. Then $f$ admits a positive definite extension to $R$ which also has a countable support.

**Proof.** Apply Theorem 2.1 when $G_1$ is the additive group generated by the support of $f$ and $G_2$ the trivial group.

Corollary 2.2 can be viewed as a generalization of a result in [13] stating that every strictly positive definite matrix-valued almost periodic Wiener class function with spectrum in the real interval $(-a, a)$ can be extended to a (pointwise) strictly positive almost periodic function on $R$ which also belongs to the Wiener class (see [16] for the scalar case).

The following result was proved in [7] for $r = 2$ by a different approach.
Corollary 2.3. Consider $\mathbb{Z}^r$ with the lexicographic order and $a \in \mathbb{Z}^r$. Then every positive definite function $f : (-a,a) \to \mathcal{L}(\mathcal{H})$ admits a positive definite extension to $\mathbb{Z}^r$.

Theorem 2.1 can be applied for a function defined on $(-a,a) \times \mathbb{Z}$, $a \in \mathbb{N}$. In this case, the function $f$ is assumed to be positive definite with respect to the set $[0,a) \times \mathbb{Z}$, which is different from the standard condition for $\mathbb{Z}^2$ and the lexicographic order, when positivity is considered with respect to the set

$$
\{(p,q) : 0 < p < a \} \cup \{ (0,n) : q \geq 0 \}.
$$

Consider a function $f$ defined in the band $\{(p,q) \in \mathbb{Z}^2 : |p| < a \}$, positive definite with respect to the set (7). Let $f_n$ be the restriction of $f$ to the interval $((-a+1,-n),(a-1,n))$ and let $F_n$ be the extension of $f_n$ to $\mathbb{Z}^2$ given by Corollary 2.3. The sequence $\{F_n\}$ has a subsequence which converges pointwise in the weak topology to a function $F$ which is a positive extension of $f$. The above can be applied similarly for $\mathbb{Z}^r$. This leads to the following conclusion.

Corollary 2.4. Consider $\mathbb{Z}^r$ with the lexicographic order and let $a \in \mathbb{N}$. Let $S = \{(m_1,m_2,\ldots,m_r) \in \mathbb{Z}^r : |m_1| < a \}$. Then every positive definite function $f : S \to \mathcal{L}(\mathcal{H})$ admits a positive definite extension to $\mathbb{Z}^r$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_r \geq 0$ be given. Consider in $\mathbb{Z}^r$ the set

$$
P = \{ (m_1,\ldots,m_r) : \alpha_1 m_1 + \cdots + \alpha_r m_r > 0 \}
$$

$$
\cup \{ (m_1,\ldots,m_r) : \alpha_1 m_1 + \cdots + \alpha_r m_r = 0, m_1 = m_2 = \cdots = m_k = 0, m_{k+1} > 0 \}.
$$

Then $P$ defines an order on $\mathbb{Z}^r$.

Corollary 2.5. Let $s > 0$ and

$$
S = \{ (m_1,\ldots,m_r) \in \mathbb{Z}^r : |\alpha_1 m_1 + \cdots + \alpha_r m_r| \leq s \},
$$

and let $k : S \to \mathcal{L}(\mathcal{H})$ be a positive definite function. Then $k$ admits a positive definite extension to $\mathbb{Z}^r$.

Proof. The result is a consequence of Theorem 2.1 combined with arguments such as those preceding Corollary 2.4.

Corollary 2.5 can be viewed as a generalization of a result in [4] stating that every matrix-valued strictly positive definite function on a set of the form (9) for $r = 2$ which belongs to the Wiener class can be extended to a (pointwise) positive function on $\mathbb{Z}^2$ which also belongs to the Wiener class.

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References


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