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THE EXTENSION OF POSITIVE DEFINITE OPERATOR-VALUED FUNCTIONS DEFINED ON A SYMMETRIC INTERVAL OF AN ORDERED GROUP

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ABSTRACT. Let G_1 be an ordered abelian group and $a \in G_1$. Let G_2 be an abelian group and f an operator-valued positive definite function on $(-a,a) \times G_2$. We prove that f admits a positive definite extension to $G_1 \times G_2$, generalizing in this way existing results for the case when $G_1 = \mathbf{R}$ and f is continuous.

1. Introduction

Let G be an abelian group and let Λ be a finite subset of G. A function $k : S = \Lambda - \Lambda \to \mathcal{L}(\mathcal{H})$ (the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$) is called *positive (semi)definite with respect to* Λ if, for every finite subset $\lambda_1, \lambda_2, \cdots, \lambda_n \in \Lambda$, the operator matrix $\{k(\lambda_i - \lambda_j)\}_{i,j=1}^n$ is positive (semi)definite. Without loss of generality, we assume in this paper that positive definite functions k have the property that $k(0) = I_{\mathcal{H}}$. Let G_1 be an ordered abelian group, $a \in G_1$, and G_2 be an abelian group. A function $k : (-a,a) \times G_2 \to \mathcal{L}(\mathcal{H})$ is referred to as positive definite if it is positive definite with respect to $[0,a) \times G_2$, unless otherwise specified. M. G. Krein proved [10] that every positive definite continuous scalar function on a real interval $(-a,a)$ admits a continuous positive definite extension to $\mathbf{R}$. A. P. Artjomenko [2] (see also Theorem 4.2.3 in [15]) provided a new proof for Krein's Extension Theorem without the continuity requirement. Y. M. Berezansky and I. M. Gali ([6], see also Theorem 5.4.4.2 in [5]) proved the following extension of Krein's Theorem: "Given a Hilbert space $\mathcal{H}$, and a positive definite function k on a layer in $\mathcal{H}$ that is J continuous at 0, then k can be extended to a positive definite function on $\mathcal{H}$ with the same property of continuity." By a similar proof it follows that every continuous positive definite function on $(-a,a) \times G$, where G is a topological group, can be extended to a continuous positive definite function on $\mathbf{R} \times G$.

J. Friedrich and L. Klotz [9] proved that, given $0 < a < \infty$ and a topological group G, any strongly continuous positive definite function $k : (-a,a) \times G \to \mathcal{L}(\mathcal{H})$ admits a positive definite extension to $\mathbf{R} \times G$. The aim of this paper is to generalize the above result by omitting the continuity requirement and by substituting $\mathbf{R}$ with an ordered abelian group. Finally, several corollaries of our main result are presented. One of them is a result recently proved in [7], and the others are

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generalizations of extension results for positive definite functions in [16], [13], and [4].

For notation and results in group theory and Fourier Analysis on groups we refer to [14] and [15]. If G is a locally compact abelian group, then its character group is denoted by Γ.

Suppose P is a semigroup of the abelian group G and that P has the properties

$$P \cap (-P) = \{0\}, \quad P \cup (-P) = G.$$

Under these conditions, P induces an order in G. If we define $x \geq y$ to mean $x - y \in P$, then the axioms of linear order are satisfied. The choice of a semigroup P with the above properties makes G into an *ordered group*.

Let K be a compact Hausdorff space and let $\mathcal{B}(K)$ denote the class of Borel measurable sets in K and $C(K)$ the set of all continuous complex functions on K. A function $F : \mathcal{B}(K) \to \mathcal{L}(\mathcal{H})$ such that $F(X) = I_{\mathcal{H}}$ is called a *semispectral measure* if for every $h \in \mathcal{H}$, $\mu(\sigma) = (F(\sigma)h, h)$ is a positive Borel measure on K. If F is a semispectral measure, define the Borel measures $\mu_{h,k}(\sigma) = (F(\sigma)h, k)$ for $\sigma \in \mathcal{B}(\mathcal{H})$ and $h, k \in \mathcal{H}$; $\{\mu_{h,k}\}_{h,k\in\mathcal{H}}$ is called the *semispectral family* associated with F. The following results are well-known facts (see, e.g., Theorem 7.1 and Proposition 9.2 in [17]).

Theorem 1.1. *Let X be a compact Hausdorff space and let $L : C(K) \to \mathcal{L}(\mathcal{H})$ be a linear operator such that $L(1) = I_{\mathcal{H}}$. Then L is positive in the sense that $L(q) \geq 0$ for every $q \in C(K)$ if and only if there exists a semispectral family $\{\mu_{h,k}\}_{h,k\in\mathcal{H}}$ on K such that*

$$(L(q)h, k) = \int_K q(x) d\mu_{h,k}(x) \tag{1}$$

for every $q \in C(K)$ and $h, k \in \mathcal{H}$.

Theorem 1.2. *Let G be an abelian group. A function $f : G \to \mathcal{L}(\mathcal{H})$ with $f(0) = I_{\mathcal{H}}$ is positive definite if and only if there exists a semispectral family $\{\mu_{h,k}\}_{h,k\in\mathcal{H}}$ on Γ such that*

$$(f(x)h, k) = \int_\Gamma \gamma(x) d\mu_{h,k}(\gamma) \tag{2}$$

for every $x \in G$ and $h, k \in \mathcal{H}$. (G is considered with the discrete topology, thus Γ is compact.)

2. Main results

The following is the main result of the paper. Its proof is modeled after Artjomenko's proof of the Krein Extension Theorem ([2], also presented in [15]).

Theorem 2.1. *Let G_1 be an ordered abelian group and $a \in G_1$. If G_2 is an abelian group, then every positive definite function $f : (-a, a) \times G_2 \to \mathcal{L}(\mathcal{H})$ admits a positive definite extension to $G_1 \times G_2$.*

Proof. Let $G = G_1 \times G_2$ and consider on G the discrete topology, and let Γ be the character group of G. Denote $V = (-a, a) \times G_2$ and let $\mathcal{P}(V)$ be the set of all functions $g : V \to \mathbf{C}$ with finite support. For $g \in \mathcal{P}(V)$ define

$$\phi(x) = \begin{cases} g(x)k(x), & x \in V, \\ 0, & x \notin V. \end{cases}$$

We prove that ϕ is positive definite on G for every positive definite $g \in \mathcal{P}(V)$. Consider $x_1, x_2, \cdots, x_n \in G$. We have to prove that the operator matrix $(\phi(x_i - x_j))_{i,j=1}^n$ is positive semidefinite. Without loss of generality, suppose that $x_i = (\lambda_i, \sigma_i)$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Consider the undirected graph $H = (V, E)$ with vertex set $V = \{1, 2, \cdots, n\}$ and edge set $E = \{(i,j) | i \neq j \text{ and } x_j - x_i \in V\}$. Then H is a so-called proper interval graph ([8]). Define the partial operator matrix

$$(A_{ij})_{i,j=1}^n = \begin{cases} k(x_i - x_j) & \text{for} \quad (i,j) \in E, \\ \text{unspecified} & \text{for} \quad (i,j) \notin E. \end{cases} \tag{3}$$

All fully specified principal submatrices of A are positive semidefinite and H is the graph associated with the pattern of A. By Corollary 3.2 in [1], A admits a positive semidefinite extension $B = (B_{ij})_{i,j=1}^n$. The matrix $(\phi(x_i - x_j))_{i,j=1}^n$ is the Schur product of B and $(g(x_i - x_j))_{i,j=1}^n$ (all unspecified entries of A correspond to zeros in $(g(x_i - x_j))_{i,j=1}^n$). By a generalized version of Schur's Theorem ([12], proof of Theorem 4.3), it follows that $(\phi(x_i - x_j))_{i,j=1}^n$ is positive semidefinite.

For $h \in \mathcal{H}$ define $\phi_h(x) = (\phi(x)h, h)$; ϕ_h is a positive definite function on G for every positive definite $g \in \mathcal{P}(V)$. Then

$$\hat{\phi}_h(1) = \int_G \phi_h(x) dx = \sum_{x \in G} \phi_h(x) \geq 0,$$

where the last inequality is a consequence of the fact that $\hat{\phi}_h$ is positive (see Theorem 1.9.8 in [15]). Consequently, $\sum_{x \in G} g(x)k(x) \geq 0$, for every positive definite $g \in \mathcal{P}(V)$.

Define $T_V = \{\hat{g} | g \in \mathcal{P}(V)\}$ and let $T_V^+ = \{p \in T_V | p \geq 0\}$. Every element of T_V^+ is the Fourier transform of a positive definite function in $P(V)$. Define $l : T_V \to \mathcal{L}(\mathcal{H})$ by $l(q) = \sum_{x \in G} \check{q}(x)k(x)$. Then l is a positive operator on T_V, which is an operator system in $C(\Gamma)$. We will prove that l is completely positive.

Let $m > 1$ and let $\mathcal{M}_m$ denote the set of all complex $m \times m$ matrices. Let $Z : G \to \mathcal{M}_m$, $Z(x) = (z_{ij}(x))_{i,j=1}^m$, $z_{ij} \in \mathcal{P}(V)$, be a positive definite function. For $i, j = 1, \cdots, m$, define

$$\Phi_{ij}(x) = \begin{cases} z_{ij}(x)k(x), & x \in V, \\ 0, & x \notin V. \end{cases}$$

We first prove that $\Phi(x) = (\Phi_{ij}(x))_{i,j=1}^m$ is a positive definite matrix function on G. Let $x_1, x_2, \cdots, x_n \in G$. Without loss of generality, suppose that $x_i = (\lambda_i, \sigma_i)$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then,

$$(\Phi(x_k - x_l))_{k,l=1}^n = [Z(x_k - x_l)]_{k,l=1}^n \odot [B \otimes J_m], \tag{4}$$

where $\odot$ denotes the Schur product, B is a positive semidefinite extension of the partial matrix A defined by (3), and J_m is the $m \times m$ matrix with all entries equal to 1. The matrix $[Z(x_k - x_l)]_{k,l=1}^n$ is positive semidefinite since $Z(x)$ is a positive definite function. Then (4) and the generalized version of Schur's Theorem ([12]) imply that $(\Phi(x_k - x_l))_{k,l=1}^n$ is positive semidefinite, thus $\Phi(x)$ is a positive definite function.

Let $Q(x) = (q_{ij}(x))_{i,j=1}^m$ be a matrix-valued function such that $q_{ij} \in T_V$ for every $i, j = 1, \cdots, m$, and $Q(x) \geq 0$ for every $x \in G$. Then $\check{Q}(x) = (\check{q}_{ij}(x))_{i,j=1}^m$ is a positive definite matrix function on G, which implies that the function $(\check{q}_{ij}(x)k(x))_{i,j=1}^m$

is also positive definite. Let $\mathbf{h} \in \mathcal{H}_n = \bigoplus_{i=1}^n \mathcal{H}$. Define $\Phi_{\mathbf{h}}(x) = (\Phi(x)\mathbf{h}, \mathbf{h})$; $\Phi_{\mathbf{h}}$ is a positive definite (scalar) function on G. Then

$$\hat{\Phi}_{\mathbf{h}}(1) = \int_G \Phi_{\mathbf{h}}(x)dx = \sum_{x \in G} \Phi_{\mathbf{h}}(x) \geq 0, \tag{5}$$

the last inequality being a consequence of Theorem 1.9.8 in [15].

Relation (5) implies that the function $\mathbf{l} : T_V \otimes \mathcal{M}_m \to \mathcal{L}(\mathcal{H}) \otimes \mathcal{M}_m$, defined by $\mathbf{l}(Q) = (\sum_{x \in G} \check{q}_{ij}(x)k(x))_{i,j=1}^m$, is positive definite for every $m \geq 1$, which means that l is completely positive.

Since T_V is an operator system in $C(\Gamma)$, by Arveson's Theorem ([3], see also Theorem 6.5 in [11]), l admits a (completely) positive extension $L : C(\Gamma) \to \mathcal{L}(\mathcal{H})$. By Theorem 1.1 there exists a semispectral family $\{\mu_{h,k}\}_{h,k \in \mathcal{H}}$ on Γ such that

$$(L(q)h, k) = \int_\Gamma q(\gamma) d\mu_{h,k}(\gamma)$$

for every $q \in C(\Gamma)$ and $h, k \in \mathcal{H}$. Define $K : G \to \mathcal{L}(\mathcal{H})$ by

$$(K(x)h, k) = \int_\Gamma \overline{\gamma(x)} d\mu_{h,k}(\gamma), \tag{6}$$

for every $x \in G$ and $h, k \in \mathcal{H}$. Since $\{\mu_{h,k}\}_{h,k \in \mathcal{H}}$ is a semispectral family, Theorem 1.2 implies that K is positive definite on G.

Let $x_0 \in V$ and consider the function $\chi_{\{x_0\}} \in \mathcal{P}(V)$. For $h \in \mathcal{H}$, we have that

$$(l(\hat{\chi}_{\{x_0\}})h, h) = \sum_{x \in G} (\chi_{\{x_0\}} k(x)h, h) = (k(x_0)h, h).$$

Also,

$$\hat{\chi}_{\{x_0\}}(\gamma) = \int_G \overline{\gamma(x)} \chi_{\{x_0\}} dx = \overline{\gamma(x_0)}.$$

Thus,

$$(L(\hat{\chi}_{\{x_0\}})h, h) = \int_\Gamma \hat{\chi}_{\{x_0\}}(\gamma) d\mu_h(\gamma) = \int_\Gamma \overline{\gamma(x_0)} d\mu_h(\gamma) = (K(x_0)h, h).$$

This implies that $(K(x_0)h, h) = (k(x_0)h, h)$ for every $h \in \mathcal{H}$, thus $K_{|(-a,a) \times G_2} = k$, and this completes the proof.

Corollary 2.2. *Let a be a positive real number and let $f : (-a, a) \to \mathcal{L}(\mathcal{H})$ be a positive definite function with a countable support. Then f admits a positive definite extension to* $\mathbf{R}$ *which also has a countable support.*

Proof. Apply Theorem 2.1 when G_1 is the additive group generated by the support of f and G_2 the trivial group.

Corollary 2.2 can be viewed as a generalization of a result in [13] stating that every strictly positive definite matrix-valued almost periodic Wiener class function with spectrum in the real interval $(-a, a)$ can be extended to a (pointwise) strictly positive almost periodic function on $\mathbf{R}$ which also belongs to the Wiener class (see [16] for the scalar case).

The following result was proved in [7] for $r = 2$ by a different approach.

Corollary 2.3. *Consider* $\mathbf{Z}^r$ *with the lexicographic order and* $a \in \mathbf{Z}^r$*. Then every positive definite function* $f : (-a, a) \to \mathcal{L}(\mathcal{H})$ *admits a positive definite extension to* $\mathbf{Z}^r$*.*

Theorem 2.1 can be applied for a function defined on $(-a, a) \times \mathbf{Z}$, $a \in \mathbf{N}$. In this case, the function f is assumed to be positive definite with respect to the set $[0, a) \times \mathbf{Z}$, which is different from the standard condition for $\mathbf{Z}^2$ and the lexicographic order, when positivity is considered with respect to the set

$$\{(p, q) : 0 < p < a\} \cup \{(0, n) : q \geq 0\}. \tag{7}$$

Consider a function f defined in the band $\{(p, q) \in \mathbf{Z}^2 : |p| < a\}$, positive definite with respect to the set (7). Let f_n be the restriction of f to the interval $((-a+1, -n), (a-1, n))$ and let F_n be the extension of f_n to $\mathbf{Z}^2$ given by Corollary 2.3. The sequence $\{F_n\}$ has a subsequence which converges pointwise in the weak topology to a function F which is a positive extension of f. The above can be applied similarly for $\mathbf{Z}^r$. This leads to the following conclusion.

Corollary 2.4. *Consider* $\mathbf{Z}^r$ *with the lexicographic order and let* $a \in \mathbf{N}$*. Let* $S = \{(m_1, m_2, \cdots, m_r) \in \mathbf{Z}^r : |m_1| < a\}$*. Then every positive definite function* $f : S \to \mathcal{L}(\mathcal{H})$ *admits a positive definite extension to* $\mathbf{Z}^r$*.*

Let $\alpha_1, \alpha_2, \cdots, \alpha_r \geq 0$ be given. Consider in $\mathbf{Z}^r$ the set

$$\begin{aligned} P = &\{(m_1, \cdots, m_r) : \alpha_1 m_1 + \cdots + \alpha_r m_r > 0\} \\ &\cup \{(m_1, \cdots, m_r) : \alpha_1 m_1 + \cdots + \alpha_r m_r = 0,\ m_1 = m_2 = \cdots m_k = 0, m_{k+1} > 0\}. \end{aligned} \tag{8}$$

Then P defines an order on $\mathbf{Z}^r$.

Corollary 2.5. *Let* $s > 0$ *and*

$$S = \{(m_1, \cdots, m_r) \in \mathbf{Z}^r : |\alpha_1 m_1 + \cdots + \alpha_r m_r| \leq s\}, \tag{9}$$

and let $k : S \to \mathcal{L}(\mathcal{H})$ *be a positive definite function. Then* k *admits a positive definite extension to* $\mathbf{Z}^r$*.*

Proof. The result is a consequence of Theorem 2.1 combined with arguments such as those preceding Corollary 2.4.

Corollary 2.5 can be viewed as a generalization of a result in [4] stating that every matrix-valued strictly positive definite function on a set of the form (9) for $r = 2$ which belongs to the Wiener class can be extended to a (pointwise) positive function on $\mathbf{Z}^2$ which also belongs to the Wiener class. ☐

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