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THE RANGE OF OPERATORS ON VON NEUMANN ALGEBRAS

TERESA BERMÚDEZ AND N. J. KALTON

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ABSTRACT. We prove that for every bounded linear operator $T : X \to X$, where X is a non-reflexive quotient of a von Neumann algebra, the point spectrum of T^* is non-empty (i.e., for some $\lambda \in \mathbb{C}$ the operator $\lambda I - T$ fails to have dense range). In particular, and as an application, we obtain that such a space cannot support a topologically transitive operator.

1. Introduction

The results in this paper are motivated by a question related to hypercyclic operators. In [8] G. Godefroy and J. Shapiro suggest an extension of the notion of a hypercyclic operator to Banach space which is not necessarily separable via the notion of topologically transitive operators (see Section 3 below). Every Hilbert space supports a topologically transitive operator (see the example due to J. Shapiro in Section 3.) Recently, it has been shown by S. Ansari [1] and L. Bernal [2] that every separable Banach space supports a hypercyclic operator, so it is natural to ask whether every Banach space supports a topologically transitive operator.

It is well-known that if T is hypercyclic, then the adjoint operator T^* has empty point spectrum, $\sigma_p(T^*)$, [13] and [14]; this extends to topologically transitive operators (Proposition 3.3). Thus we are led to the question of whether there exist complex Banach spaces so that for every operator T we have $\sigma_p(T^*) \neq \emptyset$. Such an example exists in the literature, [19] and [20]. However, we show here that there are much more natural examples. If X is any von Neumann algebra (or even a non-reflexive quotient of a von Neumann algebra), then any operator T on X has $\sigma_p(T^*) \neq \emptyset$. In particular, this holds if $X = \ell_\infty$ or $X = \mathcal{L}(\ell_2)$. We note hypercyclicity with respect to the strong-operator topology on $\mathcal{L}(\ell_2)$ has been considered in [5] and [16].

Our main result is rather stronger in that we show that if X is a non-reflexive quotient of a von Neumann algebra, then for any operator T we have that the quotient space $X/\overline{\mathcal{R}(\lambda - T)}$ contains a copy of ℓ_∞ and is in particular non-separable.

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Let us point out by way of further motivation that any operator T on ℓ_1 satisfies $\sigma_p(T^{**}) \neq \emptyset$, since if λ is in the approximate point spectrum of T, then it is in the point spectrum of T^{**} by an argument depending on the Schur property of ℓ_1 (this was shown to us by M. González). This is suggestive of the main result in the case $X = \ell_\infty$.

Our arguments depend on two Banach space concepts, which we now introduce. A projection P on a Banach space X is an L-projection if $\|x\| = \|Px\| + \|x - Px\|$ for any $x \in X$. A Banach space X is said to be *L-embedded* if there is an L-projection of X^{**} onto X, i.e., if there is a projection $\Pi : X^{**} \to X$ so that we have

$$\|x^{**}\| = \|x^{**} - \Pi x^{**}\| + \|\Pi x^{**}\| \qquad \text{for } x^{**} \in X^{**}.$$

For the basic facts on L-embedded spaces we refer to [11], Chapter IV. A Banach space X is called a *Grothendieck space* if every bounded operator $T : X \to Y$ with separable range is weakly compact. This is equivalent to requiring that if $\{x_n^*\}_{n\in\mathbb{N}}$ is a weak*-null sequence in X^*, then it is also weakly null. Any von Neumann algebra is a Grothendieck space [17] and its dual is L-embedded [21], [11]. We also recall that a Banach space X is called an *Asplund space* if every separable subspace has separable dual (this is equivalent to the original definition, [6], Theorem 5.7, p. 29).

Most of our notation is standard. We will use B_X to denote the closed unit ball of a Banach space X. If F is a subset of X, then $\langle F \rangle$ denotes its linear span.

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2. Main results

We use repeatedly the following principle:

Lemma 2.1 ([22, II.E.15]). *Let X be a Banach space and suppose $\{C_k\}_{k=1}^n$ is a finite set of convex sets. Suppose D_k is the weak*-closure of C_k in X^{**}. If $\bigcap_{k=1}^n D_k \neq \emptyset$, then for any $\epsilon > 0$ there exists $x \in C_1$ with $d(x, C_k) < \epsilon$ for $k = 2, 3 \dots, n$.*

We will also need the following well-known variant of the Hahn-Banach Theorem.

Lemma 2.2. *Let X be a Banach space and suppose F is a finite-dimensional subspace of X^*. If ψ is a linear functional on F with $\|\psi\| < 1$, then there exists $x \in X$ with $\|x\| < 1$ and $x^*(x) = \psi(x^*)$ for $x^* \in F$.*

Proof. This can be proved directly or from Lemma 2.1. Let $C_1 = \{x \in X : x^*(x) = \psi(x^*)\ \forall x^* \in F\}$ and $C_2 = \{x \in X : \|x\| \leq \|\psi\|\}$. Then, by the Hahn-Banach Theorem, the weak*-closure D_1 of C_1 is the set $\{x^{**} \in X^{**} : x^{**}(x^*) = \psi(x^*)\ \forall x^* \in F\}$. By an application of the Hahn-Banach Theorem and Goldstine's Theorem ([15], Theorem 2.6.26, p. 232) D_1 meets the weak*-closure of C_2 so that we can apply Lemma 2.1. □

Proposition 2.3. *Suppose $T : X \to Y$ is a bounded linear operator. Then the following properties are equivalent*:

(1) $\mathcal{N}(T^{**}) = \{0\}$.

(2) *If $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a bounded sequence such that $\lim_{n\to\infty} \|Tx_n\| = 0$, then $\lim_{n\to\infty} x_n = 0$ weakly.*

Proof. (1) implies (2). Clearly 0 is the only weak*-cluster point of $\{x_n\}_{n\in\mathbb{N}}$ in X^{**} and so $\lim_{n\to\infty} x_n = 0$ weakly.

(2) implies (1). Assume for some $x^{**} \neq 0$ with $\|x^{**}\| = 1$, we have $T^{**}x^{**} = 0$. Pick $x^* \in X^*$ with $x^{**}(x^*) = 1$. Then for each n the sets $C_1 = \{x : \|x\| \leq 1\}$, $C_2 = \{x : x^*(x) \geq 1\}$ and $C_3 = \{x : \|Tx\| \leq n^{-1}\}$ satisfy the conditions of Lemma 2.1, so we pick $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $\|Tx_n\| \leq n^{-1}$, $\|x_n\| \leq 2$ and $x^*(x_n) \geq \frac{1}{2}$, contradicting (2). □

Now if $T : X \to Y$ is a bounded linear operator, we denote by $\hat{T}$ the induced operator $\hat{T} : X^{**}/X \to Y^{**}/Y$.

Proposition 2.4. *Suppose $T : X \to Y$ is a bounded operator. Then the following are equivalent*:

(1) *There exists a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset X^{**}/X$ such that $\|\xi_n\| = 1$ and $\lim_{n\to\infty} \|\hat{T}\xi_n\| = 0$.*

(2) *There exists a bounded sequence $\{x_n^{**}\}_{n\in\mathbb{N}} \subset X^{**}$ such that $d(x_n^{**}, X) = 1$ and $\lim_{n\to\infty} \|T^{**}x_n^{**}\| = 0$.*

Proof. We only need to prove that (1) implies (2). Pick $w_n^{**} \in \xi_n$ with $\|w_n^{**}\| \leq 2$. Let $\epsilon_n := \|\hat{T}\xi_n\| + \frac{1}{n}$. Then there exists $u_n \in X$ with $\|T^{**}w_n^{**} - u_n\| < \epsilon_n$. We now argue that since $T^{**}w_n^{**}$ is in the weak*-closure of both $u_n + \epsilon_n B_X$ and $2T(B_X)$, then there exists $v_n \in X$ with $\|v_n\| \leq 2$ and $\|Tv_n - u_n\| \leq 2\epsilon_n$. Thus $\|T^{**}(w_n^{**} - v_n)\| \leq 3\epsilon_n$. Letting $x_n^{**} := w_n^{**} - v_n$, we are done. □

Theorem 2.5. *Suppose X is a subspace of an L-embedded Banach space V, and Y is any Banach space. Suppose $T : X \to Y$ is a bounded linear operator such that $\mathcal{N}(T^{**}) \subset X$. Then there exists $\delta > 0$ so that for all $\xi \in X^{**}/X$ we have $\|\hat{T}\xi\| \geq \delta\|\xi\|$.*

Proof. We start by proving the theorem in the special case when $\mathcal{N}(T^{**}) = \{0\}$.

Suppose the conclusion is false. Using Proposition 2.4 we produce a bounded sequence $\{x_n^{**}\}_{n\in\mathbb{N}} \subset X^{**}$ with $d(x_n^{**}, X) = 1$ but $\lim_{n\to\infty} \|T^{**}x_n^{**}\| = 0$. We can regard X^{**} as a subspace of V^{**}. Now let $\delta_n = d(x_n^{**}, V)$. For fixed n, if $\rho > \delta_n$, then x_n^{**} is in the weak*-closure of both X and $v + \rho B_V$ for some $v \in V$. Hence there is $y \in v + \rho B_V$ such that $d(y, X) \leq \rho$ by an application of Lemma 2.1 and so $d(x_n^{**}, X) \leq 2\rho$. We conclude that $\delta_n \geq \frac{1}{2}$ for each $n \in \mathbb{N}$.

Let us denote by Π the L-projection of V^{**} onto V, and let $V_s = \ker \Pi$. Let $v_n := \Pi x_n^{**}$ and $v_n^{**} := x_n^{**} - v_n$. Then $v_n^{**} \in V_s$ and $\|v_n^{**}\| = \delta_n \geq \frac{1}{2}$. Let $a := \sup_{n\in\mathbb{N}} \|x_n^{**}\|$ and $\eta_n := \|T^{**}x_n^{**}\| + \frac{1}{n}$.

We shall define inductively a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, and a sequence $\{x_n^*\}_{n\in\mathbb{N}}$ in X^* such that

$$\|x_n\| \leq a, \qquad n \in \mathbb{N}, \tag{2.1}$$

$$\|Tx_n\| < \eta_n, \tag{2.2}$$

$$\|x_n^*\| < 1, \qquad n \in \mathbb{N}, \tag{2.3}$$

$$|x_n^*(x_k)| \geq \frac{1}{8}, \quad 1 \leq k \leq n. \tag{2.4}$$

Let us suppose that $n \in \mathbb{N}$ and that $\{x_k\}_{k<n}$ and $\{x_k^*\}_{k<n}$ have been determined and satisfy (2.1), (2.2), (2.3) and (2.4); if $n = 1$, these sets are empty of course. We shall determine x_n and x_n^*.

Let $F := \langle\{x_1, \ldots, x_{n-1}, v_n\}\rangle$ and $G := \langle\{x_1, \ldots, x_{n-1}, v_n, v_n^{**}\}\rangle$. If $n > 1$, we define $\psi = \psi_n \in F^*$ by taking ψ to be a norm-preserving extension of $x_{n-1}^*|_{F\cap X}$; if $n = 1$, we simply let $\psi = 0$. Then $\|\psi\| < 1$. Let $\psi(v_n) = re^{i\theta}$ where $0 \leq \theta < 2\pi$ and $r \geq 0$. We next define $\varphi \in G^*$ to the extension of ψ such that $\varphi(v_n^{**}) = \frac{1}{4}e^{i\theta}$. We claim that $\|\varphi\| < 1$. In fact, if $u^{**} \in G$, then we can write $u^{**} = u + \mu v_n^{**}$ where $\mu \in \mathbb{C}$ and $u \in F$. Then

$$\begin{aligned}|\varphi(u^{**})| &\leq |\psi(u)| + \frac{1}{4}|\mu| \\ &\leq \|\psi\|\|u\| + \frac{1}{2}|\mu|\|v_n^{**}\| \\ &\leq \max(\frac{1}{2}, \|\psi\|)\|u^{**}\| < \|u^{**}\|.\end{aligned}$$

Now by Lemma 2.2 we can define $v^* \in V^*$ with $\|v^*\| < 1$ and $u^{**}(v^*) = \varphi(u^{**})$ for $u^{**} \in G$. Let x_n^* be the restriction of v^* to X.

Now consider the sets $C_1 = \{x : \|x\| \leq a\}$, $C_2 = \{x : \|Tx\| \leq \|T^{**}x_n^{**}\|\}$ and $C_3 = \{x : \ x_n^*(x) = x_n^{**}(x_n^*)\}$. Clearly x_n^{**} belongs to the weak*-closure of each set. By Lemma 2.1 we can find $x_n \in C_1$ with $\|Tx_n\| < \eta_n$, and so that

$$|x_n^*(x_n)| > |x_n^{**}(x_n^*)| - \frac{1}{8}.$$

It is now clear that (2.1), (2.2) and (2.3) hold. For (2.4) note that if $k < n$, we have $x_n^*(x_k) = x_{n-1}^*(x_k)$ while

$$|x_n^*(x_n)| \geq |x_n^{**}(x_n^*)| - \frac{1}{8} = (\frac{1}{4} + r) - \frac{1}{8} \geq \frac{1}{8}.$$

Now the proof is completed (for the special case $\mathcal{N}(T^{**}) = \{0\}$) by observing that, if x^* is any weak*-cluster point of the sequence $\{x_n^*\}_{n\in\mathbb{N}}$, then $|x^*(x_n)| \geq \frac{1}{8}$ for all n. Since $\lim_{n\to\infty} \|Tx_n\| = 0$, this contradicts Proposition 2.3, since x_n does not converge weakly.

To treat the general case suppose $R = \mathcal{N}(T^{**}) = \mathcal{N}(T)$. Then R is reflexive. Consider the induced map $T_0 : X/R \to Y$; clearly $\mathcal{N}(T_0^{**}) = \{0\}$. We next note that X/R embeds into V/R and V/R is L-embedded [11], p. 160. Hence $\hat{T}_0$ satisfies a lower bound on $Z = (X/R)^{**}/(X/R)$. However, it is easily seen that Z coincides with X^{**}/X and $\hat{T}_0 = \hat{T}$. □

We next need some facts about Grothendieck spaces.

Proposition 2.6. *Suppose Y is a Grothendieck space and that $T : X \to Y$ is a bounded linear operator such that T^* is one-to-one. Then T^{***} is one-to-one.*

Proof. Suppose $\{y_n^*\}_{n\in\mathbb{N}} \subset Y^*$ is a bounded sequence such that $\lim_{n\to\infty} \|T^*y_n^*\| = 0$. Let y^* be any weak*-cluster point of $\{y_n^*\}_{n\in\mathbb{N}}$. Then $T^*y^* = 0$ so that $y^* = 0$. Therefore, $\lim\limits_{n\to\infty} y_n^* = 0$ weak*. But since Y is a Grothendieck space, this implies $\lim\limits_{n\to\infty} y_n^* = 0$ weakly and we can apply Proposition 2.3. □

Proposition 2.7. *Suppose X is a Grothendieck space and Y is a subspace of X so that X/Y is reflexive. Then Y is a Grothendieck space.*

Proof. Suppose $T : Y \to c_0$ is any bounded operator. Then we may find a Banach space $E \supset c_0$ with $E/c_0 \cong X/Y$ and an extension $\tilde{T} : X \to E$. We claim E is an Asplund space. Indeed if F is a separable subspace of E, then let F' be the closure of $c_0 + F$ which is also separable. Then F'/c_0 is separable and reflexive so that since $c_0^* \cong \ell_1$ is separable, F' has separable dual. Now it follows from a deep result of Hagler and Johnson [10] (see also [7]) that B_{E^*} is weak*-sequentially compact. Hence if (e_n^*) is any sequence in B_{E^*}, there is a subsequence (f_n^*) so that $\tilde{T}^* f_n^*$ is weak* and hence weakly convergent in X^*. Thus $\tilde{T}$ is weakly compact by Gantmacher's theorem (see [15], Theorem 3.5.13, p. 343) and, in particular, T is weakly compact. □

Theorem 2.8. *Suppose X and Y are Banach spaces and Y is a Grothendieck space. Suppose $T : X \to Y$ is a bounded operator such that $Y/\overline{\mathcal{R}(T)}$ is reflexive. Then $\mathcal{N}(T^{***}) \subset Y^*$.*

Proof. Let $Y_0 = \overline{\mathcal{R}(T)}$. Then by Proposition 2.7 Y_0 is also a Grothendieck space. We write $T = JT_0$ where $J : Y_0 \to Y$ is the inclusion map and $T_0 : X \to Y_0$. Clearly $(Y/Y_0)^* \cong \mathcal{N}(T^*)$ is reflexive. We observe that T_0^* is one-to-one and by Proposition 2.6 we obtain that T_0^{***} is also one-to-one. Now, since Y/Y_0 is reflexive, this implies $\mathcal{N}(T^{***}) = \mathcal{N}(J^{***}) = \mathcal{N}(J^*) \subset Y^*$ as required. □

Theorem 2.9. *Let X be a non-reflexive complex Banach space which is a Grothendieck space such that X^* is isometric to a subspace of an L-embedded space. Suppose $T : X \to X$ is a bounded linear operator. Then there exists $\lambda \in \mathbb{C}$ so that $X/\overline{\mathcal{R}(\lambda - T)}$ is non-reflexive (and hence non-separable). In particular, the point spectrum $\sigma_p(T^*)$ is non-empty.*

Proof. Let $S = T^*$. Then since X is non-reflexive, the operator $\hat{S}$ has non-empty spectrum and furthermore for any λ in the boundary $\partial\sigma(\hat{S})$ there is a sequence $\xi_n \in X^{***}/X^*$ with $\|\xi_n\| = 1$ so that $\lim_{n\to\infty} \|(\lambda - \hat{S})\xi_n\| = 0$. This implies that for $\lambda \in \partial\sigma(\hat{S})$ we have $\mathcal{N}((\lambda - S)^{**})$ is not contained in X^* by Theorem 2.5. Then we apply Theorem 2.8 and deduce that $X/\overline{\mathcal{R}(\lambda - T)}$ is non-reflexive. By Proposition 2.6 we have that $\lambda \in \sigma_p(T^*)$. □

Our main example for Theorem 2.9 is when X is a von Neumann algebra. The fact that von Neumann algebras have the Grothendieck property is a recent result of Pfitzner [17]. In fact, slightly more follows from Pfitzner's work.

Proposition 2.10. *Let A be a von Neumann algebra and suppose $T : A \to Y$ fails to be weakly compact, then there is a closed subspace E of A such $T|_E$ is an isomorphism and E is isomorphic to ℓ_∞.*

Proof. Suppose T fails to be an isomorphism on any subspace isomorphic to ℓ_∞. Let A_0 be any maximal Abelian subalgebra of A. Then it follows from classical results of Rosenthal [18] that T is weakly compact on A_0; by Pfitzner's Theorem [17] Theorem 1 (see also Corollary 10), T is weakly compact. □

Theorem 2.11. *Let X be a non-reflexive quotient of a von Neumann algebra, and let $T : X \to X$ be any bounded linear operator. Then there exists $\lambda \in \mathbb{C}$ so that $X/\overline{\mathcal{R}(\lambda - T)}$ contains an isomorphic copy of ℓ_∞ and hence $\mathcal{N}(\lambda - T^*)$ contains an isomorphic copy of ℓ_∞^*. In particular, the point spectrum $\sigma_p(T^*)$ is non-empty.*

Proof. The dual of any C^*-algebra is L-embedded ([21], [11]) and so it follows from the work of Pfitzner [17] that X satisfies the hypotheses of Theorem 2.9. Proposition 2.10 implies that if $X/\overline{\mathcal{R}(\lambda - T)}$ is non-reflexive, then it contains a complemented isomorphic copy of ℓ_∞. Since $\left(X/\overline{\mathcal{R}(\lambda - T)}\right)^* \cong \mathcal{N}(\lambda - T^*)$, there exists in $\mathcal{N}(\lambda - T^*)$ an isomorphic copy of $(\ell_\infty)^*$. □

3. Applications to hypercyclic operators

A bounded linear operator T on a complex Banach space X is called *hypercyclic* if there is a vector $x \in X$ (called *hypercyclic vector for* T) such that $\{T^n x \ : \ n \in \mathbb{N}\}$ is dense on X. This concept is related to the problem of the existence of proper closed invariant subsets for a bounded linear operator. It is an open problem whether every bounded linear operator on a Hilbert space has a proper closed invariant subset, or equivalently if every operator has a non-zero vector which is not hypercyclic. We refer to [9] for an excellent survey.

We note that a non-separable Banach space cannot support a hypercyclic vector. An approach to obtain something similar to hypercyclicity in non-separable Hilbert and Banach spaces was given by K. Chan [5] and A. Montes and C. Romero [16], respectively. In fact, they give certain "hypercyclicity" results in $\mathcal{L}(X)$ where X is a separable Banach space, using the strong operator topology in place of the standard uniform norm topology.

It is however possible to extend the notion of hypercyclic operators to nonseparable Banach spaces in a natural way using the results of [8]. Let us say that an operator T on an arbitrary Banach space is *topologically transitive* if for every pair U, V of non-void open subsets of X, there exists a positive integer n such that $T^n(U) \cap V \neq \emptyset$. In Theorem 1.2 of [8] it is proved that if X is a separable Banach space, then T is hypercyclic if and only if T is topologically transitive.

The following proposition is immediate.

Proposition 3.1. *A bounded linear operator T is topologically transitive if and only if every proper closed invariant subset has empty interior.*

An argument similar to the result due to J. Bés and A. Peris [3] provides a sufficient condition for topological transitivity.

Proposition 3.2 (Topologically transitive criterion)**.** *Let T be a bounded linear operator on a complex Banach space X (not necessarily separable). Suppose that there exists a strictly increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ for which there are:*

1. *A dense subset $X_0 \subset X$ such that $T^{n_k}x \to 0$ for every $x \in X_0$.*
2. *A dense subset $Y_0 \subset X$ and a sequence of mappings $S_k : Y_0 \to X$ such that*
   - (a) *$S_k y \to 0$ for every $y \in Y_0$,*
   - (b) *$T^{n_k} S_k y \to y$ for every $y \in Y_0$.*

Then T is topologically transitive.

Example. The following example was suggested by J. Shapiro. Let us use Proposition 3.2 to show that there is a topologically transitive operator on any Hilbert space. If H is separable, then the result is clear by S. Ansari [1] and L. Bernal [2]. If H is a non-separable Hilbert space, we write $H = \ell_2(X)$ where X is a Hilbert space of the same density character. Define T as twice the backward shift on $\ell_2(X)$, that is,

$$T(x_1, x_2, \dots) := 2(x_2, x_3, \dots).$$

Using Proposition 3.2, we have that T is topologically transitive taking $n_k = k$,

$$X_0 := \{\text{finitely non-zero sequences in } \ell_2(X)\},$$
$$Y_0 := \ell_2(X),$$
$$S(x_1, x_2, \dots) := \frac{1}{2}(0, x_1, x_2, \dots)$$

and $S_k := S^k$.

Clearly this example can be modified to replace H by any space $\ell_p(I)$ where $1 \leq p < \infty$.

It has been shown by S. Ansari [1] and L. Bernal [2] that any separable complex Banach space supports a hypercyclic operator. Recently, J. Bonet and A. Peris gave a version for $\mathcal{F}$-spaces [4]. This suggests the corresponding problem of determining whether every complex Banach space supports a topologically transitive operator.

This question has a negative answer. In order to see this, we need to give a spectral property of topologically transitive operators, which is well-known in the case of hypercyclic operators [14] and [13].

Proposition 3.3. *Let T be a bounded linear operator on a complex Banach space. If T is topologically transitive, then $\sigma_p(T^*)$ is empty.*

Proof. If $\lambda \in \sigma_p(T^*)$ and x^* is a corresponding eigenvector, then one of the sets $\{x : |x^*(x)| \geq 1\}$ or $\{x : |x^*(x)| \leq 1\}$ is an invariant set with non-empty interior. Then use Proposition 3.1. □

As pointed out in the introduction, the examples of [19] and [20] of non-separable spaces such that every bounded operator is a perturbation of a multiple of the identity by an operator with separable range give examples of spaces which support no topological transitive operators. However, the following theorem shows that ℓ_∞ and $\mathcal{L}(\ell_2)$ are more natural examples, where $\mathcal{L}(\ell_2)$ denotes the algebra of all bounded linear operators on ℓ_2.

Theorem 3.4. *Let X be a non-reflexive quotient of a von Neumann algebra. Then X does not support a topologically transitive operator. In particular, $\mathcal{L}(\ell_2)$ and ℓ_∞ do not support a topologically transitive operator.*

Proof. Just apply Theorem 2.11 and Proposition 3.3. □

We conclude with a remark on ultrapowers. We recall some concepts about ultrapowers of Banach spaces and operators. See [12] for more information. We fix a non-trivial ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ of all positive integers. For every Banach space X, we consider the Banach space $\ell_\infty(X)$ of all bounded sequences (x_n) in X, endowed with the norm $\|(x_n)\|_\infty := \sup\{\|x_n\| : n \in \mathbb{N}\}$. Let $N_{\mathcal{U}}(X)$ be the closed subspace of all sequences $(x_i) \in \ell_\infty(X)$ which converge to 0 following $\mathcal{U}$. The *ultrapower of X following $\mathcal{U}$* is defined as the quotient

$$X_{\mathcal{U}} := \frac{\ell_\infty(X)}{N_{\mathcal{U}}(X)}.$$

The element of $X_{\mathcal{U}}$ including the sequence $(x_i) \in \ell_\infty(X)$ as a representative is denoted by $[x_i]$. Its norm in $X_{\mathcal{U}}$ is given by

$$\|[x_n]\| = \lim_{\mathcal{U}} \|x_n\|.$$

The constant sequences generate a subspace of $X_{\mathcal{U}}$ isometric to X. So we can consider the space X embedded in $X_{\mathcal{U}}$. Moreover, every operator $T \in L(X,Y)$ admits an extension $T_{\mathcal{U}} \in L(X_{\mathcal{U}}, Y_{\mathcal{U}})$, defined by

$$T_{\mathcal{U}}([x_n]) := [Tx_n], \ \ [x_n] \in X_{\mathcal{U}}.$$

An easy argument with ultrapowers gives that any ultrapower cannot be a topologically transitive operator. This fact can be obtained by the following easy argument.

Proposition 3.5. *Let $\mathcal{U}$ be an ultrafilter, X a complex Banach space and T any bounded linear operator on X. Then $T_{\mathcal{U}}$ is not topologically transitive.*

Proof. We note that any $\lambda \in \partial\sigma(T)$ is in the approximate point spectrum of T^*, i.e., there exists a sequence $\{x_n^*\}_{n\in\mathbb{N}}$ in X^* with $\|x_n^*\| = 1$ and $\lim_{n\to\infty} \|\lambda x_n^* - T^* x_n^*\| = 0$. Now let $\xi^* \in X_{\mathcal{U}}^*$ be defined by $\xi^*([x_n]) = \lim_{\mathcal{U}} x_n^*(x_n)$. Then $\lambda \in \sigma_p(T_{\mathcal{U}}^*) \neq \emptyset$, so we can apply Proposition 3.3. ☐

We conclude with the following open question: *Is there any characterization of non-separable Banach spaces which support a topologically transitive operator?*

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Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna (Tenerife), Canary Islands, Spain

E-mail address: `tbermude@ull.es`

Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65211-0001

E-mail address: `nigel@math.missouri.edu`