

## SEMIGROUPS OF MAPPINGS WITH RIGID LIPSCHITZ CONSTANT

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ABSTRACT. It is shown that two well-known uniformly fixed point free Lipschitzian semigroups of mappings have minimal Lipschitz constant on the positive part of the unit ball of  $\ell_2$ . This implies that a question raised by T. Kuczumow has a negative answer.

### 1. INTRODUCTION

Let  $(X, \|\cdot\|)$  be a Banach space, and let  $C$  be a bounded closed convex subset of  $X$ . A map  $T : C \rightarrow X$  is *k-lipschitzian* on  $C$  whenever  $\|T(x) - T(y)\| \leq k\|x - y\|$  for all  $x, y \in C$ . When  $k = 1$ ,  $T$  is also called nonexpansive.

A mapping  $T : C \rightarrow C$  is *uniformly k-lipschitzian* whenever each member of the semigroup of mappings  $\{T^n : n \in \mathbb{N}\}$  admits the Lipschitz constant  $k$ .

Since at least 1973 (for instance in [G-K-T, 1]) one can find in the literature fixed point results for uniformly  $k$ -lipschitzian semigroups of mappings.

In particular, a fixed point free  $\frac{\pi}{2}$ -uniformly Lipschitzian selfmapping of the positive part  $B_2^+$  of the closed unit ball  $B_2$  of the classical real Hilbert space  $\ell_2$  is well known. This example is often known as the Baillon mapping (see [T] and [G-K]).

In a recent paper ([Kc]), T. Kuczumov implicitly asks whether (or not) there exists a renorming  $|\cdot|$  of  $\ell_2$  that makes the Baillon mapping  $|\cdot|$ -nonexpansive on  $B_2^+$ . If this were the case, as  $B_2^+$  is a closed bounded convex subset of the superreflexive space  $(\ell_2, |\cdot|)$ , a longstanding open question in Metric Fixed Point Theory would have a negative answer. (See [G-K] for more information.)

The question raised on  $B_2^+$  by Kuczumow is deep and natural. First, because a  $|\cdot|$ -nonexpansive mapping  $T : B_2^+ \rightarrow B_2^+$  is always uniformly Lipschitzian with respect to the Euclidean norm. Second, because the set  $B_2^+$  under consideration could be at least as relevant as the norm, in order to obtain reductions of the Lipschitz constant of a mapping  $T : B_2^+ \rightarrow B_2^+$ . To illustrate this, we can regard the following celebrated example due to T.C. Lim ([Li]).

He defined the mapping  $\mathcal{T}$  in the classical space  $\ell_1$  by

$$\mathcal{T}((x_n)) := \left( 1 - \sum_{n=1}^{\infty} x_n, x_1, x_2, \dots \right).$$

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It is easy to check that  $\mathcal{T}$  has the Lipschitz constant 2 on the closed unit ball  $B_1$  of  $\ell_1$  with respect to the standard norm of  $\ell_1$ . Moreover, this value 2 is the minimum for the Lipschitz constant of  $\mathcal{T}$  on  $B_1$  under equivalent renormings of  $\ell_1$ . This follows from a straightforward application of Theorem 4 in [L].

Nevertheless, Lim showed in [Li] that on  $B_1^+ := \{x \in \ell_1^+ : \sum_{n=1}^\infty x_n \leq 1\}$  (the restriction of)  $\mathcal{T}$  is an isometry (and hence nonexpansive) with respect to the equivalent norm  $\|x\| := \max\{\|x^+\|_1, \|x^-\|_1\}$ . (Here  $x^+$ ,  $x^-$  are respectively the positive and the negative part of  $x \in \ell_1$ .)

Furthermore, P.N. Dowling and C. Lennard ([D-L]) recently defined another equivalent norm on  $\ell_1$ , namely  $|x| := |\sum_{n=1}^\infty x_n| + \sum_{n=1}^\infty |x_n|$ , in such a way that  $\mathcal{T}$  is also an  $|\cdot|$ -isometry, again on  $B_1^+$ .

These facts show that dramatic reductions of the Lipschitz constant of a mapping  $T$  are possible by renormings of the underlying space, mainly when  $T$  is restricted to a suitable  $T$ -invariant subset of its domain.

In this note we will find lower bounds for the Lipschitz constant  $k$  of two uniformly  $k$ -lipschitzian selfmappings of  $B_2^+$  after equivalent renormings of  $\ell_2$ . One of them gives a negative answer to the above-mentioned question raised by Kuczumow.

## 2. A RIGID LIPSCHITZ CONSTANT

Our main result deals with the Baillon mapping  $T : B_2 \rightarrow B_2$  given by

$$T(x) = \begin{cases} \cos\left(\frac{\pi}{2}\|x\|_2\right) e_1 + \sin\left(\frac{\pi}{2}\|x\|_2\right) \frac{1}{\|x\|_2} S(x), & x \neq 0, \\ e_1, & x = 0, \end{cases}$$

where  $\|x\|_2 = (\sum_{n=1}^\infty |x_n|^2)^{\frac{1}{2}}$ ,  $e_1 := (1, 0, \dots) \in \ell_2$  and  $S$  is the right shift operator defined by  $S(x_1, x_2, \dots) := (0, x_1, x_2, \dots)$ . It is well known that  $T$  is a  $\|\cdot\|_2$ -noncontractive uniformly  $\frac{\pi}{2}$ -lipschitzian on  $B$ . (See [B], [T].)

**Theorem 2.1.** *Let  $T : B_2^+ \rightarrow B_2^+$  be the Baillon mapping. Let  $\|\cdot\|$  be an equivalent norm on  $\ell_2$ . Then for each positive  $k < \frac{\pi}{2}$  the mapping  $T$  is not uniformly  $\|\cdot\|$ - $k$ -lipschitzian on  $B^+$ .*

*Proof.* To obtain a contradiction suppose that there exists an equivalent renorming  $\|\cdot\|$  of  $\ell_2$  for which  $T$  is  $k$  uniformly lipschitzian.

Let  $\mu := \limsup \|e_n\| > 0$ , where  $(e_n)$  is the standard Schauder basis of  $\ell_2$ . Without loss of generality we may suppose that  $\mu = 1$ . Otherwise, we can consider the norm  $\frac{1}{\mu}\|\cdot\|$ .

Choose any  $\varepsilon > 0$  so that  $\frac{2}{\pi}(1 + \varepsilon)k < 1$ . There exist  $n_0 \in \mathbb{N}$  such that  $\|e_{n_0}\| < 1 + \varepsilon$ .

For all positive integers  $n$ ,  $T^n(e_{n_0}) = e_{n+n_0}$ . Moreover, for each  $\lambda \in (0, 1)$ ,

$$T(\lambda e_{n_0}) = \cos\left(\frac{\pi}{2}\lambda\right) e_1 + \sin\left(\frac{\pi}{2}\lambda\right) \frac{1}{\lambda} S(\lambda e_{n_0}) = \cos\left(\frac{\pi}{2}\lambda\right) e_1 + \sin\left(\frac{\pi}{2}\lambda\right) e_{n_0+1}$$

and

$$T^2(\lambda e_{n_0}) = S(T(\lambda e_{n_0})) = \cos\left(\frac{\pi}{2}\lambda\right) e_2 + \sin\left(\frac{\pi}{2}\lambda\right) e_{n_0+2}.$$

A simple induction shows that for any positive integer  $n$ ,

$$T^n(\lambda e_{n_0}) = \cos\left(\frac{\pi}{2}\lambda\right) e_n + \sin\left(\frac{\pi}{2}\lambda\right) e_{n_0+n}.$$

Since  $T$  is  $k$ - $\|\cdot\|$ -uniformly lipschitzian,

$$\begin{aligned} \|\cos\left(\frac{\pi}{2}\lambda\right)e_n - (1 - \sin\left(\frac{\pi}{2}\lambda\right))e_{n+n_0}\| &= \|T^n(\lambda e_{n_0}) - T^n(e_{n_0})\| \\ &\leq k\|\lambda e_{n_0} - e_{n_0}\| \\ &= k(1 - \lambda)\|e_{n_0}\| \\ &< k(1 - \lambda)(1 + \varepsilon). \end{aligned}$$

Thus,

$$\frac{\cos\left(\frac{\pi}{2}\lambda\right)}{1 - \lambda}\|e_n\| - \frac{1 - \sin\left(\frac{\pi}{2}\lambda\right)}{1 - \lambda}\|e_{n+n_0}\| \leq k(1 + \varepsilon),$$

and letting  $\lambda \rightarrow 1^-$  we have

$$\frac{\pi}{2}\|e_n\| \leq k(1 + \varepsilon).$$

From this, we finally obtain

$$1 = \limsup_n \|e_n\| \leq \frac{2}{\pi}k(1 + \varepsilon) < 1,$$

a contradiction. □

Nonexpansive mappings are uniformly 1-lipschitzian. Combining this fact with the above Theorem, we can answer the Kuczumow question in the negative.

**Corollary 2.1.** *The mapping  $T$  cannot be nonexpansive on  $B_2^+$  with respect to any equivalent renorming of  $\ell_2$ .*

### 3. THE GOEBEL-KIRK-THELE MAPPING

It is quite natural to extend Kuczumow’s question to other well-known fixed point free uniformly lipschitzian selfmappings of  $B_2^+$ .

Perhaps the first example of a Lipschitzian fixed point free mapping  $\varphi : B_2 \rightarrow B_2$  was given by Kakutani in 1943 (see [K]) as follows:

$$\varphi(x) := \frac{1}{2}(1 - \|x\|_2)e_1 + S(x).$$

Kakutani’s construction can be generalized to the family of mappings

$$K_\varepsilon(x) := \varepsilon(1 - \|x\|_2)e_1 + S(x),$$

for  $0 < \varepsilon \leq 1$ . In fact,  $K_\varepsilon$  maps  $B_2^+$  into itself and it has the  $\|\cdot\|_2$ -Lipschitz constant  $\sqrt{1 + \varepsilon^2}$ .

A further development of Kakutani’s example is due to K. Goebel, W.A. Kirk and R.L. Thele (see [G-K-T, 2]). It is the mapping  $G : B_2^+ \rightarrow B_2^+$  given by

$$G(x) := \frac{1}{\|K_1(x)\|_2}K_1(x).$$

One can easily check that  $G$  is *uniformly-2-lipschitzian* on  $B$ . The following result shows that this uniform Lipschitz constant 2 could be smaller after a renorming, but not too much smaller.

**Theorem 3.1.** *Let  $G : B_2^+ \rightarrow B_2^+$  be the Goebel-Kirk-Thele mapping. Let  $\|\cdot\|$  be an equivalent renorming on  $\ell_2$ . If  $k$  is the uniform Lipschitz constant of the semigroup of iterates  $\{G^n : n \in \mathbb{N}\}$  with respect to the norm  $\|\cdot\|$ , then  $k \geq \sqrt{2}$ .*

*Proof.* Let  $(e_n)$  be the standard Schauder basis of  $\ell_2$ . Let  $\mu := \limsup \|e_n\| > 0$ . Fix a positive integer  $m$  and  $\varepsilon \in (0, \frac{1}{2})$ . A simple induction shows that, for any positive integer  $n$ ,

$$G^n \left( \left( \frac{1}{2} + \varepsilon \right) e_m \right) = \frac{1}{\sqrt{\left( \frac{1}{2} - \varepsilon \right)^2 + \left( \frac{1}{2} + \varepsilon \right)^2}} \left[ \left( \frac{1}{2} - \varepsilon \right) e_n + \left( \frac{1}{2} + \varepsilon \right) e_{m+n} \right],$$

$$G^n \left( \left( \frac{1}{2} - \varepsilon \right) e_m \right) = \frac{1}{\sqrt{\left( \frac{1}{2} - \varepsilon \right)^2 + \left( \frac{1}{2} + \varepsilon \right)^2}} \left[ \left( \frac{1}{2} + \varepsilon \right) e_n + \left( \frac{1}{2} - \varepsilon \right) e_{m+n} \right].$$

Since  $G$  is uniformly  $k$ -lipschitzian with respect to the norm  $\|\cdot\|$  on  $B^+$ , for any positive integer  $n$ ,

$$\frac{2\varepsilon \|e_{m+n} - e_n\|}{\sqrt{\left( \frac{1}{2} - \varepsilon \right)^2 + \left( \frac{1}{2} + \varepsilon \right)^2}} = \left\| G^n \left( \left( \frac{1}{2} + \varepsilon \right) e_m \right) - G^n \left( \left( \frac{1}{2} - \varepsilon \right) e_m \right) \right\|,$$

$$\leq 2\varepsilon k \|e_m\|,$$

that is,

$$\frac{1}{\sqrt{\left( \frac{1}{2} - \varepsilon \right)^2 + \left( \frac{1}{2} + \varepsilon \right)^2}} \|e_{m+n} - e_n\| \leq k \|e_m\|.$$

Letting  $\varepsilon \rightarrow 0^+$  we obtain, for any positive integers  $m$  and  $n$ ,

$$\sqrt{2} \|e_{m+n} - e_n\| \leq k \|e_m\|.$$

From this, as the sequence  $(e_{m+n} - e_n)_{m \geq 1}$  converges weakly to  $-e_n$ , we have

$$\sqrt{2} \| -e_n \| \leq \liminf_m \sqrt{2} \|e_{m+n} - e_n\| \leq k \limsup_m \|e_m\| = k\mu.$$

As this inequality holds for any positive integer  $n$ , we finally derive that

$$\sqrt{2}\mu = \sqrt{2} \limsup_n \| -e_n \| \leq k\mu,$$

which yields the conclusion. □

**Corollary 3.1.** *The mapping  $G$  cannot be nonexpansive on  $B_2^+$  with respect to any equivalent renorming of  $\ell_2$ .*

#### 4. FURTHER REMARKS

**4.1. About the Kakutani mapping.** We include here a result about the Kakutani mapping which may be well known, but whose proof is not easily found in the literature. We can see that the iterates of this mapping do not admit a uniform Lipschitz constant on  $B_2^+$ .

**Theorem 4.1.** *The mapping  $K_1 : B_2^+ \rightarrow B_2^+$  given by*

$$K_1(x) = (1 - \|x\|_2)e_1 + S(x)$$

*is not uniformly lipschitzian on  $B_2^+$ .*

*Proof.* For any  $x \in B_2^+$ ,  $\|K_1(x)\|_2 \geq \|x\|_2$ , that is, the sequence  $(\|K_1^n(x)\|_2)_{n=1}^\infty$  is nondecreasing. Hence, there exists  $\ell := \lim_{n \rightarrow \infty} \|K_1^n(x)\|_2 \leq 1$ .

Since

$$\|K_1^{n+1}(x)\|_2^2 = (1 - \|K_1^n(x)\|_2)^2 + \|S(K_1^n(x))\|_2^2 = (1 - \|K_1^n(x)\|_2)^2 + \|K_1^n(x)\|_2^2,$$

letting  $n \rightarrow \infty$ , we obtain  $\ell^2 = (1 - \ell)^2 + \ell^2 \Rightarrow \ell = 1$ .

Bearing this in mind, to obtain a contradiction, we suppose that  $K_1$  is uniformly  $k$ -lipschitzian on  $B_2^+$ .

Then for any positive integer  $n$  and  $\lambda \in (0, 1)$ ,

$$\|K_1^n(e_1) - K_1^n(\lambda e_1)\|_2 \leq k(1 - \lambda),$$

that is,

$$\sqrt{\|K_1^n(e_1)\|_2^2 + \|K_1^n(\lambda e_1)\|_2^2 - 2\langle K_1^n(e_1), K_1^n(\lambda e_1) \rangle} \leq k(1 - \lambda),$$

where  $\langle \cdot, \cdot \rangle$  stands for the inner product in  $\ell_2$ . As  $K_1^n(e_1) = e_{n+1}$ , it is straightforward to check that

$$\langle K_1^n(e_1), K_1^n(\lambda e_1) \rangle = \lambda.$$

Thus,

$$\sqrt{\|K_1^n(e_1)\|_2^2 + \|K_1^n(\lambda e_1)\|_2^2 - 2\lambda} \leq k(1 - \lambda).$$

Letting  $n \rightarrow \infty$ , we obtain

$$\sqrt{2 - 2\lambda} \leq k(1 - \lambda),$$

which yields the inequality

$$\frac{\sqrt{2}}{\sqrt{1 - \lambda}} \leq k$$

for all  $\lambda \in (0, 1)$ . Taking limits when  $\lambda$  tends to  $1^-$  we obtain the desired contradiction.  $\square$

**Corollary 4.1.** *The mapping  $K_1$  cannot be nonexpansive on  $B_2^+$  with respect to any equivalent norm of  $\ell_2$ .*

4.2. In [G-J-L] the authors showed that it is impossible to find in  $\ell_2$  an equivalent renorming for which some of the three above mappings  $T, G, K_\varepsilon$  could attain the Lipschitz constant 1 on  $B_2$ . Corollaries 1–3 contain the same result on  $B_2^+$ . One can note that the proofs of [G-J-L] cannot be adapted to  $B_2^+$  because this set does not have interior points.

The same happens with the results about Lipschitzian mappings in the recent paper [Ll].

### 4.3. Questions.

4.3.1. Is it possible to find a norm on  $\ell_2$  for which the (exact) uniform Lipschitz constant of  $G$  on  $B_2^+$  is  $\sqrt{2}$ ? In other words, is the bound  $\sqrt{2}$  in Theorem 2 sharp?

4.3.2. If  $U$  is any of the mappings  $K_\varepsilon, G, T$ , is  $B^+$  a minimal bounded closed convex  $U$ -invariant set? If the answer is no, it seems interesting to look for one.

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