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FROM $K(n+1)_*(X)$ TO $K(n)_*(X)$

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Abstract. Let X be a space of finite type. Set $q = 2(p-1)$ as usual, and define the mod q support of $K(n)^*(X)$ by $S(X, K(n)) = \{m \in \mathbb{Z}/q\mathbb{Z} \mid \bigoplus_{d \equiv m \bmod q} K(n)^d \neq 0\}$ for $n > 0$. Call $K(n)^*(X)$ *sparse* if there is no $m \in \mathbb{Z}/q\mathbb{Z}$ with $m, m+1 \in S(X, K(n))$.

Then we show the relation $S(X, K(n)) \subseteqq S(X, K(n+1))$ for any finite type space X with $K(n+1)^*(X)$ being sparse.

As a special case, we have $K(n+1)^{odd}(X) = 0 \Longrightarrow K(n)^{odd}(X) = 0$, and the main theorem of Ravenel, Wilson and Yagita is also generalized in terms of the mod q support.

1. Introduction

Recently, Ravenel, Wilson and Yagita [9], Bousfield [3] and Wilson [11] proved the following theorems:

Theorem 1.1 ([9, Theorem 1.2]). *If X is a space of finite type with $K(n)^*(X)$ concentrated in even dimensions for an infinite number of n, then $K(n)^*(X)$ is concentrated in even dimensions for all $n > 0$.*

Theorem 1.2 ([3], [10]). *If X is a space of finite type with $K(n+1)^*(X) = 0$ for some $n > 0$, then we also have $K(n)^*(X) = 0$.*

Whereas Bousfield's result [3] is more general than Wilson's [10], since Bousfield does not need the "finite type" assumption, we have been more influenced by [10] (cf. Remark 3.1 (ii)). Actually, this paper, which may be regarded as a pushout of these two theorems, heavily depends on the techniques of [2] and [1], just like [10] and [9].

To state our main theorem, set $q = 2(p-1)$ as usual, and define the mod q support of a generalized cohomology $E^*(X)$ by

$$S(X, E) = \{m \in \mathbb{Z}/q\mathbb{Z} \mid \bigoplus_{d \equiv m \bmod q} E^d(X) \neq 0\}.$$

Then we call $E^*(X)$ *sparse* if there is no $m \in \mathbb{Z}/q\mathbb{Z}$ with $m, m+1 \in S(X, E)$. Now our main theorem is the following.

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Main Theorem. *Let X be a space of finite type with $K(n+1)^*(X)$ being sparse for some $n > 0$. Then we have an inclusive relation of mod q supports:*

$$S(X, K(n)) \subsetneqq S(X, K(n+1)).$$

As suggested earlier, Theorem 1.1 and Theorem 1.2 are immediate corollaries. Above all, we single out the following sharpening of Theorem 1.1:

Corollary 1.3. $K(n+1)^{odd}(X) = 0 \Longrightarrow K(n)^{odd}(X) = 0$.

Example 1.4. Kriz [6] showed $K(2)^{odd}(P) \neq 0$ for P the p-Sylow subgroup of $GL_4(\mathbb{F}_p)$ at $p = 3$. Now Corollary 1.3 immediately generalizes this to $K(n)^{odd}(P) \neq 0$ for arbitrary $n \geq 2$. We note that Kriz and Lee [7] recently proved this claim at any odd prime p by some sophisticated direct calculation.

Now the proof of the Main Theorem proceeds as follows:

Step 1: When $K(n+1)^*(X)$ is sparse, show

1. $E(n, n+1)^*(X)$ has neither infinitely v_n-divisible elements nor v_n-torsion elements;
2. $E(n, n+1)^*(X)$ is also sparse; in fact,

$$S(X, E(n, n+1)) = S(X, K(n+1)).$$

Step 2: When $E(n, n+1)^*(X)$ has no infinitely v_n-divisible elements for $n > 0$, show

$$S(X, K(n)) \subsetneqq S(X, E(n, n+1)).$$

Here we recall $E(k, n+1)$ is the spectrum with $E(n+1, n+1) = K(n+1)$ and

$$E(k, n+1)_* = \begin{cases} \mathbb{Z}/p[v_k, \cdots, v_n, v_{n+1}, v_{n+1}^{-1}], & 0 < k \leq n; \\ \left(\mathbb{Z}_{(p)}[v_1, \cdots, v_n, v_{n+1}, v_{n+1}^{-1}]\right)_p^{\wedge}, & k = 0. \end{cases}$$

Then Step 1 is taken care of in §2, and Step 2 in §3. In §2, we prove a slightly stronger version of Step 1, which immediately implies:

Corollary 1.5. *Suppose $K(n+1)^*(X)$ is sparse. Then, for any $0 \leq k \leq n$, $E(k, n+1)^*(X)$ is also sparse, and*

$$S(X, E(k, n+1)) = S(X, K(n+1)).$$

We now turn our attention to $P(k)$. Like $E(0, n+1)$, let $P(0)$ stand for the p-adically completed spectrum of the usual $P(0) = BP$. Then the main theorem of Ravenel, Wilson and Yagita [9, Theorem 1.8] is generalized as follows:

Corollary 1.6. *Suppose $K(n)^*(X)$ is sparse for any $n \in I$, where $I \subsetneqq \mathbb{N}$ is an infinite subset. Then, for any $k \geq 0$, $P(k)^*(X)$ is also sparse,*

$$S(X, P(k)) \subsetneqq S_I(X, K) := \bigcup_{n \in I} S(X, K(n)),$$

and is Landweber flat.

Proof. The first two claims about the sparseness and the mod q support follow from Corollary 1.5 and the fact that any notrivial element in $P(k)^*(X)$ maps nontrivially to $E(k, n+1)$ for some n [9, Proposition 4.12].

Now the second claim implies short exact sequences (where $v_0 = p$)

$$0 \to P(l)^*(X) \xrightarrow{v_l} P(l)^*(X) \to P(l+1)^*(X) \to 0,$$

which implies the Landweber flatness by [11] and [12] (see [9, Remark 1.10]). ∎

While our techniques are not new, our results might be rather surprising. In fact, our results appear to be new even for finite complexes.

Our results were obtained while the author was visiting the Japan-United States Mathematics Institute (JAMI) at the Johns Hopkins University during the winter of 2000. The author would like to express his gratitude to JAMI for its generous hospitality and for exposing him to the beautiful mathematics of Boardman, Johnson and Wilson [1], [2], [10]. Thanks also goes to Mike Boardman, Jean-Pierre Meyer, Jack Morava and Steve Wilson for their warm hospitality and fruitful discussions. Special thanks goes to Steve Wilson, whose influence on this paper should be obvious to any reader.

2. Step 1 and the proof of Corollary 1.5

Step 1 is simply the case $l = n$ of the following:

Lemma 2.1. *Suppose $E(l+1,n+1)^*(X)$ is sparse for some $0 \le l \le n$. Then the following hold:*

1. *$E(l,n+1)^*(X)$ has neither infinitely v_l-divisible elements nor v_l-torsion elements;*
2. *$E(l,n+1)^*(X)$ is also sparse; in fact,*

$$S(X, E(l,n+1)) = S(X, E(l+1,n+1)).$$

Proof. This can be done as in [9, Lemma 5.1].

We first classify elements of $E(l,n+1)^*(X)$ into the following three types:

(i) an element which is infinitely v_l-divisible;
(ii) an element which is not infinitely v_l-divisible, but v_l torsion;
(iii) an element which is neither infinitely v_l-divisible nor v_l torsion.

First, there is no element of type (i) from [9, Corollary 4.11]. Since [9, Corollary 4.11] is our main technical tool, we record here that $P(l)^*(X)$ was used to study $E(l,n+1)^*(X)$, together with the following ingredients:

1. $E(l,n+1)$ is an exact functor spectrum with respect to $P(l)$ [11], [12];
2. the Mittag-Leffler property of $P(l)^*(X)$, and consequently the Mittag-Leffler property of $E(l,n+1)^*(X)$ [9] (it is this property, which requires the deep techniques of [2], [1]);
3. the Landweber filtration theorem for finitely presented $P(l)^*(P(l))$ modules [11], [12].

Second, we claim there is no element of type (ii). In fact, if there were such an element, then $E(l+1,n+1)^*(X)$ would not be sparse, as is easily seen from the following v_l-Bockstein triangle, where we have made everything q-graded with

$$\deg v_l = 0,\ \deg \rho = 0,\ \deg \delta = 1, \quad \overline{E}(l,n+1)^*(X) = \bigoplus_{z\in\mathbb{Z}} E(l,n+1)^{*+qz}(X),$$

$$\overline{E}(l+1,n+1)^*(X) = \bigoplus_{z\in\mathbb{Z}} E(l+1,n+1)^{*+qz}(X):$$

$$\begin{tikzcd}
\overline{E}(l,n+1)^*(X) \arrow[rr, "v_l"] & & \overline{E}(l,n+1)^*(X) \arrow[dl, "\rho"] \\
& \overline{E}(l+1,n+1)^*(X) \arrow[ul, dotted, "\delta"] &
\end{tikzcd}$$

From these considerations, we see that $E(l,n+1)^*(X)$ consists only of elements of type (iii), which implies the following:

1. The v_l-Bockstein triangle is reduced to a short exact sequence
$$0 \to E(l,n+1)^*(X) \xrightarrow{v_l} E(l,n+1)^*(X) \xrightarrow{\rho} E(l+1,n+1)^*(X) \to 0.$$
2. For any nontrivial element $x \in E(l,n{+}1)^*(X)$, there are some natural numbers d and $y \in E(l,n+1)^*(X)$ such that $v_n^d y = x$, $\rho(y) \neq 0 \in E(l+1,n+1)^*(X)$.

Now the claim follows immediately. □

Proof of Corollary 1.5. This immediately follows from Lemma 2.1 by downward induction on k. □

3. Step 2

We use some results of [4]. Given a self-map $f : \Sigma^m X \to X$, define the microscope $\mathrm{Mic}(f)$ as the homotopy inverse limit
$$\mathrm{Mic}(f) := \underleftarrow{\mathrm{holim}}(\cdots \xrightarrow{f} \Sigma^m X \xrightarrow{f} X).$$
When we apply this construction to $\times v_n : \Sigma^{2(p^n-1)} E(n,n+1) \to E(n,n+1)$, we find the following properties of $\mathrm{Mic}(v_n)$:

1. $\mathrm{Mic}(v_n)$ is a wedge of suspensions of $K(n)$. (See the text just before Corollary 2.4 in [4].)
2. In the homology Milnor sequence
$$0 \to \varprojlim{}^1 E(n,n+1)_{*+1} \to \mathrm{Mic}(v_n)_* \to \varprojlim E(n,n+1)_* \to 0,$$
we have $\varprojlim{}^1 E(n,n+1)_{*+1} \neq 0$ and $\varprojlim E(n,n+1)_* = 0$.

From these, we see that
$$\mathrm{Mic}(v_n) = \bigvee_{\lambda\in\Lambda} \Sigma^{d_\lambda} K(n),$$
$$d_\lambda \equiv -1 \bmod q, \quad \lambda \in \Lambda \neq \varnothing,$$
and thus $\mathrm{Mic}(v_n)$ has a direct summand $\Sigma^{ql-1}K(n)$ for some integer l. In particular, we see that for any space X,
$$S(X, K(n)) \subseteqq S(X, \Sigma\,\mathrm{Mic}(v_n)).$$
At this stage, we assume $E(n,n+1)^*(X)$ has no infinitely v_n-divisible elements. This implies $\varprojlim \overline{E}(n,n+1)^*(X) = 0$. Thus
$$S(X, \Sigma\,\mathrm{Mic}(v_n)) \subseteqq S(X, E(n,n+1))$$
by considering the q-graded cohomology Milnor sequence
$$0 \to \varprojlim{}^1 \overline{E}(n,n+1)^{*-1}(X) \to \overline{\mathrm{Mic}}(v_n)^*(X) \to \varprojlim \overline{E}(n,n+1)^*(X) \to 0.$$
Now the claim of Step 2 follows immediately, and so the proof of Main Theorem is now complete. □

Remark 3.1. (i) Hovey's paper [4] may be regarded as a spin-off of his effort to understand Hopkins' chromatic splitting conjecture [5], [8], and the validity of its weak form Hopkins' zeta conjecture would guarantee an existence of the dotted arrow in the following commutative diagram:

$$\begin{tikzcd} & L_{K(n+1)}S \arrow[dd, dotted] \\ S \arrow[ur, "l_{n+1}"] \arrow[dr, "l_n"'] & \\ & L_{K(n)}S, \end{tikzcd}$$

where S is the p-completed sphere spectrum, and l_{n+1} and l_n are Bousfield localization maps with respect to $K(n+1)$ and $K(n)$, respectively. In this way, we may think that [4] grew out of an effort to relate $K(n+1)$ and $K(n)$, and so it is not surprising to see its appearance in our proof.

(ii) As pointed out to us by Steve Wilson, Step 2 can be also proven along the same line as the proof of [10]. In this approach, $P(n)^*(X)$ is used to go from $E(n, n+1)^*(X)$ to $K(n)^*(X)$; the main ingredients are

1. $E(n, n+1)$ and $K(n)$ are exact functor spectra with respect to $P(n)$ [11], [12];
2. the Mittag-Leffler property of $P(n)^*(X)$, and consequently the Mittag-Leffler property of $E(n, n+1)^*(X)$ and $K(n)^*(X)$ [9];
3. the Landweber filtration theorem for finitely presented $P(n)^*(P(n))$ modules [11], [12].

In this way, this approach uses essentially the same ingredients as that of Corollary 4.11 [9], which was our main technical tool in Step 1. Since it is straightforward to translate the proof of [10] into our situation, we leave the details to an interested reader.

A small advantage of our approach is that we do not need the finite type assumption on X in Step 2, though we still cannot avoid it in Step 1.

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