PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 5, Pages 1377–1381
S 0002-9939(01)06389-4
Article electronically published on December 20, 2001

CRITICAL POINTS OF THE AREA FUNCTIONAL OF A COMPLEX CLOSED CURVE ON THE MANIFOLD OF KÄHLER METRICS

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(Communicated by Mohan Ramachandran)

ABSTRACT. We consider a compact complex manifold M of dimension n that admits Kähler metrics and we assume that $C \hookrightarrow M$ is a closed complex curve. We denote by $\mathcal{KC}(1)$ the space of classes of Kähler forms $[\omega] \in H^{1,1}(M, \mathbb{R})$ that define Kähler metrics of volume 1 on M and define $\mathbf{A}_C : \mathcal{KC}(1) \to \mathbb{R}$ by $\mathbf{A}_C([\omega]) = \int_C \omega =$ area of C in the induced metric by ω. We show how the Riemann-Hodge bilinear relations imply that any critical point of $\mathbf{A}_C$ is the strict global minimum and we give conditions under which there is such a critical point $[\omega]$: A positive multiple of $[\omega]^{n-1} \in H^{2n-2}(M, \mathbb{R})$ is the Poincaré dual of the homology class of C. Applying this to the Abel-Jacobi map of a curve into its Jacobian, $C \hookrightarrow J(C)$, we obtain that the Theta metric minimizes the area of C within all Kähler metrics of volume 1 on $J(C)$.

1. INTRODUCTION

Let M be a compact complex manifold of dimension n that admit Kähler metrics. The Kähler cone

$$\mathcal{KC} := \{[\omega] \in H^{1,1}(M, \mathbb{R}) | \omega \text{ is a Kähler form}\}$$

is an open convex cone in the vector space $H^{1,1}(M, \mathbb{R})$. On $\mathcal{KC}$ we define the function

$$\mathbf{V} : \mathcal{KC} \to \mathbb{R}^+, \quad \mathbf{V}([\omega]) = \frac{1}{n!} \int_M \omega^n = \text{Vol } (M, \omega)$$

where ω is a Kähler form representing the class $[\omega]$ and Vol (M, ω) denotes the volume of M in the metric induced by ω. Suppose that $C \hookrightarrow M$ is a closed complex curve in M. For $[\omega] \in \mathcal{KC}$, the area of C in the induced metric of ω is given by

$$\widetilde{\mathbf{A}}_C(\omega) := \int_C \omega = \text{area of } C.$$

Received by the editors November 2, 2000.
2000 *Mathematics Subject Classification.* Primary 32Q15.
Key words and phrases. Kähler form, Kähler manifold, Riemann-Hodge bilinear relations, Jacobian of a curve.

The function $\widetilde{\mathbf{A}}_C$ defines a functional in $H^{1,1}(M,\mathbb{R})$. In this paper we study the critical points of

$$\mathbf{A}_C := \widetilde{\mathbf{A}}_C|_{\mathcal{KC}(1)} : \mathcal{KC}(1) \to \mathbb{R},$$

where $\mathcal{KC}(1) := \mathbf{V}^{-1}(1)$ is the hypersurface of cohomology classes of Kähler forms that define Kähler metrics of volume 1 on M. We use the Riemann-Hodge bilinear relations to show that the set $\mathcal{D}_1 := \{[\omega] \in \mathcal{KC} | \mathbf{V}([\omega]) \geq 1\}$ is a strictly convex set where $\mathcal{KC}(1)$ is the boundary of $\mathcal{D}_1$. Let PD $[C]$ be the Poincaré dual of the homology class of C in $H^{2n-2}(M,\mathbb{R})$.

Theorem. *Let M be a compact complex manifold of dimension n that admit Kähler metrics and $C \hookrightarrow M$ be a complex closed curve in M. Then:*
(i) *any critical point of $\mathbf{A}_C$ is a strict global minimum, and*
(ii) *this critical point exists, call it $[\omega]$, if and only if*

$$PD\ ([C]) = \alpha[\omega]^{n-1} \quad \textit{for some } \alpha > 0.$$

Let C be a compact Riemann surface of genus $g \geq 2$, $J(C)$ the Jacobian of C and $C \hookrightarrow J(C)$ the Abel-Jacobi map ([1], p. 235). In this case we have from Poincaré's formula ([1], p. 350) that PD $[C] = \frac{[\omega_0]^{g-1}}{(g-1)!}$, where ω_0 is the 2-form on $J(C)$ invariant under translations representing the first Chern class of the line bundle defined by the Riemann Theta divisor of $J(C)$. In this case ω_0 defines a canonical flat Kähler metric on $J(C)$, called the Theta metric.

Corollary. *The Theta metric minimizes the area of the Abel-Jacobi curve C within all Kähler metrics of volume 1 on the Jacobian $J(C)$.*

2. The Hessian of V and the Riemann-Hodge bilinear relations

Lemma 2.1. *Let $[\omega] \in \mathcal{KC}$. Then for $[\eta] \in T_{[\omega]}\mathcal{KC} = H^{1,1}(M,\mathbb{R})$ a tangent vector, we have*

$$d\mathbf{V}_{[\omega]}([\eta]) = \int_M *_\omega \omega \wedge \eta =: \langle \omega, \eta \rangle_\omega$$

where $\langle\, , \rangle_\omega$ denotes the inner product on $H^{1,1}(M,\mathbb{R})$ and $*_\omega$ is the star operator induced by ω.

Proof. This is a simple computation:

$$\begin{aligned} d\mathbf{V}_\omega(\eta) &= \lim_{t\to 0} \frac{1}{t}(\mathbf{V}(\omega + t\eta) - \mathbf{V}(\omega)) = \frac{1}{n!} \lim_{t\to 0} \frac{1}{t}(\int_M (\omega + t\eta)^n - \int_M \omega^n) \\ &= \frac{1}{n!} \lim_{t\to 0} \frac{1}{t}(nt \int_M (\omega^{n-1} \wedge \eta) + (\binom{n}{2} \cdot t^2 \int_M (\eta^2 \wedge \omega^{n-2})) + \cdots) \\ &= \frac{n}{n!} \int_M \omega^{n-1} \wedge \eta = \frac{1}{(n-1)!} \int_M \omega^{n-1} \wedge \eta. \end{aligned}$$

Since $*_\omega \omega = \frac{\omega^{n-1}}{(n-1)!}$, we obtain that $d\mathbf{V}_\omega(\eta) = \langle \omega, \eta \rangle_\omega$. ∎

Remark. Lemma 2.1 implies that $\operatorname{Kernel} d\mathbf{V}_\omega = H_0^{1,1}([\omega])$, where $H_0^{1,1}([\omega])$ is the space of primitive cohomology classes of ω ([1], p. 122). This lemma also implies that $\mathcal{KC}(1) := \mathbf{V}^{-1}(1)$ is a smooth hypersurface and for $[\omega] \in \mathcal{KC}(1)$, $T_{[\omega]}\mathcal{KC}(1) = H_0^{1,1}([\omega])$.

For $[\omega] \in \mathcal{KC}$, we have the bilinear form $Q_\omega : H^{1,1}(M,\mathbb{R}) \times H^{1,1}(M,\mathbb{R}) \to \mathbb{R}$ defined by

$$Q_\omega(\eta_1,\eta_2) = \int_M \eta_1 \wedge \eta_2 \wedge \omega^{n-2}.$$

Riemann-Hodge bilinear relations for $H^{1,1}(M,\mathbb{R})$. The form Q_ω is negative definite on primitive cohomology $H_0^{1,1}([\omega])$ ([1], p. 123).

Lemma 2.2. *Let $[\omega] \in \mathcal{KC}$ and $[\eta],[\zeta] \in T_{[\omega]}\mathcal{KC} = H^{1,1}(M,\mathbb{R})$. Then we have*

$$\textit{Hess}\ \mathbf{V}_{[\omega]}([\eta],[\zeta]) = \frac{1}{(n-2)!}\int_M \omega^{n-2}\wedge\eta\wedge\zeta$$

and Hess $\mathbf{V}_\omega|_{H_0^{1,1}([\omega])}$ is negative definite.

Proof. By definition, Hess $\mathbf{V}_{[\omega]} : H^{1,1}(M,\mathbb{R}) \times H^{1,1}(M,\mathbb{R}) \to \mathbb{R}$ is

$$\begin{aligned}
\mathrm{Hess}\mathbf{V}_{[\omega]}([\eta],[\zeta]) &= \lim_{t\to 0}\frac{1}{t}(d\mathbf{V}_{\omega+t\eta}(\zeta) - d\mathbf{V}_\omega(\zeta)) \\
&= \frac{1}{(n-1)!}\lim_{t\to 0}((\int_M (\omega+t\eta)^{n-1}\wedge\zeta) - (\int_M \omega^{n-1}\wedge\zeta)) \\
&= \frac{(n-1)}{(n-1)!}\int_M \omega^{n-2}\wedge\eta\wedge\zeta = \frac{1}{(n-2)!}\int_M \omega^{n-2}\wedge\eta\wedge\zeta.
\end{aligned}$$

So the lemma follows from the Riemann-Hodge bilinear relations. □

3. The set $\mathcal{D}_1$ is strictly convex

Let $\mathcal{B} \subset \mathbb{R}^n$ be a set with nonempty interior: int $\mathcal{B} \neq \emptyset$. $\mathcal{B}$ is strictly convex if for all $x,y \in \mathcal{B}$, $x \neq y$, we have that $tx + (1-t)y \in \mathrm{int}\ \mathcal{B}$ for all $t \in (0,1)$.

Lemma 3.1. *Let $\mathcal{B} \subset \mathbb{R}^n$ be a strictly convex set with smooth boundary $\partial\mathcal{B}$. Let $\widetilde{T} : \mathbb{R}^n \to \mathbb{R}$ be a linear functional. Consider the restriction $T := \widetilde{T}|_{\mathcal{B}} : \mathcal{B} \to \mathbb{R}$. If $p \in \mathcal{B}$ is a strict local minimum of T, then p is the strict global minimum of T.*

Proof. This follows since $\mathcal{B} - \{p\}$ is contained in one of the half spaces obtained as the complement of the tangent hyperplane to $p \in \partial\mathcal{B}$. □

Proposition 3.2. *Let $\mathcal{B} \subset \mathbb{R}^n$ be open and convex. Let $f : \mathcal{B} \to \mathbb{R}$ be a function of class C^2 such that it satisfies the following conditions:*

(a) *for each $x \in \mathcal{B}$, $df(x) \neq 0$,*

(b) *$(Hess\ f)_x|_{Kernel\ df(x)}$ is negative definite for all $x \in \mathcal{B}$.*

Then the set $\mathcal{B}_1 := \{x \in \mathcal{B} | f(x) \geq 1\}$ is strictly convex.

Proof. Let $x,y \in \mathcal{B}_1$ be $x \neq y$, and consider the line $r(t) = tx + (1-t)y$, $t \in [0,1]$. Let $F : [0,1] \to \mathbb{R}$ be defined by $F(t) = f(r(t))$. We have the following cases:

(i) If $F'(t) > 0$ in $(0,1]$, then for all $t \in (0,1], F(t) > F(0) \geq 1$, hence $r((0,1]) \in$ int $\mathcal{B}_1$.

(ii) If $F'(t) < 0$ in $[0,1)$, then for all $t \in [0,1), F(t) > F(1) \geq 1$, hence $r([0,1)) \in$ int $\mathcal{B}_1$.

(iii) If there exists a $t_0 \in [0,1]$ such that $F'(t_0)=0$, then $x-y \in \text{Kernel } df(r(t_0))$. By hypothesis we have

$$F''(t_0) = D^2 f(r(t_0))(x-y, x-y) < 0,$$

that is, t_0 is a local maximum. Hence, at any critical point F has a local maximum, which clearly implies that t_0 is the unique global maximum of F. This implies that for all $t \in [0, t_0)$, $F'(t) > 0$, that is, F is increasing in $[0, t_0)$; and for all $t \in (t_0, 1]$, $F'(t) < 0$, then F is decreasing in $(t_0, 1]$. This implies that $F(t) > 1$ for all $t \in (0,1)$, in other words, $r((0,1)) \in \text{ int } \mathcal{B}_1$.

Then (i), (ii), (iii) prove that $\mathcal{B}_1$ is strictly convex. $\square$

Corollary 3.3. *The set* $\mathcal{D}_1 := \{[\omega] \in \mathcal{KC} | \mathbf{V}([\omega]) \geq 1\}$ *is strictly convex.*

Proof. By Lemmas 2.1 and 2.2 we have that the function $\mathbf{V}$ satisfies the hypothesis of Proposition 3.2. $\square$

4. Minimizing $\mathbf{A}_C$ on $\mathcal{D}_1$

Lemma 4.1. *If* $[\omega] \in \mathcal{KC}(1)$ *is a critical point of* $\mathbf{A}_C$*, then* $[\omega]$ *is the global minimum of* $\widetilde{\mathbf{A}}_C$ *on* $\mathcal{D}_1$*. In particular* $[\omega]$ *is the global minimum of* $\mathbf{A}_C$ *on* $\mathcal{KC}(1)$*.*

Proof. Suppose that $[\omega] \in \mathcal{KC}(1)$ is a critical point of $\mathbf{A}_C$. This implies that there exists a $\lambda_0 \in \mathbb{R}$ such that $(\lambda_0, [\omega])$ is a critical point of the Lagrange function

$$L : \mathbb{R} \times \mathcal{KC} \to \mathbb{R}, \quad L(\lambda, \omega) = \widetilde{\mathbf{A}}_C(\omega) - \lambda \mathbf{V}(\omega).$$

It is easy to see that $\lambda_0 > 0$. We have that the Hessian of $\widetilde{\mathbf{A}}_C$ is zero since it is linear. From Lemma 2.2, Hess $\mathbf{V}_\omega|_{H_0^{1,1}([\omega])}$ is negative definitive; then

$$\text{Hess } L(\lambda_0, [\omega])|_{H_0^{1,1}([\omega])} = -\lambda_0 \text{Hess } \mathbf{V}_\omega|_{H_0^{1,1}([\omega])}$$

is positive definite. Using the second derivate test criteria we have that $[\omega]$ is a local strict minimum of $\mathbf{A}_C$ on $\mathcal{KC}(1)$. This implies that there exists an open neighbourhood U of $[\omega]$ in $\mathcal{KC}(1)$ such that $\mathbf{A}_C([\omega]) < \mathbf{A}_C([\tilde{\omega}])$ for all $[\tilde{\omega}] \in U - \{[\omega]\}$. Note that $W = \{t[\tilde{\omega}] | t \geq 1, [\tilde{\omega}] \in U\}$ is an open set in $\mathcal{D}_1$. By linearity we have that $\mathbf{A}_C([\omega]) < t\mathbf{A}_C([\tilde{\omega}])$ for all $t[\tilde{\omega}] \in W$. Then $[\omega]$ is a strict local minimum of $\widetilde{\mathbf{A}}_C$ on $\mathcal{D}_1$. Applying lemma 3.1 we have that $[\omega]$ is the global minimum of $\widetilde{\mathbf{A}}_C$ on $\mathcal{D}_1$. In particular $[\omega]$ is the global minimum of $\mathbf{A}_C$ on $\mathcal{KC}(1)$. $\square$

Proof of the Theorem. Lemma 4.1 proves part (i).

To prove part (ii) suppose that PD $[C] = \alpha[\omega]^{n-1}$, $[\omega] \in \mathcal{KC}(1)$, $\alpha > 0$. Then it is easy to see that $d(\widetilde{\mathbf{A}}_C)_{[\omega]} - \lambda_0 d\mathbf{V}_{[\omega]} = 0$, $\lambda_0 = \alpha(n-1)!$. Then by Lagrange multipliers we have that $[\omega]$ is a critical point of $\mathbf{A}_C$.

Now suppose that $[\omega] \in \mathcal{KC}(1)$ is a critical point of $\mathbf{A}_C$, that is, $d\mathbf{V}_{[\omega]} \wedge (d\widetilde{\mathbf{A}}_C)_{[\omega]} = 0$. Then for $[\eta], [\zeta] \in T_{[\omega]}\mathcal{KC}$ we have

$$d\mathbf{V}_{[\omega]}([\eta]) d(\widetilde{\mathbf{A}}_C)_{[\omega]}([\zeta]) = d\mathbf{V}_{[\omega]}([\zeta]) d(\widetilde{\mathbf{A}}_C)_{[\omega]}([\eta]). \tag{4.1}$$

In particular we can take $[\zeta] = [\omega]$ and $[\eta] \in H_0^{1,1}([\omega])$, that is, $\langle \omega, \eta \rangle_\omega = 0$. By Lemma 2.1 and (4.1) we have that $d(\widetilde{\mathbf{A}}_C)_{[\omega]}([\eta]) = 0$. By linearity of $\widetilde{\mathbf{A}}_C$, $d(\widetilde{\mathbf{A}}_C)_{[\omega]}([\eta]) = \widetilde{\mathbf{A}}_C([\eta])$, then

$$0 = \int_C \eta = \int_M \eta \wedge *_\omega(*_\omega(\text{PD } [C])) = \langle \eta, *_\omega(\text{PD [C]}) \rangle_\omega$$

for all $[\eta] \in H_0^{1,1}([\omega])$. Hence there exists $r_1 \in \mathbb{R}$ such that $*_\omega \mathrm{PD}\ ([C]) = r_1[\omega]$. From the fact that C is a complex curve there exists a positive real number r_2 such that PD $[C] = r_2 \frac{[\omega]^{n-1}}{(n-1)!}$, we take $\alpha = \frac{r_2}{(n-1)!}$. $\square$

Proof of the Corollary. By Poincaré's formula we have that

$$\mathrm{PD}[C] = \frac{[\omega_0]^{n-1}}{(n-1)!}.$$

Applying the Theorem we obtain the Corollary. $\square$

Example 1. We consider the projective line $\mathbb{CP}^1$ and fix $p \in \mathbb{CP}^1$. Let $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ and define

$$\jmath : \mathbb{CP}^1 \hookrightarrow M, \quad \jmath(x) = (x, p).$$

$C := \jmath(\mathbb{CP}^1)$. We have projections $\pi_1, \pi_2 : M \to \mathbb{CP}^1$, $\pi_1(x,y) = x$, $\pi_2(x,y) = y$, and let ω be the Fubini-Study form of $\mathbb{CP}^1$. $\omega_t := t\pi_1^*(\omega) + \frac{1}{t}\pi_2^*(\omega), t > 0$, are Kähler forms on M of volume 1. Clearly $\int\limits_C \omega_t = t$. In this case we have that infimum $\{\mathbf{A}_C(\omega_t)\} = 0$. Hence $\mathbf{A}_C$ has no minimum on $\mathcal{KC}(1)$.

However when we consider the diagonal map $\triangle : \mathbb{CP}^1 \hookrightarrow M$, $\triangle(x) = (x,x)$, and $C := \triangle(\mathbb{CP}^1)$, we have PD $[C] = \pi_1^*(\omega) + \pi_2^*(\omega)$, and by the Theorem the minimum of $\mathbf{A}_C$ is 1 and is obtained in $\frac{1}{2}[\pi_1^*(\omega) + \pi_2^*(\omega)]$. This example can be generalized to $\mathbb{CP}^1 \hookrightarrow \underbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}_{k\text{-times}}$.

Example 2. Let $L \to M$ be an ample line bundle on a complex n-dimensional manifold M and $c_1(L) = [\omega]$ be the first Chern class of L. Then there exist complex closed curves $C \hookrightarrow M$ such that PD $[C] = \alpha[\omega]^{n-1}$, $\alpha > 0$: Embed $M \hookrightarrow \mathbb{CP}^N$ with a multiple of L, then intersect M with generic hyperplanes of $\mathbb{CP}^N$ untill one obtains a curve C as desired.

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