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ON AN ADJOINT FUNCTOR TO THE THOM FUNCTOR

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ABSTRACT. We construct a right adjoint functor to the Thom functor, i.e., to the functor which assigns the Thom space $T\xi$ to a vector bundle ξ.

INTRODUCTION

Let τ denote the functor which assigns the Thom space $T\xi$ to a vector bundle ξ, and similarly for maps. The goal of this paper is to construct the right adjoint functor λ to the functor τ.

To motivate this result, I remark that it is always nice to know whether a functor admits an adjoint one. However, here we have a more interesting motivation. Namely, it is useful to know when a space is the Thom space of a certain vector bundle (spherical fibration). For example, de-Thomification plays an important role in the theory of immersion of manifolds; see [BP], [C] and the survey [L]. In fact, Brown and Peterson [BP] de-Thomify a space, while Cohen [C] de-Thomifies a map. However, these de-Thomifications are very ad hoc. So, it is reasonable to want a de-Thomification machine, like the de-looping machine of May [M] or Boardman and Vogt [BV]. The following observation of Beck [B] plays the crucial role in the de-looping theory. The suspension functor S is the left adjoint to the loop functor Ω, and so there is a monad $M := \Omega S$. Clearly, every loop space is a space over M. Conversely, if a space X is a space over M, then, using the simplicial resolution of the M-space X, one can provide a de-looping of X "at the simplicial level", and then certain additional arguments enable us to lift this "simplicial de-looping" to the geometric level; see [B], [M].

Here we have a dual situation. As usual, the functor $C := \tau\lambda$ is a comonad, and every Thom space is a space over C. Conversely, if X is a space over C then, dually to what we said above, one can take the cosimplicial resolution of X and provide a de-Thomification of X "at the cosimplicial level". However, in order to do the next step, a lifting to the geometrical level, one must prove that the Thomification commutes with the functor Tot, and this problem looks quite complicated; cf. [Bo].

Summarizing, one can consider this paper as a first step in an attack on the de-Thomification problem.

Notice that the above arguments enable us to prove that a certain space is not a Thom space: it suffices to check that it is not a space over the comonad C. For

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example, we have (implicitly) used these arguments in [R1] in order to prove that the spectra k and kO are not Thom spectra.

The case of non-orientable bundles

Let O_n be the group of orthogonal transformations of the Euclidean space $\mathbb{R}^n$, let BO_n denote its classifying space, and let γ denote the universal n-dimensional vector bundle over BO_n. Given a locally trivial bundle ξ with the fiber $\mathbb{R}^n$ and structure group O_n, let $T\xi$ denote the Thom space of ξ, i.e., $T\xi := D(\xi)/S(\xi)$, where $D(\xi)$ is the total space of the unit disc bundle and $S(\xi)$ is the total space of the unit sphere subbundle of $D(\xi)$. We regard $T\xi$ as a pointed space with the base point given by $S(\xi)$.

Let $\mathcal{K}$ be the category whose objects are maps $f : B \to BO_n$, where B is a connected space and f is a map such that

$$\pi_1(B) \xrightarrow{f_*} \pi_1(BO_n) = \mathbb{Z}/2$$

is an epimorphism, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ BO_n & = & BO_n, \end{array}$$

where f and g are objects of $\mathcal{K}$. Let $\mathcal{S}$ be the category whose objects are pointed spaces X with $\pi_i(X) = 0$ for $i < n$ and $\pi_n(X) = \mathbb{Z}/2$, and whose morphisms are maps $f : X \to Y$ such that $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism.

Let $\tau : \mathcal{K} \to \mathcal{S}$ be the Thom functor which assigns the object $\tau f := T(f^*\gamma) \in \mathcal{S}$ to the object $f : X \to BO_n$ of $\mathcal{K}$.

Theorem 1. *The functor τ admits a right adjoint functor $\lambda : \mathcal{S} \to \mathcal{K}$.*

Proof. We construct λ as follows. Choose any $X \in \mathcal{S}$. Given an integer k, let $\Omega^n_k X$ be the component of $\Omega^n X$ corresponding to $k \in \pi_n(X) = \pi_0(\Omega^n X) = \mathbb{Z}/2$. The standard O_n-action on $\mathbb{R}^n$ yields the obvious O_n-action on S^n, which, in turn, induces a (right) O_n-action on $\Omega^n X = (X, *)^{(S^n,*)}$, and it is clear that every component $\Omega^n_k X, k = 0, 1$, is O_n-invariant. Convert the right O_n-action on $\Omega^n_1 X$ into a left O_n-action by setting $ga = ag^{-1}, g \in O_n, a \in \Omega^n_1 X$. Consider the locally trivial bundle

$$p : EO_n \times_{O_n} \Omega^n_1 X \to BO_n$$

which is associated with the universal principal O_n-bundle $\Gamma := \{EO_n \to BO_n\}$; cf. [PS]. We define λX to be the map p. The λ-action on morphisms is clear.

We prove that λ is right adjoint to τ, i.e., that $\mathcal{K}(f, \lambda X) = \mathcal{S}(T(f^*\gamma), X)$ for every $f : B \to BO_n$; cf. [R2]. Indeed, consider the principal O_n-bundle

$$f^*\Gamma = \{q : E \to B\},$$

and let ξ be the $\Omega^n_1 X$-bundle associated with $f^*\Gamma$, i.e.

$$\xi = \{E \times_{O_n} \Omega^n_1 X \to B\}.$$

Then ξ is induced by f from the bundle $\lambda X = \{p : EO_n \times_{O_n} \Omega^n_1 X \to BO_n\}$. So, $\mathcal{K}(f, \lambda X) = \operatorname{Sec} \xi$, where $\operatorname{Sec} \xi$ denotes the set of all sections of ξ.

For every $b \in B$ choose any O_n-equivariant map $i_b : O_n \to E$ with $qi_b(O_n) = b$. We have (the first equality can be found e.g. in [H])

$$\begin{aligned}
\operatorname{Sec}\xi &= \{O_n\text{-equivariant maps } E \to \Omega_1^n X\} \\
&= \{O_n\text{-equivariant maps } f : E \to (X,*)^{(S^n,*)} \\
&\qquad\qquad \text{such that } f(x) \in \Omega_1^n X \text{ for every } a \in E\} \\
&= \{\text{maps } f : E \times_{O_n} (S^n,*) \to (X,*) \text{ such that the map} \\
&\qquad\qquad (S^n,*) = O_n \times_{O_n} (S^n,*) \xrightarrow{i_b} E \times_{O_n} (S^n,*) \xrightarrow{f} (X,*) \\
&\qquad\qquad \text{belongs to } \Omega_1^n X \text{ for every } b\} \\
&= \mathcal{S}(T(f^*\gamma), X).
\end{aligned}$$

□

The case of orientable bundles

Let BSO_n be the classifying space for the connected component SO_n of O_n. Let $\mathcal{K}'$ be the category whose objects are maps $f : B \to BSO_n$, where B is a connected space, and whose morphisms are commutative diagrams

$$\begin{array}{ccc}
B & \xrightarrow{\varphi} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
BSO_n & = & BSO_n,
\end{array}$$

where f and g are objects of $\mathcal{K}'$. Let $\mathcal{S}'$ be the category whose objects are pairs (X, a_X), where X is a pointed space with $\pi_i(X) = 0$ for $i < n$ and a_X is a generator (one of two) of $\pi_n(X) = \mathbb{Z}$, and whose morphisms are maps $\varphi : X \to Y$ with $\varphi_*(a_X) = a_Y$.

Let γ' be the universal oriented n-dimensional vector bundle over BSO_n. There is a unique element $a \in \pi_n(T\gamma') = \mathbb{Z}$ such that $\langle u, h(a)\rangle = 1$, where $u \in H^n(T\gamma') = \mathbb{Z}$ is the orientation of γ', $h : \pi_n(T\gamma') \to H_n(T\gamma')$ is the Hurewicz homomorphism and $\langle -, -\rangle$ is the Kronecker pairing.

Given an object $f : X \to BSO_n$ of $\mathcal{K}'$, we have the canonical map $F : T(f^*\gamma') \to T\gamma'$, and $F_* : \mathbb{Z} = \pi_n(T(f^*(\gamma))) \to \pi_n(T\gamma) = \mathbb{Z}$ is an isomorphism. Now define the Thom functor $\tau' : \mathcal{K}' \to \mathcal{S}'$ by setting $\tau' f = (T(f^*\gamma), (F_*)^{-1}(a))$.

Theorem 2. *The functor τ' admits a right adjoint functor $\lambda' : \mathcal{S}' \to \mathcal{K}'$.*

Proof. Given an object (X, a_X) of $\mathcal{K}'$, consider the isomorphism $\pi_n(X) \cong \pi_0(\Omega^n X)$, and let $\Omega_1^n X$ be the component of $\Omega^n X$ which corresponds to a_X. As in §1, we have the left SO_n-action on $\Omega^n X$, and clearly the component $\Omega_1^n X$ is invariant under the SO_n-action on $\Omega^n X$. We construct a fibre bundle $p : ESO_n \times_{SO_n} \Omega_1^n X \to BSO_n$, and we define $\lambda'(X, a_X) := p$. Now the proof can be completed similarly to that of Theorem 1. □

References

[B] J. Beck, *On H-spaces and infinite loop spaces*, Category Theory, Homotopy Theory and Their Applications III. Lecture Notes in Mathematics **99**, Springer, Berlin Heidelberg New York, 1969,, pp. 139-153. MR **40:**2079

[BV] J.M. Boardman, R.M.Vogt,, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics **347**, Springer, Berlin Heidelberg New York, 1973. MR **54:**8623a

[Bo] A.K. Bousfield, *On the homology spectral sequence of a cosimplicial space*, Amer. J.Math. **109** (1987), 361–394. MR **88j:**55017

[BP] E.H. Brown, F.P. Peterson, *A universal space for normal bundles of n-manifolds*, Comment. Math. Helv. **54** (1979), 405-430. MR **80k:**57055

[C] R. Cohen, *The immersion conjecture for differentiable manifolds.*, Ann of Math **122** (1985), 237–328. MR **86m:**57030

[H] D. Husemoller, Fibre bundles. – McGraw-Hill, New York, 1966. MR **37:**4821

[L] J. Lannes, *La conjecture des immersions (d'après R.L. Cohen, E.H.Brown, F.P.Peterson et al.)*, Bourbaki Seminar **1981/1982**, Soc. Math. France, Paris, 1982, pp. 331–346. MR **84i:**57020

[M] J.P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics **271**, Springer, Berlin Heidelberg New York, 1972. MR **54:**8623b

[PS] R. Patterson, R. Stong, *Orientability of bundles*, Duke Math. J. **39** (197), 619-622. MR **47:**4248

[R1] Yu. Rudyak, *The spectra k and kO are not Thom spectra*, Group Representations: Cohomology, Group Actions and Topology, Proc. Symp. Pure Math. **63**, Amer. Math. Soc, Providence, RI, 1998, pp. 475–483. MR **98m:**55007

[R2] Yu. Rudyak, *On Thom Spectra, Orientability and Cobordism*, Springer, Berlin, 1998. MR **99f:**55001

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