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THE FUNDAMENTAL GROUPS OF ONE-DIMENSIONAL WILD SPACES AND THE HAWAIIAN EARRING

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ABSTRACT. Let X be a one-dimensional space which contains a copy C of a circle and let it not be semi-locally simply connected at any point on C. Then the fundamental group of X cannot be embeddable into a free σ-product of n-slender groups, for instance, the fundamental group of the Hawaiian earring. Consequently, any one of the fundamental groups of the Sierpinski gasket, the Sierpinski curve, and the Menger curve is not embeddable into the fundamental group of the Hawaiian earring.

1. Introduction and main result

In recent papers [4, 5], the author showed that fundamental groups of certain wild spaces are embeddable into the fundamental group of the Hawaiian earring $\mathbb{H}$, i.e. $\mathbb{H} = \bigcup_{n=1}^{\infty} \{(x,y) : (x+1/n)^2 + y^2 = 1/n^2\}$. There the following question was left open.

Question. Are the fundamental groups of the Sierpinski gasket, Sierpinski curve and the Menger curve embeddable into the fundamental group of the Hawaiian earring?

The same question was also asked by Cannon and Conner [1, Question 3.5.1]. Since they showed that an embedding between one-dimensional spaces induces an embedding between their fundamental groups, the non-embeddability of the fundamental group of the Sierpinski gasket implies those of the fundamental groups of the others. In the present paper we prove the following theorem, which implies the negative answer to the above question.

Theorem 1.1. *Let X be a one-dimensional space which contains a copy C of a circle and let it not be semi-locally simply connected at any point on C. Then the fundamental group $\pi_1(X, x_0)$ for $x_0 \in C$ cannot be embeddable into $\divideontimes^{\sigma}_{i\in I} G_i$ for n-slender groups G_i $(i \in I)$. Consequently, $\pi_1(X, x_0)$ for $x_0 \in C$ cannot be embeddable into the fundamental group of the Hawaiian earring.*

Supporting definitions will be reviewed in Sections 2 and 3.

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2. Definitions and basic facts about free σ-products and standard homomorphisms

In this section we review infinitary words, free σ-products, n-slenderness, proper sequences and standard homomorphisms from [3, 4]. Though these notions are crucial in the proof of the main result, they are not familiar ones. We refer the reader to [3] for precise definitions and more basic facts.

Infinite words and the free σ-product of groups. For groups $(G_i : i \in I)$ with $G_i \cap G_j = \{e\}$ $(i \neq j)$, a σ-word W is a map from a countable linearly ordered set $\overline{W}$ to $\bigcup\{G_i : i \in I\}$ such that $\{\alpha \in \overline{W} : W(\alpha) \in G_i\}$ is finite for each $i \in I$. The set of σ-words is denoted by $\mathcal{W}^\sigma(G_i : i \in I)$.

For a finite subset F of I, let W_F be the restriction of W to $\overline{W_F} = \{\alpha \in \overline{W} : W(\alpha) \in \bigcup_{i\in F} G_i\}$, that is, $W_F(\alpha) = W(\alpha)$ for $\alpha \in \overline{W_F}$. Then W_F is a word of finite length and is an element of a free product $*_{i\in F} G_i$. Then W is regarded as an element of the inverse limit of free products $*_{i\in F} G_i$ for finite subsets F of I. That is, words V and W are equivalent, if $V_F = W_F$ as elements of $*_{i\in F} G_i$ for all finite subsets F of I. Now the concatenation VW corresponds to the multiplication of V and W in the inverse limit, since $(VW)_F = V_F W_F$ for each finite subset F of I.

Now a free σ-product $\times^\sigma_{i\in I} G_i$ is the subgroup of the inverse limit of $*_{i\in F} G_i$ for all finite subsets F of I consisting of all the elements expressed by σ-words.

(By admitting $\overline{W}$ to be an arbitrary linear ordering, we get a free complete product $\times_{i\in I} G_i$ [3]. When the index set I is countable, the free complete product coincides with a free σ-product. Since we only deal with free σ-products in this paper, we use the notation $\times^\sigma_{i\in I} G_i$ even in case I is countable.)

We simply say a word for a σ-word from now on. We use W as a word and also an element of a group $\times^\sigma_{i\in I} G_i$. When we need to make this distinction, we express an element of a group as $[W]$. The notation $V = W$ means the equality as group elements and $V \cong W$ means the equality as words.

The combinatorial properties of infinite words. A word V is a *subword* of a word W, if $W \cong XVY$ for some X and Y. A word W is *reduced*, if $V \neq e$ for any non-empty subword V of W and a word W is *quasi-reduced*, if the reduced word of W is obtained by multiplying neighboring elements belonging to the same groups G_i ([3, Definition 1.3]).

As in the case of words of finite length, there exists a unique reduced word of each word [3, Theorem 1.4].

A word W is *cyclically reduced*, if W is a single letter, i.e. an element of some G_i, or WW is reduced.

For a word $W \in \mathcal{W}^\sigma(G_i : i \in I)$, the i-length $l_i(W)$ is the number of elements of G_i which appear in W. For an element x in the free σ-product $\times^\sigma_{i\in I} G_i$, $l_i(x)$ is $l_i(W)$ for the reduced word W for x [3, p.247].

A sequence $(x_n : n < \omega)$ of elements of free σ-product $\times^\sigma_{i\in I} G_i$ is called *proper*, if for each $i \in I$ there exists an $m < \omega$ such that elements of G_i do not appear in the reduced word for x_n $(n \geq m)$. Therefore, a sequence $(x_n : n < \omega)$ of elements of a free σ-product is proper, if and only if $(x_n : n < \omega)$ converges to the identity in the topology of the inverse limit of discrete groups $*_{i\in F} G_i$'s. For a proper sequence $(x_n : n < \omega)$ and a linear ordering $\prec$ on ω, we define an infinite multiplication [3, Proposition 1.9], i.e. there exists a word W such that:

1. W_m is a reduced word for x_m for each m;

2. $\overline{W} = \{(m, \alpha) : \alpha \in \overline{W_m}\}$ and $W(m, \alpha) = W_m(\alpha)$;
3. $(m, \alpha) < (n, \beta)$, if $m \prec n$, or $m = n$ and $\alpha < \beta$ in $\overline{W_m}$.

Homomorphisms of free σ-products. A homomorphism $h : \times^\sigma_{i\in I} G_i \to \times^\sigma_{j\in J} H_j$ is a *standard* homomorphism, if h preserves the infinite multiplication, i.e. $h(W) = W'$, where W' is defined as follows:

- V_α is a reduced word for $h(W(\alpha))$;
- $\overline{W'} = \Pi_{\alpha\in\overline{W}}\overline{V_\alpha}$;
- $W'(\alpha, u) = V_\alpha(u)$;
- $(\alpha, u) < (\beta, v)$ if $\alpha < \beta$, or $\alpha = \beta$ and $u < v$ in $\overline{V_\alpha}$.

A group G is called *n-slender*, if any homomorphism from $\times^\sigma_{n<\omega}\mathbb{Z}_n$ factors through a finitely generated free group by a projection, where $\mathbb{Z}_n$ is a copy of the integer group $\mathbb{Z}$. The fact that free groups are n-slender is a basic result due to G. Higman [7] (see [3, Appendix]). An abelian group A is n-slender if and only if A is slender and the class of all n-slender groups is closed under forming free products and restricted direct products [3, Section 3]. There is a nice characterization of slender abelian groups due to Nunke [8] which says: an abelian group A is slender if and only if A is torsionfree and does not contain copies of the direct product $\mathbb{Z}^\omega$, the rational group $\mathbb{Q}$, or the p-adic integer group $\mathbb{J}_p$ for any prime p.

The fundamental group of the Hawaiian earring. The fundamental group of the Hawaiian earring of I-many copies of the circle is isomorphic to $\times^\sigma_{i\in I}\mathbb{Z}_i$, where $\mathbb{Z}_i \simeq \mathbb{Z}$ [3].

In particular, the fundamental group of the Hawaiian earring $\pi_1(\mathbb{H}, o)$ is isomorphic to $\times^\sigma_{n<\omega}\mathbb{Z}_n$. Denote a generator of $\mathbb{Z}_n$ by δ_n. Then, δ_n corresponds to a winding to the n-th circle of $\mathbb{H}$ with the base point $o = (0, 0)$ by a canonical isomorphism. Hence, we identify $\times^\sigma_{n<\omega}\mathbb{Z}_n$ with $\pi_1(\mathbb{H}, o)$ under this correspondence.

Then, endomorphisms of the fundamental group of the Hawaiian earring $\pi_1(\mathbb{H}, o)$ are standard homomorphisms if and only if they are induced from continuous maps [4, Corollary 2.11].

Now that we have reviewed the basic definitions, it is possible for us to to indicate the lines of thought which will lead to a proof of Theorem 1.1:

In the assumption of Theorem 1.1, a given space X is not semi-locally simply connected at every point on a circle C. This bad local behavior will allow us to construct a structure similar to the Hawaiian earring at each point $x \in C$ by a map from $\mathbb{H}$ to X (Lemma 3.1).

Suppose the existence of an embedding $h : \pi_1(X, x_0) \to \times^\sigma_{i\in I} G_i$. Then by composing maps from $\pi_1(\mathbb{H}, o)$ to $\pi_1(X, x_0)$ we shall obtain an entire "circle" of homomorphisms from the free σ-product $\times^\sigma_{n<\omega}\mathbb{Z}_n = \pi_1(\mathbb{H}, o)$ to $\times^\sigma_{i\in I} G_i$. Then it all comes down to analyzing this circle of homomorphisms from $\times^\sigma_{n<\omega}\mathbb{Z}_n$ into $\times^\sigma_{i\in I} G_i$.

The general analysis of homomorphisms $h : \times^\sigma_{n<\omega}\mathbb{Z}_n \to \times^\sigma_{i\in I} G_i$ has been carried out in previous papers [3, 4].

The fundamental result, which we shall state in a moment as Lemma 2.3, is that each such homomorphism is conjugate in $\times^\sigma_{i\in I} G_i$ to a standard homomorphism. Futhermore, if the image of the homomorphism is not finitely generated (here we use the injectivity of the embedding of $\pi_1(X, x_0)$ to $\times^\sigma_{i\in I} G_i$), then the element u of $\times^\sigma_{i\in I} G_i$ which conjugates the homomorphism to a standard homomorphism is uniquely determined. The three lemmas in this section are all devoted to understanding how this element u is recognized, characterized, or controlled.

Now we have a conjugator u_x at each point x on the circle C according to the circle of homomorphisms. In Section 3, where the argument will be completed, it will be shown that as we move around the circle C the associated conjugator u_x in $\times_{i\in I}^{\sigma} G_i$, which may possibly change according to points x, does not change. But it is easy to see that, after the complete transversal of the circle, the conjugator must change precisely by multiplication of a path around the circle itself, a contradiction.

Our first lemma, whose proof is obvious, will permit the quasi-reduced factorization of an arbitrary reduced word as conjugate of a cyclically reduced word. Both the conjugator and the cyclically reduced word will play important roles in what follows.

Lemma 2.1 ([4, Lemma 2.4])**.** *For $s \in \times_{i\in I}^{\sigma} G_i$, there exist reduced words V and W such that:*

- *$a = W^{-1}VW$;*
- *V is cyclically reduced.*

If a cyclically reduced word V is not a single letter, it has the useful property that every non-zero power $V^n \cong VV\cdots V$ is reduced.

The next lemma shows that every homomorphism $h : \times_{n<\omega}\mathbb{Z}_n \to \times_{i\in I}^{\sigma} G_i$ is conjugate to a standard homomorphism, and it characterizes the conjugator in terms of the cyclic reductions $h(\delta_n) = W_n^{-1}V_nW_n$ by Lemma 2.1. In the proof of [4, Lemma 2.9], it is proved that $([W_n]u^{-1} : n < \omega)$ is proper, where u is an element appearing in the next Lemma 2.2. Therefore,

Lemma 2.2 ([4, Lemma 2.9])**.** *Let G_i $(i \in I)$ be n-slender groups. Then every homomorphism $h : \times_{n<\omega}\mathbb{Z}_n \to \times_{i\in I}^{\sigma} G_i$ is conjugate to a standard homomorphism $\overline{h}$, that is, there exists a $u \in \times_{i\in I}^{\sigma} G_i$ such that $h(x) = u^{-1}\overline{h}(x)u$ for $x \in \times_{n<\omega}\mathbb{Z}_n$. In addition if the set $\{n < \omega : h(\delta_n) \neq e\}$ is infinite, such a u is unique. Here u satisfies the following: Let $h(\delta_n) = W_n^{-1}V_nW_n$ for each n, where V_n, W_n are reduced words and $W_n^{-1}V_nW_n$ is quasi-reduced. Then, u is a limit of $[W_n]$'s in the topology of the inverse limit.*

In the proof of Theorem 1.1 in Section 3, as we move from one point x of the circle C to another, we shall need to know the precise form of certain cyclic reduction $a = W^{-1}VW$ and particularly to know that certain letters do not disappear under cancellation. The necessary result appears in the following lemma. A similar result with a similar proof appeared in [4, Theorem 4.1] (p. 303 of [4]).

Lemma 2.3. *Let $h : \times_{n<\omega}\mathbb{Z}_n \to \times_{i\in I}^{\sigma} G_i$ be conjugate to a standard homomorphism $\overline{h}$, where $h(x) = u^{-1}\overline{h}(x)u$, and U be a reduced word for u. If the set $\{n < \omega : h(\delta_n) \neq e\}$ is infinite, there exists an element y such that the reduced form of $h(y)$ is $U^{-1}WU$ for a cyclically reduced word W.*

Proof. Since $(\overline{h}(\delta_n) : n < \omega)$ is a proper sequence and the set $\{n < \omega : h(\delta_n) \neq e\}$ is infinite, by Lemma 2.1 we can obtain a proper sequence $(u_n : n < \omega)$ and reduced words W_n and V_n such that:

(1) $\overline{h}(u_n) = W_n^{-1}V_nW_n \neq e$;
(2) $W_n^{-1}V_nW_n$ is quasi-reduced;
(3) V_nV_n is reduced.

One can extract such u_n's from the set of elements of the form δ_i or $\delta_i\delta_j$ with j much larger than i.

Next we construct sequences of natural numbers $(j_n : n < \omega)$ for indices and $(k_n : n < \omega)$ for powers and a sequence of members of I $(i_n : n < \omega)$ by induction. Let $j_0 = 0$ and choose $i_0 \in I$ so that $l_{i_0}(V_0) > 0$. Let $k_0 = 1$. Suppose that we have i_n, j_n and k_n. First choose $j_{n+1} > j_n$ so that $l_{i_n}(\overline{h}(u_m)) = 0$ for any $m \geq j_{n+1}$. Next choose $i_{n+1} \in I$ so that $l_{i_{n+1}}(V_{j_{n+1}}) > 0$. Finally choose k_{n+1} so that $k_{n+1} > l_{i_{n+1}}(U)$ and $k_{n+1} > l_{i_{n+1}}(W_{j_n}) + l_{i_{n+1}}(V_{j_n})$.

We consider the cancellation in the word $V_{j_n}^{2k_n+1} W_{j_n} W_{j_{n-1}}^{-1} V_{j_{n-1}}^{2k_{n-1}+1}$. Since $l_{i_{n-1}}(\overline{h}(u_n)) = 0$, there is no case when the whole of the left most appearance of $V_{j_{n-1}}$ disappears in the cancellation. By this fact and $k_n > l_{i_n}(W_{j_{n-1}}) + l_{i_n}(V_{j_{n-1}})$, the left most appearance of $V_{j_n}^{k_n+1}$ remains after the cancellation. Also by a similar reasoning for $V_{j_{n-1}}^{2k_{n-1}+1} W_{j_{n-1}} W_{j_n}^{-1} V_{j_n}^{2k_n+1}$, we can see that the reduced word for $W_{j_n} \overline{h}(u_{j_n}^{2k_n+1} \cdots u_{j_0}^{2k_0+1} \cdots u_{j_n}^{2k_n+1}) W_{j_n}^{-1}$ is of the form

$$V_{j_n}^{k_n+1} A_n V_{j_{n-1}}^{k_{n-1}} \cdots A_1 V_{j_0} B_1 \cdots V_{j_{n-1}}^{k_{n-1}} B_n V_{j_n}^{k_n+1}.$$

Therefore, $\overline{h}(u_{j_n}^{2k_n+1} \cdots u_{j_0}^{2k_0+1} \cdots u_{j_n}^{2k_n+1})$ is expressed as a quasi-reduced word

$$W_{j_n}^{-1} V_{j_n}^{k_n} A_n V_{j_{n-1}}^{k_{n-1}} \cdots A_1 V_{j_0} B_1 \cdots V_{j_{n-1}}^{k_{n-1}} B_n V_{j_n}^{k_n+1} W_{j_n}.$$

Let W be the reduced word for

$$\overline{h}(\cdots u_{j_n}^{2k_n+1} \cdots u_{j_1}^{2k_1+1} u_{j_0}^{2k_0+1} u_{j_1}^{2k_1+1} \cdots u_{j_n}^{2k_n+1} \cdots).$$

Since $\overline{h}$ is a standard homomorphism, W is of the form

$$\cdots V_{j_n}^{k_n} \cdots V_{j_1}^{k_1} A_1 V_{j_0} B_1 V_{j_1}^{k_1} \cdots V_{j_n}^{k_n} \cdots .$$

Therefore $U^{-1} \cdots V_{j_n}^{k_n} \cdots V_{j_1}^{k_1} A_1 V_{j_0} B_1 V_{j_1}^{k_1} \cdots V_{j_n}^{k_n} \cdots U$ is the reduced word for $h(\cdots u_{j_n}^{2k_n+1} \cdots u_{j_1}^{2k_1+1} u_{j_0}^{2k_0+1} u_{j_1}^{2k_1+1} \cdots u_{j_n}^{2k_n+1} \cdots)$ by the condition $k_{n+1} > l_{j_n}(U)$.

To see that WW is reduced, consider the letter R_n of G_{i_n} which is the right most appearance in $W_{j_n}^{-1} V_{j_n}^{2k_n+1} W_{j_n}$ and the letter L_n of G_{i_n} which is the left most one in $W_{j_n}^{-1} V_{j_n}^{2k_n+1} W_{j_n}$. Then the same letters are the right most appearance and the left most one in W. Now let R_n^l and L_n^r be the corresponding appearances in the left W of WW and in the right W of WW respectively. Suppose that there is a cancellation of letters of G_{i_n} in WW. Then R_n^l and L_n^r are cancelled and consequently R_{n+1}^l and L_{n+1}^r are cancelled and also every letter between R_n^l and L_n^r disappears after the cancellation. We consider letters of $G_{i_{n+1}}$ appearing between R_n^l and L_n^r. If R_{n+1} is in $W_{j_{n+1}}$, we have a word of the next form in an intermediate stage of the cancellation: $W'^{-1} R_{n+1}^l L_{n+1}^r W'$, where $U L_{n+1}^r W' \cong W_{j_{n+1}}$ for some word U. Then a cancellation of $V_{j_{n+1}} V_{j_{n+1}}$ should occur, which is a contradiction. Otherwise, that is, when R_{n+1} is in $V_{j_{n+1}}$, we express $V_{j_{n+1}}$ as $xLyRz$, where L and R are the left most appearance and the right most appearance of letters in $G_{i_{n+1}}$. Since $V_{j_{n+1}} V_{j_{n+1}}$ is reduced, ZX is a reduced word and non-empty. Since every letter between R_n^l and L_n^r disappears in the cancellation, we have a word of the next form as an intermediate stage of the cancellation: $\cdots RZXLyR_{n+1}^l L_{n+1}^l yRZXL \cdots$. Hence, $ZXZX = e$, which contradicts that $\times_{i \in I}^{\sigma} G_i$ is torsion-free.

Now, $\cdots u_{j_n}^{2k_n+1} \cdots u_{j_1}^{2k_1+1} u_{j_0}^{2k_0+1} u_{j_1}^{2k_1+1} \cdots u_{j_n}^{2k_n+1} \cdots$ is the desired y. $\square$

3. Proof of Theorem 1.1

A path f is a continuous map from $[0,1]$ to X and a loop is a path with $f(0) = f(1)$. For a path f, we define a path f^- by $f^-(t) = f(1-t)$. For paths f and g with $f(1) = g(0)$, let $fg : [0,1] \to X$ be a path defined by

$$fg(t) = \begin{cases} f(2t) \text{ for } 0 \leq t \leq 1/2, \\ g(2t-1) \text{ for } 1/2 \leq t \leq 1. \end{cases}$$

For a loop f with the base point $x \in X$, $[f] \in \pi_1(X,x)$ denotes the homotopy class relative to $\{0,1\}$ containing f. For a path p from x to x_0, φ_p denotes the canonical isomorphism from $\pi_1(X,x) \to \pi_1(X,x_0)$, i.e. $\varphi_p([f]) = [p^- fp]$, where $[p^- fp] = [(p^- f)p] = [p^-(fp)]$. A space X is *semi-locally simply connected* at x, if there exists a neighborhood U of x such that any loop in U with the base point x is null homotopic relative to $\{0,1\}$.

Lemma 3.1. *Let X be a one-dimensional metric space and not semi-locally simply connected at a point x. Then there exists a continuous map $f : \mathbb{H} \to X$ such that $f(o) = x$ and the image of $f_* : \pi_1(\mathbb{H}, o) \to \pi_1(X,x)$ is not finitely generated and particularly $f_*(\delta_n) \neq e$ for $n < \omega$.*

Proof. There exist loops f_n with end point x such that $[f_n] \neq e$ and $\mathrm{Im}(f_n)$ converge to x. Let $f : \mathbb{H} \to X$ be the map defined by

$$f(\frac{1}{n}(\cos 2\pi t - 1), \frac{1}{n}\sin 2\pi t) = f_n(t)$$

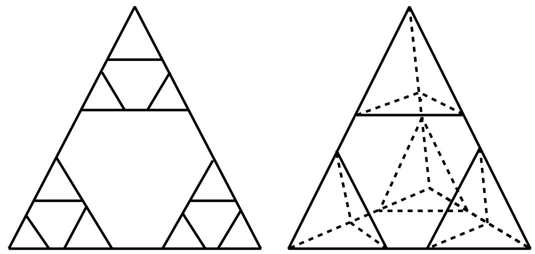
for $0 \leq t \leq 1$ and $1 \leq n < \omega$. Then $\mathrm{Im}(f_*) \leq \pi_1(X,x_0)$ is not finitely generated by [2, Lemma 3.5]. ☐

Proof of Theorem 1.1. To prove the theorem by contradiction, let $h : \pi_1(X,x_0) \to \times^\sigma_{i\in I} G_i$ be an injective homomorphism. ☐

For a path p with $p(0) = x_0$, let p_s be paths such that $p_s(t) = p(ts)$ for $0 \leq s,t \leq 1$. Suppose that X is not semi-locally simply connected at each point x on $\mathrm{Im}(p)$. We choose loops g_n with the base point $p_s(1)$ so that g_n's are not null homotopic and the images of g_n converge to $p_s(1) = p(s)$ and define $f_s : \mathbb{H} \to X$ as in the proof of Lemma 3.1. Apply Lemmas 2.2 and 3.1 to $h \cdot \varphi_{p_s^-} \cdot f_{s*}$; then we get $u_s \in \times^\sigma_{i\in I} G_i$ such that $h \cdot \varphi_{p_s^-} \cdot f_{s*}(x) = u_s^{-1}\overline{h}(x)u_s$ for a standard homomorphism $\overline{h}$.

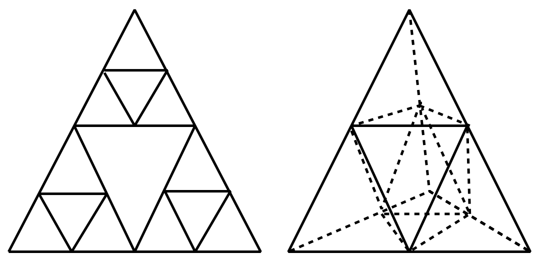
First we show that u_s does not depend on particular g_n's. Let g'_n's and u'_s be other choices. We have a homomorphism $h : \times_{n<\omega}\mathbb{Z}_n \to \pi_1(X,x_0)$ such that $h(\delta_{2n}) = [p_s g_n p_s^-]$ and $h(\delta_{2n+1}) = [p_s g'_n p_s^-]$. The restrictions of h to $\times_{n<\omega}\mathbb{Z}_{2n}$ and $\times_{n<\omega}\mathbb{Z}_{2n+1}$ give us u_s and u'_s. There is u such that h is conjugate to a standard homomorphism by u. Then, by the uniqueness in Lemma 2.2 for the restrictions of h, $u_s = u = u'_s$ holds. Therefore we have a map $s \mapsto u_s$ from $[0,1]$ to a topological group $\times^\sigma_{i\in I} G_i$.

Next we show that this map $s \mapsto u_s$ is continuous. Suppose that s_n's converge to s. Here we may assume that the images of f_{s_n} converge to s, since we may choose g_n's with small images. For each n, there is x_n such that the reduced word for $h \cdot \varphi_{p_{s_n}^-} \cdot f_{s_n *}(x_n)$ is of the form $U_n^{-1}W_nU_n$, where U_n is the reduced word for u_{s_n} by Lemma 2.3. Then there are loops g''_n with the base point $p(s_n)$ such that $[p_{s_n} g''_n p_{s_n}^-] = \varphi_{p_{s_n}^-} \cdot f_{s_n *}(x_n)$ and the images of g''_n's converge to s. By Lemma 2.2 we conclude that u_s is a limit of u_{s_n}.



(A1) (A2)

Figure A.



(B1) (B2)

Figure B.

Since the topology $\mathop{\times}^{\sigma}_{i\in I} G_i$ is 0-dimensional, $u_s = u_0$ for all $0 \leq s \leq 1$, that is, the map $s \mapsto u_s$ is constant.

Finally we deduce a contradiction from this last statement. Take $x \in C$ so that $x \neq x_0$ and also two paths p and q from x_0 to x in C so that the intersection of the images of p and q are just two points x_0 and x. Take loops l_n with the base point x so that $[l_n] \neq e$ and the images of l_n's converge to x. Then there exist standard homomorphisms $h_0, h_1 : \mathop{\times}_{n<\omega} \mathbb{Z}_n \to \mathop{\times}^{\sigma}_{i\in I} G_i$ and $u \in \mathop{\times}^{\sigma}_{i\in I} G_i$ such that $h([pl_n p^-]) = u^{-1} h_0(\delta_n) u$ and $h([ql_n q^-]) = u^{-1} h_1(\delta_n) u$ for $n < \omega$. Let $a = [pq^-]$; then $a \neq e$ and $[ql_n q^-] = [qp^-][pl_n p^-][pq^-] = a^{-1}[pl_n p^-]a$ hold. Therefore, $h_1(y) = h(a)^{-1} h_0(y) h(a)$ holds for $y \in \mathop{\times}_{n<\omega} \mathbb{Z}_n$. Since $h_0(\delta_n) \neq e$ for $n < \omega$ and $h(a) \neq e$, h_1 cannot be a standard homomorphism, which is a contradiction. □

Remark 3.2. Here, we demonstrate spaces which have similar shapes, but fundamental groups of one kind can be embeddable to that of the Hawaiian earring and on the other hand those of another kind cannot be embeddable. In Figures A and B, we have drawn only finitely many triangles and tetrahedra, but actually there are infinitely many ones and the spaces are Peano continua.

In [5], we showed that the fundamental groups of spaces in Figure A are embeddable into the one of the Hawaiian earring. Each triangle or tetrahedron is similar to the whole space respectively. On the other hand, in the present paper we have shown that the fundamental groups of spaces in Figure B, which are well-known fractals, cannot be embeddable into the fundamental group of the Hawaiian earring. There are continuous surjections from (A1) to (B1) and from (A2) to

(B2), say f_1 and f_2, which make certain intervals to one-points and patch triangles or tetrahedra. Here we show that these continuous surjections induce injective homomorphisms between fundamental groups. For a one-dimensional space X, a canonical homomorphism from the fundamental group to the first Čech homotopy group is injective [6]. On the other hand there are natural isomorphisms between the first Čech homotopy groups of (A1) and (B1), and (A2) and (B2), respectively, which are induced from f_1 and f_2 and hence commute with the induced homomorphisms f_{1*} and f_{2*}. Therefore, f_{1*} and f_{2*} are injective also between fundamental groups.

We refer the reader to [4] for other wild spaces fundamental groups which can be embeddable into that of the Hawaiian earring.

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