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SPECKER'S THEOREM FOR NÖBELING'S GROUP

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ABSTRACT. Specker proved that the group $\mathbb{Z}^{\aleph_0}$ of integer-valued sequences is far from free; all its homomorphisms to $\mathbb{Z}$ factor through finite subproducts. Nöbeling proved that the subgroup $\mathcal{B}$ consisting of the bounded sequences is free and therefore has many homomorphisms to $\mathbb{Z}$. We prove that all "reasonable" homomorphisms $\mathcal{B} \to \mathbb{Z}$ factor through finite subproducts. Among the reasonable homomorphisms are all those that are Borel with respect to a natural topology on $\mathcal{B}$. In the absence of the axiom of choice, it is consistent that all homomorphisms are reasonable and therefore that Specker's theorem applies to $\mathcal{B}$ as well as to $\mathbb{Z}^{\aleph_0}$.

1. INTRODUCTION

All groups in this paper are abelian; "free" means "free abelian." Let $\mathbb{Z}^{\aleph_0}$ be the group of all sequences $x = (x_n)_{n\in\mathbb{N}}$ of integers with the operation of componentwise addition. Specker [9] proved that $\mathbb{Z}^{\aleph_0}$ is not free. In fact, he proved a considerably stronger property of $\mathbb{Z}^{\aleph_0}$. To state his result, it is convenient to use the truncation homomorphisms

$$T_k : \mathbb{Z}^{\aleph_0} \to \mathbb{Z}^k : x = (x_n)_{n\in\mathbb{N}} \mapsto (x_n)_{0\le n<k},$$

which truncate an infinite sequence after k terms.

Theorem 1 (Specker, 1950)**.** *Every homomorphism $h : \mathbb{Z}^{\aleph_0} \to \mathbb{Z}$ factors as $h' \circ T_k$ for some finite k and some $h' : \mathbb{Z}^k \to \mathbb{Z}$.*

Specker formulated the theorem in terms of the "unit vectors" $e(j) \in \mathbb{Z}^{\aleph_0}$, where $(e(j))_n$ is 1 if $n = j$ and 0 otherwise. The theorem then says that every $h : \mathbb{Z}^{\aleph_0} \to \mathbb{Z}$ sends all but finitely many $e(j)$'s to zero and is completely determined by its values on all the $e(j)$'s.

Notice that, since $\mathbb{Z}^{\aleph_0}$ has cardinality $2^{\aleph_0}$ and therefore rank $2^{\aleph_0}$, if it were free, then it would have $2^{2^{\aleph_0}}$ homomorphisms to $\mathbb{Z}$. So by implying that the number of such homomorphisms is really only $\aleph_0$, Specker's theorem says that $\mathbb{Z}^{\aleph_0}$ is far from being free.

The situation is quite different for the subgroup $\mathcal{B} \subseteq \mathbb{Z}^{\aleph_0}$ consisting of the bounded sequences, as is shown by the following theorem of Nöbeling [6], for which a shorter proof due to G. Bergman is given in [2, Section 97].

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Theorem 2 (Nöbeling, 1968)**.** *$\mathcal{B}$ is free.*

Nöbeling's and Bergman's proofs of this theorem both make use of the axiom of choice; they well-order sets of the cardinality of the continuum. Thus, these proofs do not provide an explicit basis for $\mathcal{B}$, nor do they enable one to exhibit very many of the $2^{2^{\aleph_0}}$ homomorphisms $\mathcal{B} \to \mathbb{Z}$ whose existence the theorem guarantees.

The purpose of this note is to show that this non-constructivity is unavoidable. We shall show that, if we consider only homomorphisms $h : \mathcal{B} \to \mathbb{Z}$ that are well-behaved in a suitable sense, then they all satisfy the conclusion of Specker's theorem. Our result implies that, if one omits the axiom of choice from one's set theory, then it is consistent that *all* homomorphisms $\mathcal{B} \to \mathbb{Z}$ satisfy the conclusion of Specker's theorem, and so the number of such homomorphisms may be only $\aleph_0$ rather than $2^{2^{\aleph_0}}$.

2. Statement of results

We first review some topological concepts that will be needed for our main result.

As a set, $\mathbb{Z}^{\aleph_0}$ is the product of $\aleph_0$ copies of $\mathbb{Z}$. Equip it with the product topology obtained from the discrete topology on $\mathbb{Z}$. A basis for this topology is given by the sets

$$U(s) = \{x \in \mathbb{Z}^{\aleph_0} : T_k(x) = s\}$$

where s ranges over finite sequences of integers and k is the length of s. In particular, a neighborhood base at 0 is given by the sets $U(0^k)$, where 0^k means a sequence of zeros of length k. Notice that $\mathbb{Z}^{\aleph_0}$ is separable in the topological sense; the sequences with only finitely many nonzero terms form a countable dense subset.

There is a complete metric inducing the topology of $\mathbb{Z}^{\aleph_0}$. This follows from the general fact that the product of countably many completely metrizable spaces is again completely metrizable. It can also be verified directly, using the metric where the distance between any distinct points $x, y \in \mathbb{Z}^{\aleph_0}$ is 2^{-n} if n is the smallest number with $x_n \neq y_n$.

Topologize $\mathcal{B}$ as a subspace of $\mathbb{Z}^{\aleph_0}$. We shall also have occasion to use the smaller subspace $\{0,1\}^{\aleph_0}$, which is closed in $\mathbb{Z}^{\aleph_0}$ and hence also a complete metric space. (In fact, it is compact by Tychonoff's theorem.)

Recall that a set X in a topological space is called *meager* (or *of first Baire category*) if it can be covered by countably many closed sets with empty interiors. A set X is *comeager* if its complement is meager, i.e., if X includes the intersection of some countably many dense open sets. Obviously, the intersection of countably many comeager sets is comeager. (This "obvious" fact actually uses the axiom of choice for countable families of sets. Even when we consider models where the axiom of choice fails, we shall restrict attention to models that satisfy this countable axiom of choice. So the obvious remains true even in this context.)

The Baire category theorem (from [1], see also for example [4, Ex. 2.H.2]) asserts that a complete metric space is not meager (in itself). Equivalently, in a complete metric space a comeager set is nonempty and in fact dense.

A set X has the *Baire property* if it differs from some open set by a meager set. We say that a function from one topological space to another has the *Baire property* if the inverse image of each open set of the target space has the Baire property in the domain space. Obviously, all continuous maps have the Baire property, but so do many other maps.

We shall need the fact that, in a complete, separable, metric space, every Borel set has the Baire property. This fact goes back at least to the 1923 paper of Lusin and Sierpiński [3]; for an accessible reference, see [4, Ex. 2.H.3]. It immediately implies that a Borel-measurable function has the Baire property.

We are now in a position to state our main result. It looks somewhat technical, since it involves the Baire property, but the corollaries present the essential content in a less technical way.

Theorem 3. *Let $h : \mathcal{B} \to \mathbb{Z}$ be a homomorphism such that its restriction to $\{0,1\}^{\aleph_0}$ has the Baire property (as a map from $\{0,1\}^{\aleph_0}$ to the discrete space $\mathbb{Z}$). Then h factors through one of the truncation maps T_k.*

Corollary 4. *Every Borel-measurable homomorphism from $\mathcal{B}$ to $\mathbb{Z}$ factors through one of the truncation maps T_k.*

Proof. Restricting functions to a subspace preserves Borel measurability. So, since Borel measurability implies the Baire property, any homomorphism satifying the hypothesis of the corollary also satisfies the hypothesis of the theorem. □

Remark 5. The Baire property holds for wider classes of sets than just Borel sets. It is preserved by Souslin's operation $\mathcal{A}$ (see [4, Ex. 2.H.5]). So the corollary would remain true if, instead of Borel measurability, we assumed only measurability with respect to the smallest σ-field containing the Borel sets and closed under operation $\mathcal{A}$; in the classical literature this σ-field is sometimes called the class of $\mathcal{C}$-sets.

Remark 6. Since we are dealing with homomorphisms into the countable, discrete space $\mathbb{Z}$, to say that a homomorphism h is measurable with respect to some field of sets just means that the level sets $h^{-1}(\{n\})$ are in that field.

Various set-theoretic hypotheses, notably various determinacy axioms, are known to imply the Baire property for wide classes of sets. We limit ourselves here to stating three of the many results obtainable by applying our theorem in the context of such hypotheses. For the relevant background, see [4], particularly Ex. 6.G.11.

Corollary 7. *If there is a measurable cardinal, then every homomorphism $h : \mathcal{B} \to \mathbb{Z}$ whose level sets are in the σ-field generated by $\mathbf{\Sigma}^1_2$ sets (classically known as PCA-sets) factors through a truncation T_k.*

Corollary 8. *The axiom of projective determinacy implies that every projective homomorphism $\mathcal{B} \to \mathbb{Z}$ factors through a truncation.*

Corollary 9. *The axiom of determinacy implies that every homomorphism $\mathcal{B} \to \mathbb{Z}$ factors through a truncation.*

The axiom of determinacy [5] contradicts the axiom of choice, so the last corollary does not conflict with Nöbeling's theorem. The consistency of the axiom of determinacy (in set theory without choice) and the consistency of projective determinacy (in full set theory, with the axiom of choice) are known to be equivalent to the consistency of certain very large cardinal axioms. The conclusions of the last two corollaries, however, do not require such strong assumptions.

Corollary 10. *The statement "Every homomorphism $\mathcal{B} \to \mathbb{Z}$ that is set-theoretically definable using real and ordinal numbers as parameters factors through a truncation" is consistent relative to the usual axioms of set theory, including the axiom of choice.*

Definability with real and ordinal parameters is a very weak form of good behavior; it covers the projective hierarchy and much more.

Proof. Shelah [7] proved that the statement "Every subset of $\{0,1\}^{\aleph_0}$ that is definable from real and ordinal parameters has the Baire property" is consistent relative to the usual axioms. (Solovay [8] had previously obtained this result under the hypothesis that the existence of an inaccessible cardinal is consistent.) By our theorem, Shelah's result immeditely gives the corollary. □

Corollary 11. *It is consistent (relative to the consistency of ordinary set theory) to have*

- *all axioms of set theory except the axiom of choice,*
- *the axiom of dependent choice, and*
- *every homomorphism $\mathcal{B} \to \mathbb{Z}$ factors through a truncation.*

The axiom of dependent choice is a weak form of the axiom of choice, stronger than the axiom of countable choice and adequate for the usual development of the non-pathological properties of Baire category, Lebesgue measure, etc.

Proof. Shelah [7] proved that the conjunction of the first two items in the corollary plus "all subsets of $\{0,1\}^{\aleph_0}$ have the Baire property" is consistent relative to the consistency of ordinary set theory. (Solovay [8] had also previously obtained this consistency relative to the existence of an inaccessible cardinal.) The corollary immediately follows from Shelah's result and our theorem. □

Corollary 12. *Nöbeling's theorem that $\mathcal{B}$ is free cannot be proved from the usual axioms of set theory, minus the axiom of choice, plus the axiom of dependent choice.*

Proof. Even if the axiom of choice is replaced by the axiom of dependent choice (or even just the axiom of countable choice), a free basis for an uncountable group like $\mathcal{B}$ must have a countably infinite subset. (More is true, but we won't need it.) Since a homomorphism from a free group to any other group can be prescribed arbitrarily on a basis, any uncountable free group must have at least $2^{\aleph_0}$ homomorphisms to $\mathbb{Z}$. But only countably many homomorphisms $\mathcal{B} \to \mathbb{Z}$ factor through truncations. Therefore, in the model from the preceding corollary, $\mathcal{B}$ is not free. □

3. The proof

This section is devoted to the proof of the main theorem. Suppose that $h : \mathcal{B} \to \mathbb{Z}$ is a homomorphism and its restriction to $\{0,1\}^{\aleph_0}$ has the Baire property.

Until further notice, we work in the space $\{0,1\}^{\aleph_0}$. All notions of meagerness, Baire property, and the like will be relative to this space. We write $U'(s)$ for $U(s) \cap \{0,1\}^{\aleph_0}$, a basic open set in $\{0,1\}^{\aleph_0}$ when all terms of the finite sequence s are 0 or 1.

The hypothesis on h means that each of the sets

$$C_v = \{x \in \{0,1\}^{\aleph_0} : h(x) = v\}, \qquad v \in \mathbb{Z},$$

has the Baire property. These sets cannot all be meager, because the Baire category theorem prevents countably many meager sets from covering a complete metric space. So fix $v \in \mathbb{Z}$ such that C_v is not meager. Since it has the Baire property, C_v differs by a meager set from some open set; this open set is nonempty since C_v isn't meager, so we can find a basic open neighborhood $U'(s)$ included in it. Thus, all but a meager set of elements of $U'(s)$ are mapped by h to the same value v. Let k be the length of the sequence s.

The construction in the proof of the following lemma goes back at least to [10], but we include the proof for the reader's convenience.

Lemma 13. *There exist a partition of $\{k, k+1, k+2\dots\} = \mathbb{N} - [0, k)$ into consecutive intervals I_0, I_1, ... and functions $t_i : I_i \to \{0, 1\}$ with the following property. If $x \in \{0, 1\}^{\aleph_0}$ agrees with s on $[0, k)$ and for infinitely many i agrees with t_i on I_i, then $x \in C_v$.*

Proof. Since C_v is comeager in $U'(s)$, we can fix a countable sequence of dense open sets V_n in $U'(s)$ such that $\bigcap_n V_n \subseteq C_v$. Replacing each V_n with its intersection with all the earlier V_m's in the sequence, and remembering that the intersection of finitely many dense open sets is again dense and open, we may assume without loss of generality that $V_0 \supseteq V_1 \supseteq V_2 \supseteq \dots$. We now construct the required intervals I_i and functions t_i by induction on i. Suppose we are defining I_i and t_i, and all the earlier intervals and functions have already been defined. So we know the point $l \in \mathbb{N}$ that should be the first point in I_i, namely the point immediately after the last point in I_{i-1} (or $l = k$ if $i = 0$). List all functions $[k, l) \to \{0, 1\}$ as $f_1, \dots, f_r$ (so $r = 2^{l-k}$). We shall build I_i and t_i in r steps; I_i will be the union of r consecutive subintervals J_q $(1 \le q \le r)$, and t_i will be defined separately on these subintervals.

Proceeding by induction on q, suppose we have already defined $J_1, \dots, J_{q-1}$ and t_i on these intervals. Consider the function u that agrees with

- s on $[0, k)$,
- f_q on $[k, l)$, and
- t_i on $J_1 \cup \dots \cup J_{q-1}$.

It is defined on an initial segment of $\mathbb{N}$ up to but not including the point that should be the first point in J_q. $U'(u)$ is an open subset of $U'(s)$, so it must meet the dense open subset V_i, and the intersection must contain a basic neighborhood $U'(w)$ where w is a finite sequence extending u. Let J_q be the part of the domain of w that extends beyond the domain of u, and define t_i there to agree with w. (Technicality: If I_i turns out to be empty, enlarge it to contain one point, and define t_i arbitrarily there.)

The construction of J_q and the definition of t_i on it ensure that any $x \in \{0, 1\}^{\aleph_0}$ that agrees with s on $[0, k)$, with f_q on $[k, l)$ and with t_i on I_i must be in V_i. Since all functions $[k, l) \to \{0, 1\}$ occur among the f_q's, it follows that any $x \in \{0, 1\}^{\aleph_0}$ that agrees with s on $[0, k)$ and with t_i on I_i must be in V_i.

Finally, if $x \in \{0, 1\}^{\aleph_0}$ agrees with s on $[0, k)$ and with infinitely many of the t_i on the corresponding intervals I_i, then x is in (the corresponding) infinitely many of the V_i. But we arranged for the V_i to form a decreasing sequence, so any such x must in fact be in all the V_i. That means that it is in C_v, and the proof of the lemma is complete. □

Returning to the proof of the theorem, fix intervals I_i and functions $t_i : I_i \to \{0, 1\}$ as in the lemma. We shall prove that h factors through the truncation T_k, where k is the length of s as above. Equivalently, we show that h is identically zero on the kernel of this truncation, which is $U(0^k) \cap \mathcal{B}$. Since this kernel is generated by $U'(0^k)$, it suffices to show that h maps every $x \in U'(0^k)$ to zero.

To do this, we shall consider an arbitrary $x \in U'(0^k)$ and show how to express it in the form $x = a + b - c - d$ with $a, b, c, d \in C_v$. Then we shall have

$$h(x) = h(a) + h(b) - h(c) - h(d) = v + v - v - v = 0$$

as required.

We begin the construction of a, b, c, d by making all four of them agree with s on $[0, k)$. Then $a + b - c - d$ is identically 0 on $[0, k)$ and therefore agrees with x there. Now we show how to define these four sequences on each interval I_i.

If $i \equiv 0 \pmod 4$, then we define a to agree with t_i on I_i, and we make d identically 0 on this interval. We define b and c on this interval so as to make $a + b - c$ agree with x. That is, for any $n \in I_i$,

- if $a_n = x_n$, then we set $b_n = c_n = 0$,
- if $a_n = 0$ and $x_n = 1$, then we set $b_n = 1$ and $c_n = 0$, and
- if $a_n = 1$ and $x_n = 0$, then we set $b_n = 0$ and $c_n = 1$.

These definitions ensure that $a + b - c - d$ agrees with x on I_i.

If $i \equiv 1 \pmod 4$, then we define b to agree with t_i on I_i, we again make d identically 0 on this interval, and we define a and c so as to ensure that $a+b-c-d$ agrees with x on I_i; just interchange the roles of a and b in the preceding paragraph.

If $i \equiv 2 \pmod 4$, then we define c to agree with t_i on this interval, we again make d identically 0 on I_i, and we use a and b to achieve agreement between $a+b-c-d$ with x as follows. For any $n \in I_i$,

- if $c_n = x_n = 0$, then we set $a_n = b_n = 0$,
- if either $c_n = 0$ and $x_n = 1$ or $c_n = 1$ and $x_n = 0$, then we set $a_n = 1$ and $b_n = 0$, and
- if $c_n = x_n = 1$, then we set $a_n = b_n = 1$.

Finally, if $i \equiv 3 \pmod 4$, then we define d to agree with t_i on I_i, we define c to be identically 0 there, and we use a and b to achieve agreement between $a+b-c-d$ and x, just as in the preceding paragraph with the roles of c and d interchanged.

This completes the definition of a, b, c, d and simultaneously the verification that $a + b - c - d = x$. Furthermore, the construction ensures that each of a, b, c, d agrees with s on $[0, k)$ and agrees with t_i on I_i for infinitely many i, in fact for (at least) every fourth i. By the lemma, it follows that $a, b, c, d \in C_v$, and the proof is complete.

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