THE HOMOGENEOUS SPECTRUM OF A GRADED COMMUTATIVE RING

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Abstract. Suppose \( \Gamma \) is a torsion-free cancellative commutative monoid for which the group of quotients is finitely generated. We prove that the spectrum of a \( \Gamma \)-graded commutative ring is Noetherian if its homogeneous spectrum is Noetherian, thus answering a question of David Rush. Suppose \( A \) is a commutative ring having Noetherian spectrum. We determine conditions in order that the monoid ring \( A[\Gamma] \) have Noetherian spectrum. If rank \( \Gamma \leq 2 \), we show that \( A[\Gamma] \) has Noetherian spectrum, while for each \( n \geq 3 \) we establish existence of an example where the homogeneous spectrum of \( A[\Gamma] \) is not Noetherian.

0. Introduction

All rings we consider are assumed to be nonzero, commutative and with unity. All the monoids are assumed to be torsion-free cancellative commutative monoids. Let \( \Gamma \) be a monoid such that the group of quotients \( G \) of \( \Gamma \) is finitely generated, and let \( R = \bigoplus_{\gamma \in \Gamma} R_{\gamma} \) be a commutative \( \Gamma \)-graded ring. A goal of this paper is to answer in the affirmative a question mentioned to one of us by David Rush as to whether \( \text{Spec} \ R \) is necessarily Noetherian provided the homogeneous spectrum, \( h-\text{Spec} \ R \), is Noetherian.

If \( I \) is an ideal of a ring \( R \), we let \( \text{rad}(I) \) denote the radical of \( I \), that is \( \text{rad}(I) = \{ r \in R : r^n \in I \text{ for some positive integer } n \} \). We say that \( I \) is a radical ideal if \( \text{rad}(I) = I \). A subset \( S \) of the ideal \( I \) generates \( I \) up to radical if \( \text{rad}(I) = \text{rad}(SR) \). The ideal \( I \) is radically finite if it is generated up to radical by a finite set.

We recall that a ring \( R \) is said to have Noetherian spectrum if the set \( \text{Spec} \ R \) of prime ideals of \( R \) with the Zariski topology satisfies the descending chain condition on closed subsets. In ideal-theoretic terminology, \( R \) has Noetherian spectrum if and only if \( R \) satisfies the ascending chain condition (a.c.c.) on radical ideals. Thus a Noetherian ring has Noetherian spectrum and each ring having only finitely many prime ideals has Noetherian spectrum. As is shown in [8, Prop. 2.1], \( \text{Spec} \ R \) is Noetherian if and only if each ideal of \( R \) is radically finite. It is well known that \( R \) has Noetherian spectrum if and only if \( R \) satisfies the two properties: (i) a.c.c. on prime ideals, and (ii) every ideal of \( R \) has only finitely many minimal prime ideals [6, Theorem 88, page 59 and Ex. 25, page 65].

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In analogy with the result of Cohen that a ring $R$ is Noetherian if each prime ideal of $R$ is finitely generated, it is shown in \cite{HeinZ} Corollary 2.4] that $R$ has Noetherian spectrum if each prime ideal of $R$ is radically finite. It is shown in \cite{HeinZ} Theorem 2.5] that Noetherian spectrum is preserved under polynomial extension in finitely many indeterminates. Thus finitely generated algebras over a ring with Noetherian spectrum again have Noetherian spectrum.

In Section 1 we prove that if $R$ is a $\Gamma$-graded ring, where $\Gamma$ is a monoid with finitely generated group of quotients, and if $h$-Spec $R$ is Noetherian, then Spec $R$ is Noetherian (Theorem \ref{thm1.7}). In Section 2 we deal with monoid rings. It turns out that if $M$ is a monoid with finitely generated group of quotients and $k$ is a field, then the homogeneous spectrum of the monoid ring $k[M]$ is not necessarily Noetherian (Example \ref{ex2.9}). On the positive side, $h$-Spec $A[M]$ is Noetherian if $A$ is a ring with Noetherian spectrum and $M$ is a monoid of torsion-free rank $\leq 2$.

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1. The homogeneous spectrum

The homogeneous spectrum, $h$-Spec $R$, of a graded ring $R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$ is the set of homogeneous prime ideals of $R$. The most common choices for the commutative monoid $\Gamma$ are the monoid $\mathbb{N}$ of nonnegative integers or its group of quotients $\mathbb{Z}$. A standard technique using homogeneous localization shows the following: if $R = \bigoplus_{n \in \mathbb{Z}} R_n$ is a $\mathbb{Z}$-graded integral domain, if $t$ is a nonzero element of $R_1$, and if $H$ is the multiplicative set of nonzero homogeneous elements of $R$, then the localization $R_H$ of $R$ with respect to $H$ is the graded Laurent polynomial ring $K_0[t, t^{-1}]$, where $K_0$ is a field \cite{Nagata} page 157. This implies the following remark.

Remark 1.1. Suppose $R = \bigoplus_{n \in \mathbb{Z}} R_n$ is a graded integral domain and $P$ is a nonzero prime ideal of $R$. If zero is the only homogeneous element contained in $P$, then the localization $R_P$ is one-dimensional and Noetherian.

If $R = \bigoplus_{n \in \mathbb{Z}} R_n$ is a graded ring with no nonzero homogeneous prime ideals, then $R_0$ is a field and either $R = R_0$, or $R$ is a Laurent polynomial ring $R_0[x, x^{-1}]$ \cite{HeinZ} page 83.

Every ring can be viewed as a graded ring with the trivial gradation that assigns degree zero to every element of the ring. Thus Nagata in \cite{Nagata} Section 8] develops primary decomposition for graded ideals in a graded Noetherian ring. It is not surprising that there is an interrelationship among the Noetherian properties of Spec $R$, $h$-Spec $R$, Spec $R[X]$ and $h$-Spec $R[X]$.

Proposition 1.2 is useful in considering the Noetherian property of spectra. It follows from induction from \cite{HeinZ} Prop. 2.2 (ii), but we prefer to prove it directly.

**Proposition 1.2.** Let $I$ be an ideal of a ring $R$. Let $J$ be an ideal of $R$ and $S$ a subset of $J$ such that $J = \text{rad}SR$. If $I + J$ is radically finite and if for each $s \in S$ the ideal $IR[1/s]$ is radically finite, then $I$ is radically finite.

**Proof.** Since $I + J$ is radically finite and since $J = \text{rad}SR$, there exist finite sets $F \subseteq I$ and $G \subseteq S$ such that $\text{rad}(I + J) = \text{rad}((F + GR)R)$. For each $g \in G$ there exists a finite subset $T_g$ of $I$ such that $\text{rad}(IR[1/g]) = \text{rad}(T_gR[1/g])$. Let $I' = (F \cup \bigcup_{g \in G} T_g)R$, thus $I' \subseteq I$. Suppose $P \in \text{Spec} R$ and $I' \subseteq P$. If $G \subseteq P$, then $I \subseteq P$ since $\text{rad}(I' + GR) = \text{rad}(I + J)$. Otherwise, we have $g \notin P$ for some element $g \in G$. Therefore $\text{rad}(I'R[1/g]) = \text{rad}(IR[1/g]) \subseteq PR[1/g]$. Since $P$ is the
preimage in $R$ of $PR[1/g]$, we have $\text{rad}(I) \subseteq P$. Therefore $\text{rad}(I') = \text{rad}(I)$, so $I$ is radically finite.

For Corollary 1.3 we use that the (homogeneous) spectrum of a graded ring $R$ is Noetherian if each (homogeneous) ideal of $R$ is radically finite.

**Corollary 1.3.** (1) Let $S$ be a finite subset of a ring $R$. If $\text{Spec}(R/SR)$ is Noetherian and for each $s \in S$, $\text{Spec}(R[1/s])$ is Noetherian, then $\text{Spec} R$ is Noetherian.

(2) Let $S$ be a finite set of homogeneous elements of a graded ring $R$. If $h\text{-Spec}(R/SR)$ is Noetherian and for each $s \in S$, $h\text{-Spec}(R[1/s])$ is Noetherian, then $h\text{-Spec} R$ is Noetherian.

The hypotheses in Proposition 1.2 and Corollary 1.3 concerning the set $S$ may be modified as follows and still give the same conclusion:

**Proposition 1.4.** Let $I$ be an ideal of a ring $R$, and let $S$ be a finite subset of $R$. Let $U$ be the multiplicatively closed subset of $R$ generated by $S$.

(1) If $I + sR$ is radically finite for each $s \in S$ and $IR_U$ is radically finite, then $I$ is radically finite.

(2) If $\text{Spec}(R/sR)$ is Noetherian for each $s \in S$ and $\text{Spec} R_U$ is Noetherian, then $\text{Spec} R$ is Noetherian.

(3) If $R$ is a $\Gamma$-graded ring for some monoid $\Gamma$, each $s \in S$ is homogeneous with $h\text{-Spec}(R/sR)$ Noetherian and if $h\text{-Spec} R_U$ is Noetherian, then $h\text{-Spec} R$ is Noetherian.

The next corollary is a special case of Proposition 1.2.

**Corollary 1.5.** Suppose $S$ is a subset of a ring $R$ that generates $R$ as an ideal and let $I$ be an ideal of $R$. If $IR[1/s]$ is radically finite for each $s \in S$, then $I$ is radically finite.

If $R[1/s]$ has Noetherian spectrum for each $s \in S$, then $R$ has Noetherian spectrum.

In analogy with Corollary 1.3, it is a standard result in commutative algebra that if $SR = R$ and $R[1/s]$ is a Noetherian ring for each $s \in S$, then $R$ is a Noetherian ring. However, the analogue of Corollary 1.3 for the Noetherian property of a ring is false: There exists a non-Noetherian ring $R$ and an element $s \in R$ such that $R/sR$ and $R[1/s]$ are Noetherian. For example, let $X, Y$ be indeterminates over a field $k$, let $R := k[X, \{Y/X^n\}_{n=0}^\infty]$ and let $s = X$. Then $P = (\{Y/X^n\}_{n=0}^\infty)$ is a nonfinitely generated prime ideal of $R$, so $R$ is not Noetherian, although both $R/XR = k$ and $R[1/X] = k[1, Y, 1/X]$ are Noetherian. Incidentally, both the ideal $(P + XR)/XR = (0)$ of $R/XR$ and the ideal $PR[1/X]$ of $R[1/X]$ are principal.

Proposition 1.6 is the graded analogue of [8, Theorem 2.5].

**Proposition 1.6.** Suppose $\Gamma$ is a torsion-free cancellative commutative monoid with a group of quotients $G$ and $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ is a $\Gamma$-graded commutative ring. Fix $g \in G$, and consider the polynomial ring $R[X]$ as a graded extension ring of $R$ uniquely determined by defining $X$ to be a homogeneous element of degree $g$. If $h\text{-Spec} R$ is Noetherian, then $h\text{-Spec} R[X]$ is Noetherian.

**Proof.** Assume that $h\text{-Spec} R$ is Noetherian, but $h\text{-Spec} R[X]$ is not Noetherian. Then there exists a homogeneous prime ideal $P$ of $R[X]$ that is maximal with respect
to not being radically finite. Since $P \cap R = p$ is a homogeneous prime ideal of $R$ and h-Spec $R$ is Noetherian, we may pass from $R[X]$ to $R[X]/p[X] \cong (R/p)[X]$ and assume that $P \cap R = (0)$. Then $R$ is a graded domain and h-Spec $R$ is Noetherian.

Choose an element $f \in P$ having minimal degree $d$ as a polynomial in $R[X]$. By replacing $f$ by one of its nonzero homogeneous components, we may assume that $f = a_dX^d + a_{d-1}X^{d-1} + \cdots + a_0$, where the elements $a_i \in R$ are homogeneous elements of $R$ with $a_d \neq 0$. Since $P \cap R = (0)$, we have $d > 0$ and $a_d \notin P$. The maximality of $P$ with respect to not being radically finite implies $(P, a_d)R[X]$ is radically finite. Since $a_d^{-1}f$ is a polynomial of minimal degree in $PR[1/a_d][X]$ and since this polynomial is monic in $R[1/a_d][X]$, we see that $PR[1/a_d][X] = (f)$. But Proposition 1.4 (1) then implies that $P$ is radically finite, a contradiction.

We use Proposition 1.6 in the proof of Theorem 1.7.

**Theorem 1.7.** Let $R = \oplus_{\gamma \in \Gamma} R_\gamma$ be a $\Gamma$-graded commutative ring, where $\Gamma$ is a nonzero torsion-free cancellative commutative monoid such that its group of quotients $G$ is finitely generated. If h-Spec $R$ is Noetherian, then Spec $R$ is also Noetherian.

**Proof.** Up to a group isomorphism, we have $G \cong \mathbb{Z}^d$ for some positive integer $d$. Hence we may assume $G = \mathbb{Z}^d$. For $1 \leq i \leq d$, let $g_i$ be the element of $G$ having $1$ as its $i$-th coordinate and zeros elsewhere. Consider the graded polynomial extension $R[X] = R[X_1, \ldots, X_d]$ obtained by defining $X_i$ to be a homogeneous element of degree $g_i$ for $i = 1, \ldots, d$. Proposition 1.6 implies that h-Spec $R[X]$ is Noetherian.

We associate with each nonzero element $r \in R$ a homogeneous element $\bar{r} \in R[X]$ such that $\deg(\bar{r}) = (c_1, \ldots, c_d)$, where $c_j$ is the maximum of the $j$-th coordinates of the degrees of the nonzero homogeneous components of $r \in R$ as follows: let $r = r_1 + \cdots + r_k$ be the homogeneous decomposition of $r$; set $\bar{r} = \sum_{i=1}^k r_iX^{m_i}$, where $m_i = (c_1, \ldots, c_d) - \deg r_i$ for each $i$ and $X^{(a_1, \ldots, a_d)} = \prod_{i=1}^d X_i^{a_i}$ for each sequence $(a_1, \ldots, a_d)$ in $\mathbb{Z}^d$. We define $0 = 0$. With each ideal $I$ of $R$, let $\bar{I}$ denote the homogeneous ideal of $R[X]$ generated by $\{\bar{r} : r \in I\}$ ($\bar{r}$ is the homogenization of $r$ and $\bar{I}$ is the homogenization of $I$).

Let $\phi : R[X] \to R$ denote the $R$-algebra homomorphism defined by $\phi(X_i) = 1$ for $i = 1, \ldots, n$. Since $\phi$ is an $R$-algebra homomorphism and $\phi(\bar{r}) = r$ for each $r \in R$, for each ideal $I$ of $R$, we have $\phi(\bar{I}) = I$ (the meaning of $\phi$ is dehomogenization). Therefore the map $I \to \bar{I}$ is a one-to-one inclusion preserving correspondence of the set of ideals of $R$ into the set of homogeneous ideals of $R[X]$.

Let $I$ be an ideal of $R$. Since h-Spec $R[X]$ is Noetherian there exists a finite set $S$ such that $\text{rad} \bar{I} = \text{rad}(SR[X])$. We have $\text{rad} I = \text{rad} \phi(\bar{I}) = \text{rad}(\phi(S))R$, thus $I$ is radically finite. Therefore Spec $R$ is Noetherian.

The following corollary is immediate from Theorem 1.7.

**Corollary 1.8.** Let $R$ be an $\mathbb{N}$-graded or a $\mathbb{Z}$-graded ring. If h-Spec $R$ is Noetherian, then Spec $R$ is Noetherian.

Without the assumption in Theorem 1.7 that the group of quotients of $\Gamma$ is finitely generated, it is possible to have h-Spec $R$ be Noetherian and yet Spec $R$ not be Noetherian. For example, if $K$ is an algebraically closed field of characteristic zero and $\Gamma = \mathbb{Q}$, then $(0)$ is the only homogeneous prime ideal of the group ring $R := K[\mathbb{Q}]$ so h-Spec $R$ is Noetherian, but as we note in Theorem 2.4 below, Spec $R$ is not Noetherian.
2. The Noetherian spectra of monoid rings

Suppose \( A \) is a ring and \( M \) is a cancellative torsion-free commutative monoid. We consider the monoid ring \( A[M] \) as a graded ring with its natural \( M \)-grading where the nonzero elements of \( A \) are of degree zero. The monoid \( M \) is naturally identified with a subset of \( A[M] \). We write \( X^m \) for \( m \in M \subseteq A[M] \). Note that \( 0 \in M \) is identified with \( 1 \in A[M] \).

A \( \mathbb{Q} \)-monoid in a \( \mathbb{Q} \)-vector space \( V \) is an additive submonoid of \( V \) that is closed under multiplication by positive rationals. A subset of a \( \mathbb{Q} \)-monoid \( W \) is called a \( \mathbb{Q} \)-ideal of the \( \mathbb{Q} \)-monoid \( W \) if it is an ideal of the monoid \( W \) that is closed under multiplication by positive (that is, strictly positive) rationals.

If \( M \) is a cancellative torsion-free monoid with group of quotients \( G \), we denote by \( M^{(G)} \) the \( \mathbb{Q} \)-monoid generated by \( M \) in \( G \otimes \mathbb{Q} \); thus \( M = \{ qm | q > 0 \text{ in } \mathbb{Q}, m \in M \} \).

Remark 2.1. Let \( S \) be a subset of a monoid \( M \), let \( R \) be a ring, and let \( I \) be a homogeneous ideal of \( R[M] \) containing \( S \) and generated by monomials in \( M \). Then \( S \) generates \( I \) up to radical if \( S \) generates the \( \mathbb{Q} \)-ideal \( (I \cap M)^{(\mathbb{Q})} \) of \( M^{(\mathbb{Q})} \).

Remark 2.2. Suppose \( M \) is a cancellative torsion-free commutative monoid and \( k \) is a field. There is a natural one-to-one inclusion preserving correspondence between the homogeneous radical ideals of the monoid domain \( k[M] \) and the \( \mathbb{Q} \)-ideals of the \( \mathbb{Q} \)-monoid \( M^{(\mathbb{Q})} \). Indeed, to each \( \mathbb{Q} \)-ideal \( L \) of \( M^{(\mathbb{Q})} \) (which is generated by \( L \cap M \)) we make correspond the ideal of \( k[M] \) generated by \( L \cap M \).

Lemma 2.3. Suppose \( M \) is a torsion-free cancellative commutative monoid, \( A \) is a ring with Noetherian spectrum, and \( P \) is a homogeneous prime ideal of the monoid ring \( A[M] \). Then the following two conditions are equivalent:

1. The prime ideal \( P \) is radically finite in \( A[M] \).
2. The \( \mathbb{Q} \)-ideal \( (P \cap M)^{(\mathbb{Q})} \) of \( M^{(\mathbb{Q})} \) is finitely generated.

Proof. Since \( P \) is prime and homogeneous, it is generated by \( (P \cap A) \cup (P \cap M) \). Since \( \text{Spec } A \) is Noetherian, we see that \( P \) is radically finite iff the ideal in \( A[M] \) generated by \( P \cap M \) is radically finite iff the \( \mathbb{Q} \)-ideal \( (P \cap M)^{(\mathbb{Q})} \) of \( M^{(\mathbb{Q})} \) is finitely generated (Remark 2.1). This proves Lemma 2.3. \( \square \)

The following is an immediate corollary to Lemma 2.3.

Corollary 2.4. Let \( M \) be a torsion-free cancellative commutative monoid and let \( A \) be a ring with Noetherian spectrum. Then the following two conditions are equivalent:

1. The monoid ring \( A[M] \) has Noetherian homogeneous spectrum.
2. Each \( \mathbb{Q} \)-ideal in the \( \mathbb{Q} \)-monoid \( M^{(\mathbb{Q})} \) is finitely generated.

We denote the torsion-free rank of a monoid \( M \) by \( \text{rank } M \).

Proposition 2.5. Suppose \( A \) is a ring and \( M \) is a cancellative torsion-free commutative monoid.

1. If \( \text{Spec } A[M] \) is Noetherian, then \( \text{Spec } A \) is Noetherian and \( \text{rank } M \) is finite.
2. If \( \text{Spec } A \) is Noetherian and if \( \text{rank } M \leq 2 \), then \( h\text{-Spec } A[M] \) is Noetherian.

Proof. (1) \( \text{Spec } A \) is Noetherian since every ideal \( I \) of \( A \) satisfies \( I = IA[M] \cap A \), and if \( I \) is a radical ideal of \( A \), then \( IA[M] \) is a radical in \( A[M] \). On the other hand,
if rank $M$ is infinite, let $B$ be an infinite set of elements in $M$ which are linearly independent over $\mathbb{Q}$ in the $\mathbb{Q}$-vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$, where $G$ is the group of quotients of $M$. Then the ideal of $A[M]$ generated by the elements $X^b - 1$ for $b \in B$ is not radically finite. Therefore $h$-$\text{Spec } A[M]$ is not Noetherian.

(2) By Lemma 2.3, it suffices to show that each $\mathbb{Q}$-ideal in $M^{(\mathbb{Q})}$ is finitely generated. We may assume that $M \subseteq \mathbb{Q}^2$. Let $W$ be a nonempty $\mathbb{Q}$-ideal of $M^{(\mathbb{Q})}$. We show that $W$ is a finitely generated ideal of $\bar{W} := W \cup \{0\}$. If $W$ spans a one-dimensional subspace and $v$ is a nonzero element of $W$, then the $\mathbb{Q}$-ideal $W$ is generated by 0 if $v \in W$, and by $v$ otherwise. If $W$ spans $\mathbb{Q}^2$, then choose two linearly independent vectors in $W$. By changing coordinates, we may assume that these vectors are $(1, 0)$ and $(0, 1)$. If $W$ contains a vector $v$ with both coordinates strictly negative, then $W$ is generated by 0 as a $\mathbb{Q}$-ideal. Otherwise, define vectors $u$ and $v$ as follows: if $a = \min\{|y| : (1, y) \in W\}$ exists, let $u = (1, a)$; if the minimum does not exist, let $u = (1, 0)$. Similarly, define a vector $v$ with second coordinate 1. Then $u$ and $v$ generate $W$ as a $\mathbb{Q}$-ideal of $\bar{W}$. \hfill \qed

**Theorem 2.6.** Let $A$ be a ring with Noetherian spectrum and $M$ be a cancellative torsion-free commutative monoid. If the group of quotients of $M$ is finitely generated and if rank $M \leq 2$, then the monoid ring $A[M]$ has Noetherian spectrum.

On the other hand, if $A[M]$ has Noetherian spectrum and if $A$ contains an algebraically closed field of zero characteristic, then the group of quotients of $M$ is finitely generated.

**Proof:** Assume that the group of quotients of $M$ is finitely generated and that rank $M \leq 2$. By Proposition 2.3 (2), $A[M]$ has Noetherian homogeneous spectrum. By Theorem 1.7, $\text{Spec } A[M]$ is Noetherian.

For the second statement, assume that the group of quotients of $M$ is not finitely generated. By Proposition 2.5 (1), we may assume that $M$ has finite rank. It follows that there exists an element $s \in M$ that is divisible by infinitely many positive integers. Since $A$ contains all roots of unity and they are distinct, we obtain that over the element $X^s - 1 \in A[M]$ there are infinitely many minimal primes. Therefore $\text{Spec } A[M]$ is not Noetherian. \hfill \qed

With regard to Theorem 2.6, if the monoid $M$ is finitely generated, then it follows from [8, Theorem 2.5], that $\text{Spec } A[M]$ is Noetherian if $\text{Spec } A$ is Noetherian.

**Example 2.7.** Over a field $k$ of characteristic $p > 0$, there exists a monoid $M$ for which the group of quotients is not finitely generated and yet the monoid domain $k[M]$ has Noetherian spectrum. For example, if $M := \mathbb{Z}[[1/p^n]_{n=1}^\infty]$, then $k[M]$ is an integral purely inseparable extension of $k[\mathbb{Z}]$ and $\text{Spec } (k[M])$ is Noetherian.

A prime $\mathbb{Q}$-ideal of a $\mathbb{Q}$-monoid $M$ is a $\mathbb{Q}$-ideal $Q$ of $M$ that is a prime ideal, that is, if $a + b \in Q$, then either $a \in Q$ or $b \in Q$.

Let $S$ be a subset of a vector space over $\mathbb{Q}$. $S$ is $\mathbb{Q}$-convex if for any points $p, q$ in $S$ and rational $0 \leq t \leq 1$ we have $tp + (1 - t)q \in S$.

**Remark 2.8.** Let $M$ be a $\mathbb{Q}$-monoid in a vector space over $\mathbb{Q}$, and let $I$ be a subset of $M$ that is closed under addition and under multiplication by positive rationals; thus $I$ is a $\mathbb{Q}$-convex set. Then $I$ is an ideal of $M$ iff for any two points $p \in I$ and $q \in M$ and any rational $0 < t < 1$, we have $tp + (1 - t)q \in I$. Moreover, for $I$ as above, if $I$ is an ideal, then $I$ is prime iff the set $M \setminus I$ is $\mathbb{Q}$-convex.
We denote by $C$ the unit circle in $\mathbb{R}^2$, that is, $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. We let $C_\mathbb{Q} = C \cap \mathbb{Q}^2$. For any subset $S$ of $\mathbb{R}^n$ we denote by $S^+$ the set of points in $S$ with nonnegative coordinates.

**Example 2.9.** Let $n \geq 3$. Then there exists a cancellative torsion-free commutative monoid $M$ of rank $n$ such that the group of quotients of $M$ is finitely generated, but for any ring $A$ the homogeneous spectrum of $A[M]$ is not Noetherian. Furthermore, the monoid $M$ is completely integrally closed. Hence, if $A$ is an integrally closed (completely integrally closed) domain, then $A[M]$ is an integrally closed (completely integrally closed) domain.

First let $n = 3$. Let $W$ be the $\mathbb{Q}$-submonoid of $\mathbb{Q}^3$ generated by the set $\{(x, y, 1) : (x, y) \in C_\mathbb{Q}\}$. We claim that the $\mathbb{Q}$-ideal $W \setminus \{0\}$ of $W$ is not finitely generated; moreover, if $(p, 1) \in C_\mathbb{Q} \times \{1\}$, then $(p, 1)$ does not belong to the $\mathbb{Q}$-ideal of $W$ generated by $C_\mathbb{Q} \times \{1\} \setminus \{(p, 1)\}$. Indeed, by Remark 2.8, the set of points $(x, y, z) \in W$ such that $\frac{1}{z}(x, y) \neq p$ is a $\mathbb{Q}$-ideal of $W$ which does not contain $(p, 1)$.

Now let $A$ be any ring. Since the $\mathbb{Q}$-ideal generated by $W \setminus \{0\}$ in $W$ is not finitely generated, we obtain by Lemma 2.3 that the ideal in $A[M]$ generated by the nonzero elements of $M$ is not radically finitely closed; thus $\text{h-Spec } A[M]$ is not Noetherian.

If $n > 3$ let $\widetilde{M} = M \oplus \mathbb{Z}^{n-3}$, where $M$ is the monoid defined above. Then $\text{rank } \widetilde{M} = n$ and $\widetilde{M}$ satisfies our requirements.

Clearly, $M$ is a completely integrally closed monoid. Thus the assertions on $A[M]$ follow from [2, Corollary 12.7 (2) and Corollary 12.11 (2)].

We now elaborate on Example 2.9 but with $W$ replaced by $W^+$. As seen in Example 2.9, $R$ is a completely integrally closed domain, and $\text{h-Spec } R$ is not Noetherian. Moreover, $R = k[M]$ is a subring of the polynomial ring $k[X, Y, Z]$ and has fraction field $k(X, Y, Z)$. By [2, Theorem 21.4], $\dim R = 3$. It is interesting that the maximal homogeneous ideal $N$ of $R$ has height 3, but its homogeneous height (defined using just homogeneous prime ideals) is 2. Indeed, let $P \neq N$ be a nonzero prime homogeneous ideal of $R$. Let $Q$ be the $\mathbb{Q}$-ideal of $W$ generated by the points $(a, b, c)$ in $\mathbb{Q}^3$ such that $X^aY^bZ^c \in P$. Since $Q$ is a prime $\mathbb{Q}$-ideal of $W$ and since $C_\mathbb{Q}^+$ is dense in $C^+$, by Remark 2.8 we easily obtain that $Q$ contains $C_\mathbb{Q}^+ \times \{1\}$ except one point. Thus the homogeneous height of $N$ is at most 2. Since the $\mathbb{Q}$-ideal of $W$ generated by $C_\mathbb{Q}^+ \times \{1\}$ with one point removed is prime, we see that the homogeneous height of $N$ is 2.

On the other hand, $\text{ht } N = 3$. More generally, if $R$ is a $k$-subalgebra of a polynomial ring $k[\mathbf{X}] := k[X_1, \ldots, X_n]$, and $f$ the homogeneous ideal of $k(\mathbf{X})$ over a field $k$ with the same fraction field $k(\mathbf{X}_I)$, then $\text{ht}(\mathbf{X}_I k[\mathbf{X}] \cap R) = n$. Indeed, this prime ideal has height at most $n$ since $k(\mathbf{X}_I)$ has transcendence degree $n$ over $k$. Moreover, each nonzero ideal of $k[\mathbf{X}]$ has a nonzero intersection with $R$. Since the primes of height $n - 1$ of $k[\mathbf{X}]$ contained in $(\mathbf{X}_I k[\mathbf{X}] \cap R)$ intersect in zero, there exists such a prime ideal $P_{n-1}$ of $k[\mathbf{X}]$ such that $P_{n-1} \cap R \subseteq (\mathbf{X}_I k[\mathbf{X}] \cap R$. Repeating this argument, we find a strictly descending chain of prime ideals contained in $R$: $(\mathbf{X}_I k[\mathbf{X}] \cap R \supseteq (P_{n-1} \cap R) \supseteq \cdots \supseteq P_0 = (0)$.

This behavior where the dimension of the homogeneous spectrum of a graded integral domain $R$ is less than $\dim R$ also occurs in the case where $R$ is an $\mathbb{N}$-graded integral domain. For example, if $A$ is a one-dimensional quasilocal integral domain
such that the polynomial ring $\mathcal{A}[X]$ has dimension three \cite{9}, then the homogeneous spectrum of $\mathcal{A}[X]$ in its natural $\mathbb{N}$-grading has dimension two.

**References**


\[4\] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, Boston, 1985. MR 86e:14001


