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ON ONE PROBLEM OF UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING SMALL FUNCTIONS

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Abstract. In this paper, we show that if two non-constant meromorphic functions f and g satisfy $\overline{E}(a_j, k, f) = \overline{E}(a_j, k, g)$ for $j = 1, 2, \ldots, 5$, where a_j are five distinct small functions with respect to f and g, and k is a positive integer or ∞ with $k \geq 14$, then $f \equiv g$. As a special case this also answers the long-standing problem on uniqueness of meromorphic functions concerning small functions.

1. Introduction and main result

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane $\mathbb{C}$. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [1]. For any non-constant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f))$$

for $r \to \infty$ except possibly a set of r of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r, a) = S(r, f)$. Let $S(f)$ be the set of meromorphic functions in the complex plane $\mathbb{C}$ which are small functions with respect to f. Note that $\mathbb{C} \in S(f)$ and $S(f)$ is a field (see [2]).

If $f(z)$ is a non-constant meromorphic function, $a(z) \in S(f) \cup \{\infty\}$, and k is a positive integer or ∞, we denote by $\overline{E}(a, k, f)$ the set of distinct zeros of $f(z) - a(z)$ with multiplicities $\leq k$, where $f(z) - \infty$ means $1/f(z)$ (see [10, p.195]). In particular, we denote by $\overline{E}(a, \infty, f)$ the set of distinct zeros of $f(z) - a(z)$, and we denote it simply by $\overline{E}(a, f)$.

Let $f(z)$ and $g(z)$ be non-constant meromorphic functions and let $a(z) \in \{S(f) \cap S(g)\} \cup \{\infty\}$. We denote by $\overline{N}_0(r, a, f, g)$ the counting function of common zeros of $f(z) - a(z) = 0$ and $g(z) - a(z) = 0$ (ignoring multiplicities), each point counted only once. Let

$$\overline{N}_{12}(r, a, f, g) := \overline{N}(r, a, f) + \overline{N}(r, a, g) - 2\,\overline{N}_0(r, a, f, g). \tag{1.1}$$

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Then $\overline{N}_{12}(r, a, f, g)$ denotes the counting function of different solutions to $f(z) - a(z) = 0$ and $g(z) - a(z) = 0$ (see [7, p.107]). If

$$\overline{N}(r, a, f) - \overline{N}_0(r, a, f, g) = 0 \quad \text{and} \quad \overline{N}(r, a, g) - \overline{N}_0(r, a, f, g) = 0,$$

we say $f(z)$ and $g(z)$ share $a(z)$ **IM**. If

$$\overline{N}(r, a, f) - \overline{N}_0(r, a, f, g) = S(r, f) \quad \text{and} \quad \overline{N}(r, a, g) - \overline{N}_0(r, a, f, g) = S(r, g),$$

we say $f(z)$ and $g(z)$ share $a(z)$ **"IM"** (see [10, p.254]). It is obvious that if $f(z)$ and $g(z)$ share $a(z)$ **IM**, then $\overline{E}(a, f) = \overline{E}(a, g)$, and $\overline{N}_{12}(r, a, f, g) = 0$; if $f(z)$ and $g(z)$ share $a(z)$ **"IM"**, then $\overline{N}_{12}(r, a, f, g) = S(r, f) + S(r, g)$.

In the 1920s, R. Nevanlinna established the following famous second fundamental theorem:

Theorem A ([7, p.70])**.** *Let $f(z)$ be a non-constant meromorphic function, and let $a_1, a_2, \cdots, a_q$ be q (≥ 3) distinct elements in $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)\,T(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, a_j, f) + S(r, f).$$

In 1929, R. Nevanlinna proved the following well-known theorem on the uniqueness of meromorphic functions by making use of Theorem A:

Theorem B ([7, p.109], see also [1, p.48])**.** *If f and g are meromorphic functions sharing a_j **IM** for $j = 1, 2, \ldots, 5$, where $a_1, a_2, \ldots, a_5$ are five distinct elements in $\mathbb{C} \cup \{\infty\}$, then $f \equiv g$.*

In [7, p.111], R. Nevanlinna proved that Theorem B is sharp. It is natural to ask the following problem, which is the long-standing one:

Problem A (see [7, p.77 and p.109], see also [2]–[6], [8]–[11])**.** *Does Theorem B hold if $a_1, a_2, \ldots, a_5$ are five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$?*

In recent years, some partial results were obtained on Problem A (see [2]–[6], [8]–[11]). In this paper, we give a positive answer to Problem A. In fact, we prove more generally the following theorem:

Theorem 1. *Let f and g be non-constant meromorphic functions and let a_j $(j = 1, 2, \ldots, 5)$ be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If*

$$\overline{E}(a_j, k, f) = \overline{E}(a_j, k, g) \quad (j = 1, 2, \ldots, 5), \tag{1.2}$$

where k is a positive integer or ∞ with $k \geq 14$, then $f \equiv g$.

By Theorem 1, we obtain the following corollary:

Corollary 1. *Let f and g be non-constant meromorphic functions and let a_j $(j = 1, 2, \ldots, 5)$ be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_j **IM** $(j = 1, 2, \ldots, 5)$, then $f \equiv g$.*

It is obvious that Corollary 1 answers the above Problem A in the affirmative.

2. Some lemmas

Lemma 1 ([11], see also [2, Lemma 2] or [10, p.185]). *Let $f(z)$ be a non-constant meromorphic function and let $a_1, a_2, \ldots, a_5$ be five distinct elements in $S(f) \cup \{\infty\}$. Then*

$$2\,T(r,f) \le \sum_{j=1}^{5} \overline{N}(r,a_j,f) + S(r,f). \tag{2.1}$$

Let $f(z)$ be a non-constant meromorphic function, $a(z) \in S(f) \cup \{\infty\}$, and k be a positive integer. We denote by $\overline{N}_{k)}(r,a,f)$ the counting function of distinct zeros of $f(z) - a(z)$ with multiplicities $\le k$, by $\overline{N}_{(k+1}(r,a,f)$ the counting function of distinct zeros of $f(z) - a(z)$ with multiplicities $\ge k+1$, each point in these counting functions counted only once (see [10, p.190]).

Lemma 2. *Let $f(z)$ be a non-constant meromorphic function and let $a_1, a_2, \ldots, a_5$ be five distinct elements in $S(f) \cup \{\infty\}$, and k a positive integer. Then*

$$\sum_{j=1}^{5} \overline{N}_{(k+1}(r,a_j,f) \le \frac{3}{k}\,T(r,f) + S(r,f). \tag{2.2}$$

Proof. By Lemma 1, we can obtain (2.1). Noting that for $j = 1, 2, \ldots, 5$

$$k\,\overline{N}_{(k+1}(r,a_j,f) + \overline{N}(r,a_j,f) \le N(r,a_j,f) \le T(r,f) + S(r,f),$$

we have from (2.1)

$$k \sum_{j=1}^{5} \overline{N}_{(k+1}(r,a_j,f) + 2\,T(r,f) \le 5\,T(r,f) + S(r,f).$$

From this we get (2.2).

Lemma 3. *Let h be a non-constant meromorphic function, and let $a \in S(h)$ and $a \not\equiv 0$. Then*

$$m(r, \frac{a'h - ah'}{h - a}) = S(r,h), \tag{2.3}$$

$$m(r, \frac{a'h - ah'}{h(h - a)}) = S(r,h). \tag{2.4}$$

Proof. Noting

$$\frac{a'h - ah'}{h - a} = a' - \frac{a(h' - a')}{h - a}, \tag{2.5}$$

$$\frac{a'h - ah'}{h(h - a)} = \frac{h'}{h} - \frac{h' - a'}{h - a}, \tag{2.6}$$

and the lemma of the logarithmic derivative, we obtain Lemma 3.

The following lemma plays an important role in the proof of Theorem 1.

Lemma 4. *Let f and g be non-constant meromorphic functions and let $a_1, a_2, \ldots, a_5$ be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If $f \not\equiv g$, then*

$$\overline{N}_0(r,a_5,f,g) \le \sum_{j=1}^{4} \overline{N}_{12}(r,a_j,f,g) + S(r,f) + S(r,g). \tag{2.7}$$

3. Proof of Lemma 4

If $\overline{N}_0(r,a_5,f,g)=S(r,f)+S(r,g)$, (2.7) obviously holds. In the following we suppose

$$\overline{N}_0(r,a_5,f,g)\neq S(r,f)+S(r,g). \tag{3.1}$$

Set

$$L(w):=\frac{(w-a_1)(a_3-a_2)}{(w-a_2)(a_3-a_1)}. \tag{3.2}$$

Let $F(z):=L(f(z)),G(z):=L(g(z)),b_j:=L(a_j)\quad(j=1,2,\ldots,5)$. From (3.2) we have $b_1=0$, $b_2=\infty$, $b_3=1$,

$$T(r,F)=T(r,f)+S(r,f),\quad T(r,G)=T(r,g)+S(r,g). \tag{3.3}$$

Since $a_1,a_2,\ldots,a_5$ are five distinct elements in $\{S(f)\cap S(g)\}\cup\{\infty\}$, from (3.2) we know that $b_1,b_2,\ldots,b_5$ are five distinct elements in $\{S(F)\cap S(G)\}\cup\{\infty\}$. Thus, $b_4,b_5\not\equiv 0,1,\infty$ and $b_4\not\equiv b_5$. Noting $f\not\equiv g$, we have

$$F\not\equiv G. \tag{3.4}$$

From (3.1) and (3.3), we get

$$\overline{N}_0(r,b_5,F,G)\neq S(r,F)+S(r,G). \tag{3.5}$$

Set

$$H:=\frac{F'(a'G-aG')(F-G)}{F(F-1)G(G-a)}-\frac{G'(a'F-aF')(F-G)}{G(G-1)F(F-a)}, \tag{3.6}$$

where $a=b_4\,(\not\equiv 0,1)$. Then we have from (3.6)

$$H=\frac{(F-G)Q}{F(F-1)(F-a)G(G-1)(G-a)}, \tag{3.7}$$

where

$$Q=F'(a'G-aG')(G-1)(F-a)-G'(a'F-aF')(F-1)(G-a). \tag{3.8}$$

By a simple computation,

$$\begin{aligned} Q=\;&a'FF'G^2-a'FF'G-a(a-1)FF'G'-aa'F'G^2+aa'F'G\\ &-a'F^2GG'+a'FGG'+a(a-1)F'GG'+aa'F^2G'-aa'FG'. \end{aligned} \tag{3.9}$$

Suppose that $H\equiv 0$. From (3.4) and (3.6) we obtain

$$\frac{F'(a'G-aG')}{(F-1)(G-a)}\equiv\frac{G'(a'F-aF')}{(G-1)(F-a)}. \tag{3.10}$$

If a is a constant, noting $a\neq 1$, from (3.10) we get $F\equiv G$, which contradicts (3.4). Thus, a is not a constant. From (3.10) we have

$$\frac{F'(a'G-aG')}{G'(a'F-aF')}-1\equiv\frac{(F-1)(G-a)}{(G-1)(F-a)}-1.$$

Thus,

$$\frac{a'[(F'-G')G-(F-G)G']}{G'(a'F-aF')}\equiv\frac{(1-a)(F-G)}{(G-1)(F-a)}.$$

From this we get

$$\frac{F'-G'}{F-G} \equiv \frac{(1-a)G'(a'F-aF')}{a'G(G-1)(F-a)} + \frac{G'}{G}. \tag{3.11}$$

By (3.5), we know that there is a point z_0 such that z_0 is a common zero of $F-b_5$ and $G-b_5$ that is not a zero or a pole of a, a', b_5, b_5-1, b_5-a. It is obvious that z_0 is a pole of the left-hand side of (3.11), and not a pole of the right-hand side of (3.11), which is a contradiction. Thus,

$$H \not\equiv 0. \tag{3.12}$$

Suppose that z_n is a common zero of $F-b_5$ and $G-b_5$ that is not a zero or pole of a, b_5, b_5-1, b_5-a. It is obvious that z_n is a zero of $F-G$, and z_n is not a pole of

$$\frac{Q}{F(F-1)(F-a)G(G-1)(G-a)},$$

where Q is given by (3.8). By (3.7), we know that z_n is a zero of H. Again by (3.12) we obtain

$$\begin{aligned}\overline{N}_0(r,b_5,F,G) &\leq N(r,0,H)+S(r,F)+S(r,G)\\ &\leq m(r,H)+N(r,H)+S(r,F)+S(r,G).\end{aligned} \tag{3.13}$$

From (3.6) we have

$$\begin{aligned}H =& \frac{F'}{F-1}\cdot\frac{a'G-aG'}{G(G-a)} - \left(\frac{F'}{F-1}-\frac{F'}{F}\right)\cdot\frac{a'G-aG'}{G-a}\\ &-\left(\frac{G'}{G-1}-\frac{G'}{G}\right)\cdot\frac{a'F-aF'}{F-a}+\frac{G'}{G-1}\cdot\frac{a'F-aF'}{F(F-a)}.\end{aligned} \tag{3.14}$$

Again by Lemma 3 and the lemma of the logarithmic derivative we obtain

$$m(r,H) = S(r,F)+S(r,G). \tag{3.15}$$

Substituting (3.15) into (3.13) we have

$$\overline{N}_0(r,b_5,F,G) \leq N(r,H)+S(r,F)+S(r,G). \tag{3.16}$$

Next we estimate on $N(r,H)$.

By (3.6), we know that the poles of H only possibly occur from the zeros of F, G, $F-1$, $G-1$, $F-a$ and $G-a$, the poles of F, G and a. Let S_0 be the set of all zeros, 1-points and poles of a, and let for $j=1,2,3,4$

$$A_j := \{z|F(z)-b_j(z)=0\}\setminus S_0, \qquad B_j := \{z|G(z)-b_j(z)=0\}\setminus S_0,$$

where $b_1=0$, $b_2=\infty$, $b_3=1$ and $b_4=a$. Thus, the poles of H occur possibly only from the set

$$\bigcup_{1\leq p\leq 4} A_p \bigcup_{1\leq q\leq 4} B_q \bigcup S_0.$$

Let

$$S_1 := \bigcup_{1\le p\le 4} \{A_p \bigcap B_p\},$$
$$S_2 := \{\bigcup_{1\le p\le 4} A_p\} \setminus \{\bigcup_{1\le q\le 4} B_q\},$$
$$S_3 := \{\bigcup_{1\le q\le 4} B_q\} \setminus \{\bigcup_{1\le p\le 4} A_p\},$$
$$S_4 := \bigcup_{\substack{1\le p\le 4\\ 1\le q\le 4\\ p\ne q}} \{A_p \bigcap B_q\}.$$

It is clear that S_1 is a set of the common zeros of $F-b_j$ and $G-b_j$ $(j=1,2,3,4)$; S_2 is a set of the zeros of $F-b_j$ $(j=1,2,3,4)$ that is not the zeros of $G-b_k$ $(k=1,2,3,4)$; S_3 is a set of the zeros of $G-b_j$ $(j=1,2,3,4)$ that is not the zeros of $F-b_k$ $(k=1,2,3,4)$; S_4 is a set of the zeros of $F-b_j$ that is the zeros of $G-b_k$, where $1\le j,k\le 4$ and $j\ne k$. From this we have

$$\bigcup_{1\le j\le 4} S_j = \bigcup_{1\le p\le 4} A_p \bigcup_{1\le q\le 4} B_q.$$

Thus, the poles of H occur possibly only from the set

$$\bigcup_{1\le j\le 4} S_j \bigcup S_0.$$

Since b_1, b_2, b_3 and b_4 are four distinct elements in $\{S(F)\cap S(G)\}\cup\{\infty\}$, the contribution of S_0 to $N(r,H)$ is at most $S(r,F)+S(r,G)$. We next estimate the contribution of $\bigcup_{1\le j\le 4} S_j$ to $N(r,H)$. We discuss four cases:

Case 1. The contribution of S_1 to $N(r,H)$.

We distinguish four subcases:

Subcase 1.1. Suppose that $z_{11}\in A_1\cap B_1$, and assume that z_{11} is a zero of F of order p_1 and G of order q_1. Then from (3.9), we know that z_{11} is a zero of Q of order at least p_1+q_1-1. Noting that z_{11} is a zero of $F-G$, from (3.7) we deduce that z_{11} is not a pole of H.

Subcase 1.2. Suppose that $z_{12}\in A_2\cap B_2$, and assume that z_{12} is a pole of F of order p_2 and G of order q_2. From (3.9), we know that z_{12} is a pole of Q of order at most $2p_2+2q_2+1$. Noting that z_{12} is a pole of $F-G$ of order at most $\max\{p_2,q_2\}$, from (3.7), we have that z_{12} is not a pole of H.

Subcase 1.3. Suppose that $z_{13}\in A_3\cap B_3$. Noting that z_{13} is a zero of $F-G$, a simple pole of $\frac{F'}{F-1}$ and $\frac{G'}{G-1}$, from (3.6) we have that z_{13} is not a pole of H.

Subcase 1.4. Suppose that $z_{14}\in A_4\cap B_4$. From (2.6), we know that z_{14} is a simple pole of $\frac{a'F-aF'}{F(F-a)}$ and $\frac{a'G-aG'}{G(G-a)}$. Noting that z_{14} is a zero of $F-G$, from (3.6) we see that z_{14} is not a pole of H.

From the above, it follows that the points in S_1 are not poles of H. Thus, the contribution of S_1 to $N(r,H)$ is 0.

Case 2. The contribution of S_2 to $N(r, H)$.

We distinguish four subcases:

Subcase 2.1. Suppose that $z_{21} \in A_1$ and $z_{21} \notin \bigcup_{1 \le q \le 4} B_q$. Then z_{21} is a zero of F, not a zero of G, $1/G$, $G-1$ and $G-a$. From (3.6), we have that z_{21} is a pole of H of order at most 1.

Subcase 2.2. Suppose that $z_{22} \in A_2$ and $z_{22} \notin \bigcup_{1 \le q \le 4} B_q$. Then z_{22} is a pole of F, not a zero of G, $1/G$, $G-1$ and $G-a$. From (3.6), we have that z_{22} is a pole of H of order at most 1.

Subcase 2.3. Suppose that $z_{23} \in A_3$ and $z_{23} \notin \bigcup_{1 \le q \le 4} B_q$. Then z_{23} is a zero of $F-1$, not a zero of G, $1/G$, $G-1$ and $G-a$. From (3.6), we have that z_{23} is a pole of H of order at most 1.

Subcase 2.4. Suppose that $z_{24} \in A_4$ and $z_{24} \notin \bigcup_{1 \le q \le 4} B_q$. Then z_{24} is a zero of $F-a$, not a zero of G, $1/G$, $G-1$ and $G-a$. From (2.6) and (3.6), we have that z_{24} is a pole of H of order at most 1.

From the above, we know that the points in S_2 are poles of H of order at most 1.

Case 3. The contribution of S_3 to $N(r, H)$.

As with Case 2, we have that the points in S_3 are poles of H of order at most 1.

Case 4. The contribution of S_4 to $N(r, H)$.

Suppose that $z_4 \in S_4$. Then $z_4 \in A_p$ and $z_4 \in B_q$, where $1 \le p, q \le 4$ and $p \ne q$. Without loss of generality we can assume that $z_4 \in A_1$ and $z_4 \in B_2$. Then, z_4 is a zero of F, and a pole of G. From (2.6) and (3.6), we have that z_4 is a pole of H of order at most 2. Thus, the points in S_4 are poles of H of order at most 2.

Noting that each point of S_2 and S_3 is counted once, each point of S_4 is counted twice in

$$\sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G).$$

From the above and (1.1) we obtain that the contribution of $\bigcup_{1 \le j \le 4} S_j$ to $N(r, H)$ is at most

$$\sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G).$$

Thus,

$$N(r, H) \le \sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G). \tag{3.17}$$

Substituting (3.17) into (3.16) we have

$$\overline{N}_0(r, b_5, F, G) \le \sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G),$$

i.e.,

$$\overline{N}_0(r, a_5, f, g) \le \sum_{j=1}^{4} \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g),$$

and Lemma 4 is proved.

4. Theorem 2 and its proof

Using Lemma 4, we can prove the following Theorem:

Theorem 2. *Let f and g be non-constant meromorphic functions and let a_j $(j = 1, 2, \ldots, 5)$ be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If $f \not\equiv g$, then*

$$2T(r, f) + 2T(r, g) \leq 9 \sum_{j=1}^{5} \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g). \tag{4.1}$$

Proof. By Lemma 4 we have

$$\overline{N}_0(r, a_5, f, g) \leq \sum_{j=1}^{4} \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g). \tag{4.2}$$

Noting

$$\overline{N}(r, a_5, f) + \overline{N}(r, a_5, g) = 2\overline{N}_0(r, a_5, f, g) + \overline{N}_{12}(r, a_5, f, g),$$

from this and (4.2) we get

(4.3)

$$\overline{N}(r, a_5, f) + \overline{N}(r, a_5, g) \leq 2 \sum_{j=1}^{5} \overline{N}_{12}(r, a_j, f, g) - \overline{N}_{12}(r, a_5, f, g) + S(r, f) + S(r, g).$$

In the same manner as above, we have for $i = 1, 2, 3, 4$

(4.4)

$$\overline{N}(r, a_i, f) + \overline{N}(r, a_i, g) \leq 2 \sum_{j=1}^{5} \overline{N}_{12}(r, a_j, f, g) - \overline{N}_{12}(r, a_i, f, g) + S(r, f) + S(r, g).$$

By Lemma 1, we have

$$2T(r, f) + 2T(r, g) \leq \sum_{i=1}^{5} \overline{N}(r, a_i, f) + \sum_{i=1}^{5} \overline{N}(r, a_i, g) + S(r, f) + S(r, g). \tag{4.5}$$

Substituting (4.3) and (4.4) into (4.5) we obtain (4.1), and Theorem 2 is thus proved. □

5. Proof of Theorem 1

Suppose that $f \not\equiv g$. By Theorem 2, we can obtain (4.1).

If $k = \infty$, by hypothesis we have $\overline{E}(a_j, \infty, f) = \overline{E}(a_j, \infty, g)$ $(j = 1, 2, \ldots, 5)$. Thus,

$$\overline{N}_{12}(r, a_j, f, g) = 0 \quad (j = 1, 2, \ldots, 5). \tag{5.1}$$

Substituting (5.1) into (4.1) we obtain

$$2T(r, f) + 2T(r, g) = S(r, f) + S(r, g).$$

This is a contradiction. Next, we assume that k is a positive integer.

By hypothesis we have

$$\overline{E}(a_j, k, f) = \overline{E}(a_j, k, g) \quad (j = 1, 2, \ldots, 5).$$

From this we obtain for $j = 1, 2, \ldots, 5$

$$\overline{N}_{12}(r, a_j, f, g) \leq \overline{N}_{(k+1}(r, a_5, f) + \overline{N}_{(k+1}(r, a_5, g) + S(r, f) + S(r, g). \tag{5.2}$$

Substituting (5.2) into (4.1) we obtain

(5.3)

$$2T(r,f)+2T(r,g)\leq 9\sum_{j=1}^{5}\overline{N}_{(k+1}(r,a_j,f)+9\sum_{j=1}^{5}\overline{N}_{(k+1}(r,a_j,g)+S(r,f)+S(r,g).$$

By Lemma 2, we have

$$\sum_{j=1}^{5}\overline{N}_{(k+1}(r,a_j,f)\leq \frac{3}{k}\,T(r,f)+S(r,f), \tag{5.4}$$

$$\sum_{j=1}^{5}\overline{N}_{(k+1}(r,a_j,g)\leq \frac{3}{k}\,T(r,g)+S(r,g). \tag{5.5}$$

Substituting (5.4) and (5.5) into (5.3) we obtain

$$2T(r,f)+2T(r,g)\leq \frac{27}{k}T(r,f)+\frac{27}{k}T(r,g)+S(r,f)+S(r,g),$$

which contradicts the assumption $k\geq 14$. Thus, $f\equiv g$.

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