DIRECT SUMS OF LOCAL TORSION-FREE ABELIAN GROUPS

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Abstract. The category of local torsion-free abelian groups of finite rank is known to have the cancellation and $n$-th root properties but not the Krull-Schmidt property. It is shown that 10 is the least rank of a local torsion-free abelian group with two non-equivalent direct sum decompositions into indecomposable summands. This answers a question posed by M.C.R. Butler in the 1960’s.

1. Introduction

Let $TF$ denote the category of local torsion-free abelian groups of finite rank, where an abelian group $G$ is local if there is a fixed prime $p$ with $qG = G$ for each prime $q \neq p$. Each $M$ in $TF$ has the cancellation property (if $M \oplus N$ is isomorphic to $M \oplus K$ in $TF$, then $N$ is isomorphic to $K$), and the $n$-th root property (if the direct sum $M^n$ of $n$ copies of $M$ is isomorphic to $N^n$ for some $N$ in $TF$, then $M$ is isomorphic to $N$) [Lady 75]. An $M$ in $TF$ is a Krull-Schmidt group if any two direct sum decompositions of $M$ into indecomposable summands are equivalent, i.e. unique up to isomorphism and order of summands.

M.C.R. Butler, in an unpublished note dating from the 1960’s, constructed an example of a local torsion-free abelian group of rank 16 that is not a Krull-Schmidt group (see [Arnold 82]) and asked for the smallest such rank. An example of a rank-10 local torsion-free abelian group that is not a Krull-Schmidt group is given in [Arnold 01].

This paper is devoted to showing that 10 is the minimum such rank, i.e. if $M \in TF$ with rank $M \leq 9$, then $M$ is a Krull-Schmidt group. Many arguments in this paper carry over directly to torsion-free modules of finite rank over valuation domains, keeping in mind that the existence and minimal rank of a non-Krull-Schmidt module depends on the structure of the valuation domain; see [Goldsmith May 99] and references.

The quasi-isomorphism category $TF_\mathbb{Q}$ of $TF$ is an additive category with objects those of $TF$ but with morphism sets $\mathbb{Q} \otimes \text{Hom}(M, N)$ for $M, N \in TF$ and $\mathbb{Q}$ the rational numbers. The category $TF_\mathbb{Q}$ is a Krull-Schmidt category in that each object in $TF_\mathbb{Q}$ can be written uniquely, up to isomorphism in $TF_\mathbb{Q}$ and order, as a finite direct sum of indecomposable objects in $TF_\mathbb{Q}$; see [Walker 64]. This is because

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an indecomposable object $M$ in $TF_Q$ has a local endomorphism ring $\mathbb{Q}EndM$ in $TF_Q$. Indecomposable objects in $TF_Q$ are called strongly indecomposable groups, isomorphism in $TF_Q$ is called quasi-isomorphism, and summands of groups in $TF_Q$ are called quasi-summands.

Each $M \in TF$ is a torsion-free $\mathbb{Z}(p)$-module, where $\mathbb{Z}(p)$ is the localization of the integers at the prime $p$. The rank of $M$ as a group is equal to the rank of $M$ as a $\mathbb{Z}(p)$-module, and $\text{Hom}_{\mathbb{Z}}(M, N) = \text{Hom}_{\mathbb{Z}(p)}(M, N)$ for each $M, N \in TF$. Hence, rank $M = 1$ if and only if $M$ is isomorphic to either $\mathbb{Z}(p)$ or $\mathbb{Q}$. Define $p$-rank $M$ to be the $\mathbb{Z}/p\mathbb{Z}$-dimension of $M/pM$, a finite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space. Notice that $p$-rank $M \leq$ rank $M$, $M$ is divisible if and only if $p$-rank $M = 0$, and $M$ is isomorphic to a free $\mathbb{Z}(p)$-module if and only if $p$-rank $M = \text{rank} M$. Moreover, if $N$ is a $\mathbb{Z}(p)$-submodule of $M$, then the $p$-rank of the pure submodule of $M$ generated by $N$ is less than or equal to the $p$-rank of $N$ and if $M$ is quasi-isomorphic to $N \oplus K$, then $p$-rank $M = p$-rank $N + p$-rank $K$ [Arnold 72]. If $M$ is reduced (no proper divisible subgroups), then $M$ is isomorphic to a pure subgroup of $M^*$, the completion of $M$ in the $p$-adic topology. Moreover, $M^*$ is a free $\mathbb{Z}^*$-module with rank equal to the $p$-rank of $M$, where $\mathbb{Z}^*$ is the $p$-adic completion of $\mathbb{Z}(p)$. Each endomorphism of $M$ lifts to a unique $\mathbb{Z}^*$-endomorphism of $M^*$, whence $End M$ is a pure subring of $End_{\mathbb{Z}^*}M^*$.

2. Uniqueness of direct sums

A group $M \in TF$ has the one-sided UDS property if whenever $M \oplus N$ is isomorphic to $K_1 \oplus ... \oplus K_n \in TF$ with $M$ quasi-isomorphic to each $K_j$, then $M$ is isomorphic to some $K_j$. The group $M$ has the UDS property if whenever $N_1 \oplus ... \oplus N_m$ is isomorphic to $K_1 \oplus ... \oplus K_n \in TF$ with $M$ quasi-isomorphic to each $N_i$ and $K_j$, then $m = n$ and there is a relabelling of indices with each $N_i$ isomorphic to $K_i$.

Given a strongly indecomposable $M$ in $TF$, there is a faithful $G_M \in TF$ quasi-isomorphic to $M$ such that $End G_M/NEnd G_M$ is a maximal order in the division algebra $\mathbb{Q}End M/J\mathbb{Q}End M$, where $J\mathbb{Q}End M$ is the Jacobson radical of the finite dimensional $\mathbb{Q}$-algebra $\mathbb{Q}End M$, $NEnd G_M = End G_M \cap J\mathbb{Q}End M$ is a nilpotent ideal of $End M$, and $G_M$ is faithful if $IGM \neq G_M$ for each maximal right ideal $I$ of $End G_M$ [Arnold 01]. A maximal right ideal $J$ of $End G_M/pEnd G_M$ has the unique maximal condition if whenever $I$ is a non-zero right ideal and $J$ is a unique maximal right ideal of $End G_M/pEnd G_M$ containing $I$, then $I = J$.

The first lemma is the local version of [Arnold 01] Theorem 1.5).

**Lemma 1.** The following statements are equivalent for a strongly indecomposable $N$ in $TF$, where $G_N$ is as defined above:

(i) $N$ has the UDS property.

(ii) Each group in $TF$ quasi-isomorphic to $N$ has the one-sided UDS property.

(iii) Either $End G_N$ is a local ring or else $End G_N$ has exactly two maximal right ideals $M_1$ and $M_2$ such that $M_1$ is a principal right ideal of $End G_N$, $G_N/M_1G_N \cong \mathbb{Z}/p\mathbb{Z}$, and $M_1/pEnd G_N$ has the unique maximal condition in $End G_N/pEnd G_N$.

Following are non-trivial examples of groups in $TF$ with the UDS property.

**Example 1.** If $N \in TF$ is strongly indecomposable with $p$-rank $N \leq 2$, then $N$ has the (one-sided) UDS property.
Proof. It suffices to confirm the conditions of Lemma 1(iii). If $p$-rank $N \leq 1$, then
$p$-rank $G_N \leq 1$, since $G_N$ is quasi-isomorphic to $N$. Thus, either $p$-rank $G_N = 0$
and $G_N \cong \mathbb{Q}$ or else $p$-rank $G_N = 1$, $G_N \cong \mathbb{Z}^*$, and $\text{End} G_N$ is isomorphic to a pure
subring of $\mathbb{Z}^* \cong \text{End}_\mathbb{Z} \mathbb{Z}^*$. In either case, $\text{End} G_N$ is a local ring, as desired.

Now assume that $p$-rank $N = p$-rank $G_N = 2$ and $\text{End} G_N$ is not a local ring.
Let $M_1, \ldots, M_n$ be distinct maximal right ideals of $\text{End} G_N$ with $n \geq 2$. Then
$G_N/(M_1 \cap \ldots \cap M_n)G_N \cong G_N/M_1G_N \oplus \ldots \oplus G_N/M_nG_N$ and $p\text{End} G_N \subseteq J\text{End} G_N \subseteq M_1 \cap \ldots \cap M_n$. Since $p$-rank $G_N = 2$ and $\text{End} G_N$ is faithful, it follows that
$n = 2$, $p\text{End} G_N = M_1 \cap M_2 = J\text{End} G_N$, each $G_N/M_iG_N \cong \mathbb{Z}/p\mathbb{Z}$, and each
$M_i/p\text{End} G_N$ has the unique maximal condition. Finally, each $M_i$ is principal as
an application of Nakayama’s Lemma, because $p\text{End} G_N = J\text{End} G_N$ and $\text{End} G_N/p\text{End} G_N$ is finite.

The next lemma is used for an induction step in the proof of the main theorem.

**Lemma 2.** Assume that $M = N \oplus N' = K_1 \oplus \ldots \oplus K_n \in TF$. There are subgroups
$K'_i$ of $K_i$ with $N \oplus N' = N \oplus K'_1 \oplus \ldots \oplus K'_n$ if either
(a) [Warfield 72] $\text{End} N$ is a local ring or
(b) [Arnold Lady 75] $N$ and $N'$ have no quasi-summands in common.

In this case, $N'$ is isomorphic to $K'_1 \oplus \ldots \oplus K'_n$.

An indecomposable $M \in TF$ is purely indecomposable if $p$-rank $M = 1$. In this case, $\text{End} M$ is a local ring, being a pure subring of $\mathbb{Z}^* \cong \text{End}_\mathbb{Z} \mathbb{Z}^*$. Dually, $M$ is co-purely indecomposable if $M$ is indecomposable with rank $M = p$-rank $M + 1$.
There is a contravariant duality $F$ on $TF_\mathbb{Q}$ sending a purely indecomposable group $M$ to a co-purely indecomposable group $F(M)$ [Arnold 72] (see [Lady 77] for an alternate definition of the duality). Hence, $\mathbb{Q}\text{End} F(M)$ is isomorphic to $\mathbb{Q}\text{End} M$, a subring of the $p$-adic rationals $\mathbb{Q}^*$.

Following are some elementary properties of purely indecomposable and co-
purely indecomposable groups that are consequences of the definitions and the
duality $F$.

**Proposition 1** ([Arnold 72]). Let $M \in TF$.

(a) If $M$ is purely indecomposable, then:
(i) $\text{End} M$ is a pure subring of $\mathbb{Z}^*$;
(ii) each pure subgroup of $M$ is strongly indecomposable;
(iii) if $K \in TF$ is a homomorphic image of $M$ with rank $K < \text{rank} M$, then
$K$ is divisible; and
(iv) two purely indecomposable groups $M$ and $N$ in TF are isomorphic if and
only if $\text{rank} M = \text{rank} N$ and $\text{Hom}(M, N) \neq 0$; equivalently $M$ and $N$
are quasi-isomorphic.

(b) If $M$ is co-purely indecomposable, then:
(i) $\text{End} M$ is isomorphic to a subring of $\mathbb{Q}^*$, hence an integral domain;
(ii) each torsion-free homomorphic image of $M$ is strongly indecomposable;
(iii) if $K$ is a pure subgroup of $M$ with rank $K < \text{rank} M$, then $M$ is a free
$\mathbb{Z}_p^*$-module; and
(iv) two co-purely indecomposable groups $M$ and $N$ are quasi-isomorphic if
and only if $\text{rank} M = \text{rank} N$ and $\text{Hom}(M, N) \neq 0$.

(c) If $M$ is indecomposable with rank $\geq 2$ and $N$ is co-purely indecomposable with
rank $M < \text{rank} N$, then $\text{Hom}(M, N) = 0$. 

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(d) If \( M \) is purely indecomposable with rank \( \geq 3 \) and \( N \) is co-purely indecomposable with rank \( M = \text{rank} \, N \), then \( \text{Hom}(M, N) = 0 \).

**Remark 1.** There is a co-purely indecomposable \( M \in TF \) with \( p \)-rank 3 and rank 4 that does not have either UDS property. In this case \( \text{End} \, M \) has 3 maximal right ideals and \( M \) is the summand of a non-Krull-Schmidt group of rank 12 [Arnold 01]. In view of the following lemma, this group cannot be a summand of a non-Krull-Schmidt group of rank 8 = 2(rank \( M \)). On the other hand, if \( M \in TF \) and \( \text{End} \, M \) has at least 4 maximal right ideals, then \( M \) is a summand of a non-Krull-Schmidt group of rank equal to 2(rank \( M \)).

The next lemma is used in the proof of the main theorem. In view of Proposition 1(b)(i), the hypotheses are satisfied if \( N \) is co-purely indecomposable.

**Lemma 3.** Assume that \( N \in TF \) with \( \text{End} \, N \) an integral domain and \( M = N \oplus N' = K_1 \oplus K_2 \in TF \) with each \( K_i \) indecomposable. If \( p \)-rank \( N \leq 3 \), then \( N \) is isomorphic to some \( K_i \).

**Proof.** The proof is a variation on a proof given in [Arnold 01]. Let \( \pi \) be a projection of \( M \) onto \( N \) with kernel \( N' \), and \( \pi_i \) a projection of \( M \) onto \( K_i \) for each \( i \) with \( 1_M = \pi_1 + \pi_2 \). Then \( 1_N = \beta_1 + \beta_2 \), where \( \beta_i \in \text{End} \, N \) is the restriction of \( \pi_i \) to \( N \) and \( \beta_i(N) \) is contained in a subgroup \( \pi(K_i) \) of \( N \). Since \( \text{End} \, N \) is an integral domain, \( \mathbb{Q} \text{End} \, N \) is a field and each \( \beta_i \) is a unit in \( \mathbb{Q} \text{End} \, N \).

For each \( 1 \leq i \leq 2 \), let \( I_i = \beta_i \text{End} \, N \), a right ideal of \( \text{End} \, N \). Then \( \text{End} \, N = I_1 + I_2 \), since \( 1_N = \beta_1 + \beta_2 \). Each \( (\text{End} \, N)/I_i \) is bounded by a power of \( p \) since \( \beta_i \) is a unit in \( \mathbb{Q} \text{End} \, N \). Moreover, \( I_i N \) is contained in \( A_i = \pi(K_i) \) so that \( [N : A_i] \) is finite. It now suffices to prove that \( N \cong A_i \) for some \( i \), in which case \( N \cong K_i \).

If some \( [N : A_i] = 1 \), then \( N = A_i \) and the proof is complete. The next step is to assume that each \( [N : A_i] \neq 1 \) and reduce to the case that each \( [N : A_i] = p \). Suppose, by way of induction, that \( [N : A_i] \neq p \). Choose \( x \in N \setminus A_i \) such that \( px \in A_i \). Then \( A_i \subset A_i + Zx \). If \( N \) and \( A_i + Zx \) are not isomorphic, then replace \( A_i \) by \( A'_i = A_i + Zx \). If \( N \cong A_i + Zx \), say \( f \in \text{End} \, N \) with \( f(N) = A_i + Zx \), then replace \( A_i \) by \( A'_i = f^{-1}(A_i) \). In either case, \( [N : A'_i] \) is a proper divisor of \( [N : A_i] \).

The substitution of \( A'_i \) for \( A_i \) doesn’t change the hypothesis that \( \text{End} \, N = I_1 + I_2 \) for right ideals \( I_i \) of bounded index with \( I_i \) contained in \( A_i \). In particular, \( I'_i = f^{-1}I_i \) is an ideal of \( \text{End} \, N \) (since \( I_i N \) is a subgroup of \( f(N) \)), \( I'_i N \) is contained in \( A'_i \), and \( I'_i + \Sigma\{I_j : j \neq i\} \cong \text{End} \, N \) (since \( f \text{End} \, N = \Sigma fI_i \) is contained in \( I_1 + \Sigma\{fI_j : j \neq i\} \)). If \( N \cong A'_i \), then, by the construction of \( A'_i \), \( N \cong A_i \). By induction, and the fact that \( [N : A'_i] \) is a proper divisor of \( [N : A_i] \), the \( A_i \)’s can be chosen with each \( [N : A_i] = p \).

At this stage, \( \text{End} \, N = I_1 + I_2 \) for right ideals \( I_i \) of finite index in \( \text{End} \, N \) with \( I_i N \) contained in a subgroup \( A_i \) of \( N \) and \( [N : A_i] = p \) for each \( i \). Replace \( I_i \) by \( I_i + p\text{End} \, N \), if necessary, to guarantee that \( p\text{End} \, N \) is contained in \( I_i \) for each \( i \). But \( p \)-rank \( N \leq 3 \), \( p\text{End} \, N \subseteq J\text{End} \, N \), \( \text{End} \, N \) is an integral domain, and \( N/(M_1 \cap \ldots \cap M_n)N \cong N/M_1 N \oplus \ldots \oplus N/M_n N \) for maximal ideals \( M_i \) of \( \text{End} \, N \). Hence, \( \text{End} \, N \) has at most 3 maximal right ideals \( M_1, M_2, \) and \( M_3 \) and \( p\text{End} \, N = M_1^{i_1}M_2^{i_2}M_3^{i_3} \) with \( i_1 + i_2 + i_3 \leq 3 \). Furthermore, \( pN \subseteq (I_1 \cap I_2)N \), and \( N/(I_1 \cap I_2)N \cong N/I_1 N \oplus N/I_2 N \). After relabelling subscripts, if necessary, \( I_1 = M_1, N/I_1 N = \mathbb{Z}/p\mathbb{Z} \), and \( I_1 N = A_1 \). Finally, \( I_1 \) is principal by Nakayama’s
Lemma, since $p\text{End } N \subseteq j\text{End } N$ and $I_1/p\text{End } N$ is principal. This shows that $A_1$ is isomorphic to $N$, as desired. \hfill \square

The point of the next lemma, as used in the proof of the main theorem, is that Lemma 2(a) applies to a group quasi-isomorphic to a direct sum of two purely indecomposable groups of the same rank.

**Lemma 4.** If $N \in TF$ is indecomposable and quasi-isomorphic to $A \oplus B$ for purely indecomposable groups $A$ and $B$ in $TF$ with rank $A = \text{rank } B$, then $\text{Hom}(A, B) = 0 = \text{Hom}(B, A)$ and $\text{End } N$ is a local ring.

*Proof.* Choose purely indecomposable pure subgroups $A$ and $B$ of $N$ and some least positive integer $i$ with $p^iN \subset A \oplus B \subset N$. Since $p^2$-rank $N = 2$, $N/p^iN \cong \mathbb{Z}/p^j\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z}$ for some $1 \leq j \leq i$. Because $A$ and $B$ are purely indecomposable pure subgroups of $N$, $N/(A \oplus B) \cong \mathbb{Z}/p^j\mathbb{Z}$, say $N = A \oplus B + \mathbb{Z}(a, b)(1/p^j)$ for some $a \in A \setminus pA$ and $b \in B \setminus pB$.

If $\text{Hom}(A, B) \neq 0$ or $\text{Hom}(B, A) \neq 0$, then $A$ and $B$ are isomorphic by Proposition 1(iv). Moreover, $C = N/A$ is purely indecomposable and quasi-isomorphic to $B$. Hence, $C \cong A \cong B$ and $\text{Hom}(C, N)C = N$. By Baer’s Lemma [Arnold 82], $A$ is a summand of $N$, a contradiction to the assumption that $N$ is indecomposable.

Now assume that $\text{Hom}(A, B) = 0 = \text{Hom}(B, A)$. Then $A$ and $B$ are fully invariant subgroups of $N = A \oplus B + \mathbb{Z}(a, b)(1/p^j)$. Thus, $\text{End } N$ is the pullback of a homomorphism $A \to \mathbb{Z}/p^j\mathbb{Z}$ with kernel $p^jA$ and a homomorphism $B \to \mathbb{Z}/p^j\mathbb{Z}$ with kernel $p^jB$. It follows that $\text{End } N/p^j\text{End } N \cong \mathbb{Z}/p^j\mathbb{Z}$, whence $\text{End } N$ is a local ring. \hfill \square

3. The main theorem

**Theorem 1.** If $M \in TF$ and rank $M \leq 9$, then $M$ is a Krull-Schmidt group.

*Proof.* Let $N$ be an indecomposable summand of $M$ of minimal rank and $M = N \oplus N_1 \oplus \ldots \oplus N_m = K_1 \oplus \ldots \oplus K_n$ with each $N_i$ and $K_j$ indecomposable. Then rank $N \leq 4$, rank $N \leq \text{rank } N_j$, and rank $N \leq \text{rank } K_i$ for each $i$ and $j$, since rank $N \leq 9$ and $N$ is an indecomposable summand of $M$ of minimal rank.

If $p\text{-rank } N \leq 1$, then $\text{End } N$ is a local ring, as noted above. In this case, by Lemma 2(a), $N_1 \oplus \ldots \oplus N_m$ is isomorphic to $K_1 \oplus \ldots \oplus K_n$ for subgroups $K_i$ of $K_i$. It follows, by an induction on the rank of $M$, that $M$ is a Krull-Schmidt group. In particular, if $p\text{-rank } N \leq \text{rank } N$, then $N$ is free and cyclic, hence of $p\text{-rank } 1$.

If $N$ and $N_1 \oplus \ldots \oplus N_m$ have no quasi-summands in common, then, by Lemma 2(b), the proof is completed by an induction on the rank of $M$.

In view of the preceding remarks, it is now sufficient to assume that $M$ is reduced, $2 \leq p\text{-rank } N < \text{rank } N \leq 4$ for each indecomposable summand $N$ of minimal rank, and if $M = N \oplus N'$, then $N$ and $N' = N_1 \oplus \ldots \oplus N_m$ have a quasi-summand in common. Under these assumptions, $M$ has no rank-1 quasi-summands. This is because the only rank-1 groups in $TF$ are $\mathbb{Z}_{(p)}$ and $\mathbb{Q}$ and, since $M$ is reduced, any rank-1 quasi-summand must actually be a summand isomorphic to $\mathbb{Z}_{(p)}$. The strategy of the remainder of the proof is to show that $N$ must be isomorphic to some $K_i$, in which case the cancellation property for $N \in TF$ and an induction on the rank of $M$ shows that $M$ is a Krull-Schmidt group.

First assume that rank $N = 4$, $p\text{-rank } N = 3$. Then $N$, being indecomposable, is co-purely indecomposable, hence strongly indecomposable by Proposition 1. Thus,
$N_1 \oplus \ldots \oplus N_m$ is quasi-isomorphic to $N \oplus L$ for some $L$ of rank $\leq 1$. To see this, recall that $N$ and $N_1 \oplus \ldots \oplus N_m$ have a quasi-summand in common, $N$ is strongly indecomposable, rank $N \geq \text{rank } N_i$, and rank $N + \sum \text{rank } N_i = 4 + \sum \text{rank } N_i \leq 9$. Since $M$ has no rank-1 quasi-summands, $L = 0$, $m = 1$, and $n = 2$. But $TF_2$ is a Krull-Schmidt category so that $M = N \oplus N_1 = K_1 \oplus K_2$ has rank 8 with $N$ quasi-isomorphic to $N_1$, $K_1$, and $K_2$. By Lemma 3, $N$ is isomorphic to either $K_1$ or $K_2$, as desired.

Next, consider the case that rank $N = 4$ and $p$-rank $N = 2$. If $N$ is strongly indecomposable, then, as above, $M = N \oplus N_1 = K_1 \oplus K_2$ has rank 8 and $N$ is quasi-isomorphic to $N_1$, $K_1$, and $K_2$. By Example 1, $N$ has the UDS property so that $N$ is isomorphic to either $K_1$ or $K_2$, as desired. If $N$ is not strongly indecomposable, then $N$ is quasi-isomorphic to $A \oplus B$, where $A$ and $B$ are purely indecomposable groups with $p$-rank 1 and rank 2. This is because $M$ has no rank-1 quasi-summands. Now apply Lemmas 2 and 4 and induction on the rank of $M$ to see that $M$ is a Krull-Schmidt group.

The only remaining case is that $p$-rank $N = 2$, rank $N = 3$. In this case $N$ is co-purely indecomposable, hence strongly indecomposable by Proposition 1. Since $N$ and $N_1$ have a quasi-summand in common, $N_1$ is quasi-isomorphic to $N \oplus A$ for some pure subgroup $A$ of $N_1$ with $1 \leq p$-rank $A < \text{rank } A \leq 3$. This is because $M$ has no rank-1 quasi-summands and rank $M \leq 9$.

If $A$ has $p$-rank 1, then $\text{Hom}(A, N) = 0$ by Proposition 1(c) and (d), since $A$ is purely indecomposable with $2 \leq \text{rank } A \leq 3 = \text{rank } N$, and $N$ is co-purely indecomposable. In this case, $\text{Hom}(A, M) = \text{Hom}(A, A)$. It follows that $A$ is a purely invariant subgroup, hence equal to a subgroup of some $K_i$, say $K_1$. Thus, $N \oplus (N_1/A)$ is isomorphic to $(K_1/A) \oplus K_2 \oplus K_3$ and induction on the rank of $M$ completes the proof.

Finally, assume that $A$ has $p$-rank 2. Then rank $A = 3 = \text{rank } N$ and $A$ and $N$ are both co-purely indecomposable. If $\text{Hom}(A, N) = 0$, then, as above, $M$ is a Krull-Schmidt group. Finally, if $\text{Hom}(A, N) \neq 0$, then $A$ is quasi-isomorphic to $N$, since $A$ and $N$ are both co-purely indecomposable modules with the same rank $\text{rank } N$. Hence, $M = N \oplus N_1 = K_1 \oplus K_2 \oplus \ldots \oplus K_n$ has rank 9 with $n \leq 3$. If $n = 3$, then $N$ is quasi-isomorphic to $K_1$, $K_2$ and $K_3$ by the minimality of the rank of $N$. In this case, Example 1 yields $N$ isomorphic to some $K_i$. If $n = 2$, then, by Lemma 3, $N$ is isomorphic to some $K_i$, as desired. \hfill \Box

**Example 2** ([Arnold 01]). There is a rank-10 group in $TF$ that is not a Krull-Schmidt group.

**Proof.** The argument is briefly outlined. There is $M \in TF$ of $p$-rank 4 and rank 5 such that $M \cong \text{End } M$, a subring of an algebraic number field with exactly four maximal ideals $M_1$, $M_2$, $M_3$, and $M_4$, and $pM = p\text{End } M = M_1 \cap M_2 \cap M_3 \cap M_4$. Furthermore, there are subgroups $A_1$ and $A_2$ of $M$ not isomorphic to $M$ with $(M_1 \cap M_2)M \subset A_1$ and $(M_3 \cap M_4)M \subset A_2$. It follows that there is $B \in TF$ with $M \oplus B = A_1 \oplus A_2$, a rank 10 group in $TF$ that is not a Krull-Schmidt group. \hfill \Box

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