PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 6, Pages 1797–1803
S 0002-9939(01)06249-9
Article electronically published on October 24, 2001

ON THE CONTINUITY OF BICONJUGATE CONVEX FUNCTIONS

J. M. BORWEIN AND J. D. VANDERWERFF

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We show that a Banach space is a Grothendieck space if and only if every continuous convex function on X has a continuous biconjugate function on X^{**}, thus also answering a question raised by S. Simons. Related characterizations and examples are given.

1. INTRODUCTION

In [12, Example 2.8], Simons showed that

$$f(x) = \sum_{n=1}^{\infty} x_n^{2n} \tag{1.1}$$

is a continuous convex function on c_0 whose Fenchel biconjugate f^{**} fails to be continuous as an extended real-valued function on ℓ_∞. In this note, if a function is said to be continuous without any further qualifications, it is assumed that the function is real-valued. However, lower semicontinuous functions may map into the extended real numbers $\mathbb{R} \cup \{\infty\}$. In the case a function f may take the value ∞ and $f(x_n) \to f(x)$ in the usual topology on $\mathbb{R} \cup \{\infty\}$ whenever $x_n \to x$ in norm, we will say that f is *continuous as an extended real-valued function*. For example, the biconjugate f^{**} of f from (1.1) is not continuous as an extended real-valued function because [12, Example 2.8] produces a sequence $x_n \to x_0$ in ℓ_∞ such that $f^{**}(x_n) = \infty$ while $f^{**}(x_0) < \infty$. Let us note that any continuous convex function whose biconjugate fails to be continuous must be unbounded on some bounded set [12, Equation (2.7.1)], and that the Josefson-Nissensweig theorem ensures that each infinite dimensional Banach space admits a continuous convex function that is unbounded on some bounded set [2, Theorem 2.2]. It is asked in [12, Problem 2.9] whether a continuous convex function whose biconjugate fails to be continuous can be constructed on each nonreflexive Banach space. The purpose of this note is to answer this question negatively and to present some related results concerning extensions of convex functions.

We shall work in real Banach spaces X with closed unit ball and unit sphere denoted by B_X and S_X respectively; the dual of X is denoted by X^*. Unless specified otherwise, the continuity of a function is with respect to the norm topology. Given

Received by the editors September 11, 2000 and, in revised form, January 9, 2001.
2000 *Mathematics Subject Classification.* Primary 46B20, 52A41.
Key words and phrases. Continuous convex function, conjugate function, Grothendieck space.
The first author's research was supported by an NSERC grant.

a convex function f on X, its conjugate f^* is defined by $f^*(\phi) = \sup\{\langle \phi, x\rangle - f(x) : x \in X\}$ for $\phi \in X^*$. The *biconjugate function* f^{**} is defined on X^{**} as the conjugate function of f^*. Then f^* is weak*-lower semicontinuous as a supremum of weak*-continuous functions, and f^{**} is the supremum of the weak*-lower semicontinuous convex (or affine) functions that are majorized by f. While f^{**} is the largest weak*-lower semicontinuous convex extension of a continuous convex function f to X^{**}, in general, the set of weak*-lower semicontinuous convex extensions has no least element [8]. See also [12] for further discussion on conjugate functions.

2. Continuity of biconjugates and convex extensions

Recall that a Banach space is called a *Grothendieck space* if weak and weak* convergence coincide for sequences in its dual. The nonreflexive Grothendieck spaces include $C(K)$ spaces where K is compact quasi-Stonean; see [11, Theorem II.10.4]. Thus, in particular, $\ell_\infty(\Gamma)$ where Γ is infinite is a Grothendieck space (see [6, p. 156]). Furthermore, biduals of any AM-space—and hence biduals of $C(K)$ spaces—are Grothendieck spaces because these biduals are $C(K)$ spaces where K is compact Stonean (see [11, p. 121]). On the other hand, the Eberlein-Šmulian theorem ensures that a non-reflexive Grothendieck space cannot have a weak*-sequentially compact dual ball and so cannot be a Gâteaux differentiability space; see [7, Theorem 2.1.2]. Nonetheless, Haydon has constructed a nonreflexive Grothendieck space $C(K)$ space that fails to contain a copy of ℓ_∞ [9], and assuming the continuum hypothesis Talagrand has constructed such a space that neither contains ℓ_∞ nor has it as a quotient [13].

Theorem 2.1. *For a Banach space X, the following are equivalent*:

(a) *X is a Grothendieck space.*
(b) *For each continuous convex function f on X, every weak*-lower semicontinuous convex extension of f to X^{**} is continuous.*
(c) *For each continuous convex function f on X, f^{**} is continuous on X^{**}.*
(d) *For each continuous convex function f on X, there is at least one weak*-lower semicontinuous convex extension of f to X^{**} that is continuous.*
(e) *For each Fréchet differentiable convex function f on X, there is at least one weak*-lower semicontinuous convex extension of f to X^{**} that is continuous.*

Proof. (a) $\Rightarrow$ (b): Suppose f is continuous and convex on X, but some weak*-lower semicontinuous convex extension, say $\tilde{f}$, is discontinuous on X^{**}. Because $\tilde{f}$ is lower semicontinuous, it would be continuous if it were finite valued everywhere. Thus we choose $\Phi \in X^{**}$ so that $\tilde{f}(\Phi) = \infty$. Let the weak*-closed convex sets C_n be defined by $C_n = \{\Lambda \in X^{**} : \tilde{f}(\Lambda) \le n\}$. Now $\Phi \notin C_n$ for all n and so we choose $\phi_n \in S_{X^*}$ such that

$$\sup_{C_n} \phi_n \le \langle \phi_n, \Phi\rangle \le \|\Phi\|. \tag{2.1}$$

Now there is some $\delta > 0$ such that $\sup_{C_n} \phi_n \ge \delta > 0$ because f is continuous at the origin in X, and where without loss of generality, we have assumed $f(0) < n$ for all n. Therefore $\langle \phi_n, \Phi\rangle \ge \delta$ for all n and so $\phi_n \not\to_w 0$ in X^*.

However, if $\phi_n \not\to_{w^*} 0$, then we can choose $x_0 \in X$ such that $\langle \phi_n, x_0\rangle > \|\Phi\|$ for all n, by passing to a subsequence as necessary. Then $x_0 \notin \{x : f(x) \le n\}$. Indeed, since $\tilde{f}(x) = f(x)$ for all $x \in X$, it follows that $C_n \cap X = \{x : f(x) \le n\}$. Consequently (2.1) implies that $f(x_0) > n$ for each n, a contradiction.

Therefore, we have constructed a sequence $\{\phi_n\}$ in X^* that converges weak* to 0 but does not converge weakly to 0, and so X is not a Grothendieck space.

Now (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) is trivial.

(e) $\Rightarrow$ (a): Suppose X is not a Grothendieck space. Let $\{x_n^*\}$ be a sequence in S_{X^*} such that $x_n^* \to_{w^*} 0$, but $x_n^* \not\to_w 0$. We will construct a function f that is Fréchet differentiable and convex on X such that all of its weak*-lower semicontinuous extensions to X^{**} fail to be continuous as extended real-valued functions. Choose $\Phi_0 \in X^{**}$ such that $\limsup\langle x_n^*, \Phi_0\rangle > 1$. Passing to a subsequence, we may assume $\langle x_n^*, \Phi_0\rangle \geq 1$ for each n. Let $a_n = \langle x_n^*, \Phi_0\rangle \geq 1$ and define f on X by $f(x) = \sum_{n=1}^{\infty} f_n(\langle x_n^*, x\rangle)$ where $f_n : \mathbb{R} \to [0, \infty)$ is defined by $f_n(t) = 0$ if $|t| \leq a_n$ and $f_n(t) = n^2(|t| - a_n)^2$ otherwise. Then f is a Fréchet differentiable convex function on X because it is a locally finite sum of Fréchet differentiable convex functions (cf. [3, Lemma 2.1]). Now [8, Proposition 5] implies that every weak*-lower semicontinuous extension of f to X^{**} is Fréchet differentiable on X. Then [8, Proposition 7] implies that there is only one weak*-lower semicontinuous convex extension of f to X^{**} (see Theorem 3.1 below for a proof of [8, Proposition 7]). This extension is the canonical extension $\tilde{f}$ of f to X^{**} where for $\Phi \in X^{**}$ we write $\tilde{f}(\Phi) = \sum_{n=1}^{\infty} f_n(\langle x_n^*, \Phi\rangle)$. Then $\tilde{f}$ is weak*-lower semicontinuous because it is a sum of positive weak*-lower semicontinuous functions. Now let $\Phi_n = (1 + \frac{1}{n})\Phi_0$ and observe that $\tilde{f}(\Phi_0) = 0$ while $\tilde{f}(\Phi_n) \geq \sum_{k=n}^{\infty} k^2(1/n)^2 = \infty$ for each n. Thus we have shown $\tilde{f}$ (and hence f^{**} which equals $\tilde{f}$) is not continuous as an extended real-valued function. $\square$

Some other characterizations of Grothendieck spaces in terms of the weak*-lower semicontinuous convex extensions preserving points of Gâteaux differentiability are given in Godefroy's paper [8]. One may further ask whether the existence of convex extensions to X^{**} that are merely continuous in Theorem 2.1(d) would still provide a characterization of Grothendieck spaces. Part (a) of the following example observes that this is not the case, while (b) illustrates the necessity of having f continuous on all of X.

Example 2.2. (a) Let J be the James space. Then J is not a Grothendieck space, and so by Theorem 2.1(d), there is a continuous convex function f on J that has no continuous convex weak*-lower semicontinuous extension to all of J^{**}. However, $J^{**} = J \times \mathbb{R}$, and in particular, there is a continuous linear projection P from J^{**} onto J. Then $\tilde{f} = f \circ P$ is a continuous convex extension of f to J^{**}. Similar examples can be constructed on L_1 which is not a Grothendieck space, but is complemented as a subspace of its bidual L_∞^* (cf. [10, p. 218]).

(b) Let X be any nonreflexive Banach space. Then James' theorem [4, Theorem I.3.2] ensures there is a $\phi \in S_{X^*}$ such that ϕ does not attain its supremum on B_X. Define $f(x)$ by $f(x) = 1/(1 - \langle\phi, x\rangle)$ if $\langle\phi, x\rangle < 1$ and $f(x) = \infty$ otherwise. Then f is lower semicontinuous convex and continuous on B_X. Now $f^{**}(\Phi) = 1/(1 - \langle\phi, \Phi\rangle)$ if $\Phi \in X^{**}$ and $\langle\phi, \Phi\rangle < 1$, while $f^{**}(\Phi) = \infty$ otherwise. Therefore, $f^{**}(\Phi_0) = \infty$ for $\Phi_0 \in B_{X^{**}}$ such that $\langle\Phi_0, \phi\rangle = 1$. This shows that f^{**} is not continuous on $B_{X^{**}}$ although f is continuous on B_X. However, f^{**} is continuous as an extended real-valued function on X^{**}.

On the other hand, both [8, p. 372] and [12, Example 2.8] present explicit constructions of continuous convex functions on c_0 whose canonical weak*-lower semicontinuous extensions to ℓ_∞ fail to be continuous. It is natural to ask whether there are *any* convex extensions of those functions that are continuous on ℓ_∞. The following result which contrasts with Example 2.2(a) shows that this is not the case. To present this result somewhat more generally, we say, as in [2], that a Banach space X has the *DP*-property* if $\langle x_n^*, x_n \rangle \to 0$ whenever $x_n^* \to_{w^*} 0$ in X^* and $x_n \to_w 0$ in X, or equivalently if weak* and Mackey convergence agree sequentially in X^*. Banach spaces with the Grothendieck and Dunford-Pettis properties—the latter being possessed by all $C(K)$ spaces [6, p. 177]—have the DP*-property (see [2] for more information). Consequently, Grothendieck $C(K)$ spaces such as $\ell_\infty(\Gamma)$ and biduals of any $C(K)$ space have the DP*-property (see also [11, §II.9]). Therefore, the following theorem applies, in particular, when X is a $C(K)$ space not containing ℓ_1, that is, when K is a scattered compact set [4, Lemma VI.8.3].

Theorem 2.3. *Let $X \not\supset \ell_1$ be such that X^{**} has the DP*-property, and suppose that f is a continuous convex function on X. Then the following are equivalent*:

(a) *All weak*-lower semicontinuous convex extensions of f to X^{**} are bounded on bounded subsets of X^{**}.*
(b) *f^{**} is continuous on X^{**}.*
(c) *There is a continuous convex function on X^{**} which is an extension of f.*
(d) *f is bounded on weakly compact subsets of X.*
(e) *f is bounded on bounded subsets of X.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d): Let $\tilde{f}$ be a continuous convex extension of f. It follows from [3, Theorem 2.7] that $\tilde{f}$ is bounded on weakly compact subsets of X^{**}, and thus f, its restriction to X, is bounded on weakly compact subsets of X.

(d) $\Rightarrow$ (e): This is a direct consequence of [3, Theorem 2.4].

(e) $\Rightarrow$ (a): This follows from [8, Proposition 3], or also from [12, Equation (2.7.1)]. $\square$

In particular, applying this theorem when $X = c_0$ shows that the function in (1.1) has no continuous convex extension to ℓ_∞. Inter alia, this recovers the result that c_0 is not complemented in ℓ_∞. By contrast, Sobczyk's theorem [5, Theorem 4, p. 71] shows c_0 is complemented in every separable superspace. Thus, one can extend arbitrary continuous convex functions on c_0 to continuous convex functions on arbitrary separable superspaces of c_0.

Observe further that Example 2.2(a) shows that the condition X^{**} has the DP*-property cannot be dropped in Theorem 2.3 (for the (c) implies (b) implication) because the James space J has a separable dual and so does not contain ℓ_1. On the other hand, [3, Theorem 2.4] ensures that the condition $X \not\supset \ell_1$ is needed in the (d) implies (e) implication of Theorem 2.3. Furthermore, we next observe that the equivalence of (b) and (e) in the previous theorem actually characterizes Banach spaces whose duals have the Schur property (a Banach space is said to have the *Schur property* if weakly convergent sequences are norm convergent).

Proposition 2.4. *For a Banach space X, the following are equivalent*:

(a) *X^* has the Schur property.*
(b) *If f is any continuous convex function on X such that f^{**} is continuous on X^{**}, then f is bounded on bounded sets.*

Proof. (a) $\Rightarrow$ (b): This follows because [3, p. 67] shows that any finite valued weak*-lower semicontinuous function on X^{**} must be bounded on bounded sets when X^* has the Schur property.

(b) $\Rightarrow$ (a): Suppose X^* does not have the Schur property. Then there is a sequence $\{x_n^*\} \subset X^*$ that converges weakly to 0 while $\|x_n^*\| = 1$ for all n. Proceeding as in the proof of Theorem 2.1, we can construct a continuous convex function $\tilde{f}$ on X^{**} that is weak*-lower semicontinuous and Fréchet differentiable on X^{**} and unbounded on some bounded set. Now let f be the restriction of $\tilde{f}$ to X. Reasoning as in Theorem 2.1, we get that $\tilde{f} = f^{**}$, and so f^{**} is continuous, but unbounded on a bounded set because $\tilde{f}$ is on a bounded set. According to [12, Equation (2.7.1)] f is unbounded on some bounded subset of X. □

Our final result makes some observations related to Theorem 2.3 concerning the extension of continuous convex functions to arbitrary superspaces of spaces not containing ℓ_1.

Theorem 2.5. *Suppose $X \not\supset \ell_1$ and f is a continuous convex function on X. Then the following are equivalent*:

(a) *There is a continuous convex extension of f to $\ell_\infty(\Gamma)$ where Γ is such that $X \subset \ell_\infty(\Gamma)$.*
(b) *f is bounded on weakly compact subsets of X.*
(c) *f is bounded on bounded subsets of X.*
(d) *If Z is any superspace of X, then f can be extended to a convex function that is bounded on bounded subsets of Z.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is proved as in Theorem 2.3 (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

(c) $\Rightarrow$ (d): This is an immediate consequence of the following general fact which is not restricted to Banach spaces not containing ℓ_1. □

Fact. *Suppose Y is a closed subspace of a Banach space Z and suppose f is a convex function on Y. Then there is a convex lower semicontinuous extension $\tilde{f}$ of f such that $\sup_{rB_Z} \tilde{f} = \sup_{rB_Y} f$ for all $r > 0$.*

Proof. For each $y \in Y$, let $\phi_y \in \partial f(y)$. Let $\tilde{\phi}_y$ be a norm preserving extension of ϕ_y to all of Z. Now define h_y on Z by $h_y(x) = f(y) + \langle \tilde{\phi}_y, x - y\rangle$. We define the desired extension $\tilde{f}$ of f by $\tilde{f}(x) = \sup_{y \in Y} h_y(x)$. Clearly $\tilde{f}$ is lower semicontinuous and convex as the supremum of such functions, and also $\tilde{f}|_Y = f$ by the subgradient inequality. If $\sup_{rB_Y} f = \infty$ there is nothing further to show, so we suppose $\sup_{rB_Y} f = M_r$. Now let $y_0 \in Y$ and observe that

$$f(y_0) + \langle \phi_{y_0}, x - y_0\rangle \le f(x) \le M_r \quad \text{for all } x \in rB_Y.$$

Therefore,

$$\sup_{x \in rB_Y} \langle \phi_{y_0}, x\rangle \le M_r - f(y_0) + \langle \phi_{y_0}, y_0\rangle.$$

Because $\tilde{\phi}_{y_0}$ is a norm preserving extension, we have

$$\sup_{x \in rB_Z} \langle \tilde{\phi}_{y_0}, x\rangle \le M_r - f(y_0) + \langle \tilde{\phi}_{y_0}, y_0\rangle.$$

It follows that $\sup_{x \in rB_Z} h_{y_0}(x) \le M_r$. Because y_0 was arbitrary in Y, we conclude that $\tilde{f}(x) \le M_r$ on rB_Z, and the fact is proved.

(d) $\Rightarrow$ (a): is obvious. □

Let us mention that using [3, Theorem 2.7] as in Theorem 2.3, one can show that if Y is a subspace of X and Y fails the DP*-property while X has it, then there is a continuous convex function on Y that cannot be extended to a continuous convex function on X. In particular, this applies when Y is not Schur and B_{Y^*} is weak*-sequentially compact [2, Corollary 3.4], and X is some $\ell_\infty(\Gamma)$ space. However, even the case of extending continuous convex functions from separable spaces to separable superspaces is not clear to us. In this direction, R. Girgensohn has shown us a construction that enables one to extend $f : \ell_1 \to \mathbb{R}$ defined by $f(x) = \sum x_n^{2n}$ to a continuous convex function on $C[0,1]$. Unfortunately, we do not know how generally his construction will work.

3. Appendix

Because the proof of [8, Proposition 7] was not included in [8], we are including a proof here of a version that is suitable for our purposes. We are grateful to both G. Godefroy and S. Simons for helpful correspondence and suggestions concerning this result and its proof.

Theorem 3.1 (Godefroy). *Suppose f is a continuous convex function on X. If f^{**} is Gâteaux differentiable at all points of X, then it is the unique convex weak*-lower semicontinuous extension of f to X^{**}.*

Proof. Now $f = f^{**}|_X$ is Gâteaux differentiable on X. Moreover, $f'(x) \in \partial f^{**}(x)$ because f^{**} is the largest weak*-lower semicontinuous convex extension of f to X^{**}. Therefore, the Gâteaux differentiability of f^{**} at $x \in X$ implies that $\partial f^{**}(x) = \{f'(x)\}$ which is in X^*. Now we define $\check{f}$ by

$$\check{f}(x^{**}) = \sup\{f(x) + \langle f'(x), x^{**} - x\rangle : x \in X\}.$$

The subgradient inequality implies $\check{f}|_X = f$. Now let g be any weak*-lower semicontinuous convex extension of f to X^{**}. Then for $x \in X$, $\partial g(x) \neq \emptyset$ because g is continuous there. Now f^{**} majorizes g and f^{**} agrees with g on X, thus the Gâteaux differentiability of f^{**} at each $x \in X$ implies that g is Gâteaux differentiable at $x \in X$ with $g'(x) = (f^{**})'(x)$. Hence $f'(x) \in \partial g(x)$ for all $x \in X$, and so $g \geq \check{f}$. Therefore $\check{f}$ is the smallest weak*-lower semicontinuous convex extension of f.

It remains to show that $\check{f} = f^{**}$. For this, we will use a result of Attouch and Beer which states that if $\phi \in \operatorname{dom}(f^*)$, then there is a sequence $\{\phi_n\}$ in the range of ∂f such that $\|\phi_n - \phi\| \to 0$ and $f^*(\phi_n) \to f^*(\phi)$ [1, Proposition 4.3]. Therefore

$$\begin{aligned} f^{**}(x^{**}) &= \sup\{\langle x^{**}, x^*\rangle - f^*(x^*) : x^* \in \operatorname{dom}(f^*)\} \\ &= \sup\{\langle x^{**}, f'(x)\rangle - f^*(f'(x)) : x \in X\} \qquad \text{(by [1, Proposition 4.3])} \\ &= \sup\{\langle x^{**}, f'(x)\rangle - (\langle f'(x), x\rangle - f(x)) : x \in X\} \\ &= \sup\{\langle x^{**} - x, f'(x)\rangle + f(x) : x \in X\} = \check{f}. \end{aligned}$$

Therefore $f^{**} = \check{f}$ and we are done. ∎

Because the Fréchet differentiability of a convex function f is preserved at points in $X \subset X^{**}$ by any weak*-lower semicontinuous extension of f to X^{**} (see [8, Proposition 5]), Theorem 3.1 implies that Fréchet differentiable convex functions on X have unique weak*-lower semicontinuous convex extensions to X^{**}. Finally, we wish to thank the referee for several helpful suggestions concerning this note.

References

1. H. Attouch and G. Beer, On the convergence of subdifferentials of convex functions, *Arch. Math.* **60** (1993), 389–400. MR **94b:**49018
2. J. Borwein, M. Fabian and J. Vanderwerff, Characterizations of Banach spaces via convex and other locally Lipschitz functions, *Acta. Vietnamica* **22** (1997), 53–69. MR **98k:**46019
3. J. Borwein, S. Fitzpatrick and J. Vanderwerff, Examples of convex functions and classifications of normed spaces, *J. Convex Anal.* **1** (1994), 61–73. MR **96e:**46012
4. R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitnam Monographs and Surveys in Pure and Applied Mathematics **64**, Longman, 1993. MR **94d:**46012
5. J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics **92**, Springer–Verlag, New York, Berlin, Heidelberg, Tokyo, 1984. MR **85i:**46020
6. J. Diestel and J.J. Uhl, Jr., *Vector Measures*, AMS Mathematical Surveys **15**, Providence, 1977. MR **56:**12216
7. M.J. Fabian, *Gâteaux Differentiability of Convex Functions and Topology*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., 1997. MR **98h:**46009
8. G. Godefroy, Prolongement de fonctions convexes définies sur un espace de Banach E au bidual E'', *C. R. Acad. Sci. Paris* **292** (1981), 371–374. MR **82b:**46022
9. R. Haydon, A non-reflexive Grothendieck space that does not contain ℓ_∞, *Israel J. Math.* **40** (1981), 65–73. MR **83a:**46028
10. R.B. Holmes, *Geometric Functional Analysis and its Applications*, Graduate Texts in Mathematics **24**, Springer–Verlag, New York, Heildelberg, Berlin, 1975. MR **53:**14085
11. H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer–Verlag, New York, Heidelberg, Berlin, 1974. MR **54:**11023
12. S. Simons, Maximal monotone multifunctions of Brønsdsted–Rockafellar type, *Set-Valued Analysis* **7** (1999), 255–294. MR **2001b:**47100
13. M. Talagrand, Un nouveau $C(K)$ qui possède la propriété de Grothendieck, *Israel J. Math.* **37** (1980), 181–191. MR **82g:**46029

Department of Mathematics & Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

E-mail address: jborwein@cecm.sfu.ca

Department of Mathematics, La Sierra University, Riverside, California 92515

E-mail address: jvanderw@lasierra.edu