SUBSPACES WITH NORMALIZED TIGHT FRAME WAVELETS IN \( \mathbb{R} \)

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Abstract. In this paper we investigate the subspaces of \( L^2(\mathbb{R}) \) which have normalized tight frame wavelets that are defined by set functions on some measurable subsets of \( \mathbb{R} \) called Bessel sets. We show that a subspace admitting such a normalized tight frame wavelet falls into a class of subspaces called reducing subspaces. We also consider the subspaces of \( L^2(\mathbb{R}) \) that are generated by a Bessel set \( E \) in a special way. We present some results concerning the relation between a Bessel set \( E \) and the corresponding subspace of \( L^2(\mathbb{R}) \) which either has a normalized tight frame wavelet defined by the set function on \( E \) or is generated by \( E \).

§1. Introduction and preliminaries

The concept of frames first appeared in the late 40’s and early 50’s ([1], [10], [11]). There has recently been renewed interest in frames due to the development of wavelet theory since wavelets and frames are tightly related ([4], [5], [6] and [9]). A sequence of elements \( \{x_j\} \) in \( \mathcal{H} \) is called a Bessel sequence if there exists a positive constant \( B \) such that \( \sum_j |\langle x, x_j \rangle|^2 \leq B\|x\|^2 \) for all \( x \in \mathcal{H} \). If in addition, there exists a positive constant \( A \) such that \( A\|x\|^2 \leq \sum_j |\langle x, x_j \rangle|^2 \) for all \( x \in \mathcal{H} \), then \( \{x_j\} \) is called a frame for \( \mathcal{H} \). The supremum of all such numbers \( A \) and the infimum of all such numbers \( B \) are called the frame bounds of the frame and denoted by \( A_0 \) and \( B_0 \) respectively. The frame is called a tight frame when \( A_0 = B_0 \) and is called a normalized tight frame when \( A_0 = B_0 = 1 \). The kind of frames that are related to wavelets and of special interest to mathematicians are of the form \( \{\psi_{j,k}(x) = 2^j \psi(2^j x - k) : j, k \in \mathbb{Z}\} \), where \( \psi \in L^2(\mathbb{R}) \). The function \( \psi \) is called a normalized tight frame wavelet (NTF wavelet) if this frame is a normalized tight frame (NTF). Similarly, \( \psi \) is called a tight frame wavelet (TF wavelet) if this frame is a tight frame. From an operator theoretic point of view, NTF wavelets are simply the frame vectors for the unitary system \( \mathcal{U} = \{D^nT^l : l, n \in \mathbb{Z}\} \), where \( D \) and \( T \) are the unitary dilation and translation operators on \( L^2(\mathbb{R}) \) (see [7]) defined by \((Df)(x) = \sqrt{2}f(2x)\) and \((Tf)(x) = f(x - 1)\) for any \( f \in L^2(\mathbb{R}) \). A characterization of the NTF wavelets is obtained in [8].

In this paper, we are mainly interested in the case when \( \psi \) is defined by \( \tilde{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E \), where \( E \) is a Lebesgue measurable set with finite measure. We will call \( E \) a normalized tight frame wavelet set (NTF wavelet set) when \( \psi \) is an NTF wavelet.
A tight frame wavelet set (TF wavelet set) and a frame wavelet set are defined similarly. Likewise, \( E \) is called a Bessel set if \( \{ \psi_{j,k}(x) : j, k \in \mathbb{Z} \} \) is a Bessel sequence. All these concepts can be applied to the subspaces of \( L^2(\mathbb{R}) \) (to be formally defined in the next section) and many questions can be asked. A closed subspace \( X \) of \( L^2(\mathbb{R}) \) is called a reducing subspace of \( D \) and \( T \) (or simply reducing subspaces in this paper) if \( TX = X \) and \( DX = X \). In [2], it is shown that the reducing subspaces are pertinent to the study of wavelets in subspace. It is shown there that such a subspace always has an orthonormal wavelet. It is natural to ask the question about the existence of NTF wavelets for an arbitrary subspace of \( L^2(\mathbb{R}) \). One may even try to characterize all the NTF wavelets for a given subspace of \( L^2(\mathbb{R}) \). As a first step, perhaps it will shed some light on the whole picture by looking at the NTF wavelets which are generated by some subsets of \( \mathbb{R} \).

In this paper, we show that if \( X \) is a reducing subspace with an NTF wavelet \( \psi \) whose Fourier transform is \( \frac{1}{\sqrt{2\pi}} \chi_E \), then \( E \) can be completely characterized (Theorem 1). On the other hand, we also show that if \( X \) admits an NTF wavelet \( \psi \) whose Fourier transform is \( \frac{1}{\sqrt{2\pi}} \chi_E \) for a Bessel set \( E \), then \( X \) must be a reducing subspace (Theorem 3). Furthermore, we prove that a maximal reducing subspace exists in the subspace \( X_E \) generated (see Definition 3) by a Bessel set \( E \) and this maximal reducing subspace is completely characterized (Theorem 2).

We will use \( \mu \) to denote the Lebesgue measure. The inner product of two functions \( f \) and \( g \) in \( L^2(\mathbb{R}) \) is denoted by \( \langle f, g \rangle \). The Fourier transform of a function \( f \) in \( L^2(\mathbb{R}) \) is denoted by \( \mathcal{F}(f) \) or simply \( \hat{f} \). Define \( \hat{T} = \mathcal{F}TF^{-1}, \hat{D} = \mathcal{F}DF^{-1} \). We have \( \hat{D} = D^{-1} = D^* \) and \( \hat{T}(f) = e^{-inx}f \) for \( f \in L^2(\mathbb{R}) \).

Let us first introduce some set theoretic notations.

Let \( E \) be a set. Define \( \tau(E) = \bigcup_{n \in \mathbb{Z}} (E \cap [2n\pi, 2(n+1)\pi] - 2n\pi) \). If this is a disjoint union, we say that \( E \) is a \( 2\pi \)-translation equivalent to \( \tau(E) \), which is always a subset of \([0, 2\pi) \). If \( E \) and \( F \) are \( 2\pi \)-translation equivalent to the same subset in \([0, 2\pi) \), then we say \( E \) and \( F \) are \( 2\pi \)-translation equivalent. This defines an equivalence relation and is denoted by \( \tilde{=} \). It is clear that \( \mu(E) \geq \mu(\tau(E)) \). The equality holds if and only if \( E \tilde{=} \tau(E) \). If \( E \tilde{=} F \), then \( \mu(E) = \mu(F) \). Two points \( x, y \in E \) are said to be \( 2\pi \)-translation equivalent in \( E \) if \( x - y = 2m\pi \) for some integer \( m \). The \( 2\pi \)-translation redundancy index of a point \( x \) in \( E \) is the number of elements in its equivalence class. We write \( E(\tau, k) \) for the set of all points in \( E \) with \( 2\pi \)-translation redundancy index \( k \). Of course, \( E(\tau, k) \) could be an empty set, a proper subset of \( E \), or the set \( E \) itself. For \( k \neq m \), \( E(\tau, k) \cap E(\tau, m) = \emptyset \), so \( E = E(\tau, \infty) \cup \bigcup_{n \in \mathbb{N}} E(\tau, n) \) is a disjoint union.

Similarly, let \( E \) be a set and define

\[
\delta(E) = \bigcup_{n \in \mathbb{Z}} 2^{-n}(E \cap [\{ -2^{n+1}\pi, -2^n\pi \} \cup [2^n\pi, 2^{n+1}\pi])).
\]

If this union is disjoint, we say that \( E \) is a \( 2 \)-dilation equivalent to \( \delta(E) \), which is a subset \([-2\pi, -\pi) \cup [\pi, 2\pi) \), the Littlewood-Paley wavelet set. If \( E \) and \( F \) are \( 2 \)-dilation equivalent to the same subset in \([-2\pi, -\pi) \cup [\pi, 2\pi) \), then we say \( E \) and \( F \) are \( 2 \)-dilation equivalent. This also defines an equivalence relation and is denoted by \( \tilde{=} \). Two non-zero points \( x, y \in E \) are said to be \( 2 \)-dilation equivalent if \( \log_2 \frac{x}{y} \in \mathbb{Z} \). The \( 2 \)-dilation redundancy index of a point \( x \) in \( E \) is the number of elements in its equivalence class. The set of all points in \( E \) with \( 2 \)-dilation redundancy
redundancy index $k$ is denoted by $E(\delta, k)$. For $k \neq m$, $E(\delta, k) \cap E(\delta, m) = \emptyset$ and $E = E(\delta, \infty) \cup \bigcup_{n \in \mathbb{N}} E(\delta, n)$ is a disjoint union.

When $E$ is a measurable set, one can prove that $E(\tau, m)$ and $E(\delta, m)$ are also measurable for any $m \geq 1$. Furthermore, each $E(\tau, m)$ can be decomposed into $m$ disjoint measurable subsets $E^{(j)}(\tau, m)$ ($j = 1, 2, \ldots, m$) such that each $E^{(j)}(\tau, m)$ is $2\pi$-translation equivalent to a subset of $[0, 2\pi)$. Similarly, each $E(\delta, m)$ can be decomposed into $m$ disjoint measurable subsets $E^{(j)}(\delta, m)$ ($j = 1, 2, \ldots, m$) such that each $E^{(j)}(\delta, m)$ is $2\pi$-translation equivalent to a subset of $[-2\pi, -\pi) \cup [\pi, 2\pi)$. The proof of these facts is elementary and is left to the reader. Notice that the decompositions of $E(\tau, m)$ and $E(\delta, m)$ here are not unique, but we will choose one and stay with it throughout the paper.

**Definition 1.** A measurable set $E$ of finite measure is called a basic set if there exists $M > 0$ such that $\mu(E(\tau, m)) = \mu(E(\delta, m)) = 0$ for all $m > M$.

Let $X$ be a subspace of $L^2(\mathbb{R})$. A function $\psi \in X$ is an orthonormal wavelet for $X$ if the set $\{D^nT^\ell \psi : n, \ell \in \mathbb{Z}\}$ forms an orthonormal basis for $X$. A measurable set $E \subset \mathbb{R}$ is a wavelet set for $X$ if the Fourier inverse transform of $\frac{1}{\sqrt{2\pi}} \chi_E$, namely $F^{-1}\left(\frac{1}{\sqrt{2\pi}} \chi_E\right)$, is an orthonormal wavelet for the subspace $X$. It is shown in [2] that $X$ is a reducing subspace if and only if there exists a subset $\Omega$ of $\mathbb{R}$ with the property $\Omega = 2\Omega$ such that $\hat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$, where $\hat{X}$ is the set of all Fourier transforms of elements in $X$ (which is itself a subspace of $L^2(\mathbb{R})$).

**Definition 2.** Let $X$ be a closed subspace of $L^2(\mathbb{R})$ and let $\psi$ be a function in $L^2(\mathbb{R})$. Then $\psi$ is called an NTF wavelet for $X$ if $D^nT^\ell \psi \in X$ for each $n, \ell \in \mathbb{Z}$ and

$$f = \sum_{n, \ell \in \mathbb{Z}} \langle f, D^nT^\ell \psi \rangle D^nT^\ell \psi, \forall f \in X.$$  

A measurable set $E$ is called an NTF wavelet set for $X$ if the function $\psi_E$ defined by $\hat{\psi}_E = \frac{1}{\sqrt{2\pi}} \chi_E$ is an NTF wavelet for $X$.

**Remark.** The convergence in (1) is unconditional in norm topology of $L^2(\mathbb{R})$. The same is true for all other similar occasions unless otherwise specified.

Let $E$ be a Lebesgue measurable set with finite measure. Let $\psi$ be the function defined by $\hat{\psi}(s) = \frac{1}{\sqrt{2\pi}} \chi_E$. Consider the formal sum

$$G_E(f) = \sum_{n, \ell \in \mathbb{Z}} \langle f, D^nT^\ell \psi \rangle D^nT^\ell \psi,$$

where $f$ is any function in $L^2(\mathbb{R})$. If $G_E(f)$ converges (in $L^2(\mathbb{R})$ norm) for all $f \in L^2(\mathbb{R})$, then it defines a bounded linear operator on $L^2(\mathbb{R})$ (by the Banach-Steinhaus Theorem). Apparently, when this is the case, $E$ has to be a Bessel set. By definition, a frame wavelet set is always a Bessel set but not vice versa. The following proposition links the Bessel sets and basic sets.

**Proposition 1.** Let $E$ be a measurable set of finite measure. Then the following statements are equivalent:

(i) $E$ is a Bessel set.
(ii) $E$ is a basic set.
(iii) $G_E$ is a bounded linear operator.
The proof is omitted since (iii) \( \Rightarrow \) (i) is trivial, (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) can be found in Theorem 1 and its proof in [3].

Denote the formal sums
\[
\sum_{\ell \in \mathbb{Z}} \langle \hat{f}, \hat{D}^k \hat{T}^\ell \rangle \frac{1}{\sqrt{2\pi}} \chi_E \hat{D}^k \hat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E
\]
and
\[
\sum_{k, \ell \in \mathbb{Z}} \langle \hat{f}, \hat{D}^k \hat{T}^\ell \rangle \frac{1}{\sqrt{2\pi}} \chi_E \hat{D}^k \hat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E
\]
by \( H_E^k(\hat{f}) \) and \( H_E(\hat{f}) \) respectively. Notice that \( A\|f\|^2 \leq \langle G_E(f), f \rangle \leq B\|f\|^2 \) is equivalent to \( A\|\hat{f}\|^2 \leq \langle H_E(\hat{f}), \hat{f} \rangle \leq B\|\hat{f}\|^2 \).

**Definition 3.** Let \( E \) be a Bessel set. Then \( X_E = \{ f \in L^2(\mathbb{R}) : f = G_E(f) \} \) is called the subspace generated by \( E \).

**Note.** In general, the function \( \psi \) defined by \( \hat{\psi} = \frac{1}{\sqrt{2\pi}} \chi_E \) may not belong to \( X_E \), so \( E \) may not be an NTF wavelet set for \( X_E \). But if \( E \) is an NTF wavelet set for a closed subspace \( X \), then \( X = X_E \) by the definitions. What we are interested in is that under what conditions will \( E \) be an NTF wavelet set of \( X_E \) (or any other subspace \( X \))? Furthermore, under what conditions will \( X_E \) be a reducing subspace? Some partial answers to these questions are obtained in this paper.

Let \( E \) be a Lebesgue measurable set in \( \mathbb{R} \) with finite positive measure. For any \( f \in L^2(\mathbb{R}) \), let \( \hat{f}^k_{m_j} \) be the \( 2^{k+1}\pi \) periodic extension of \( \hat{f} \cdot \chi_{2^k E(\mathbb{R})} \) over \( \mathbb{R} \). The following proposition is the main tool we need in proving our results. Its proof can be found in [3].

**Proposition 2.** Let \( E \) be a Bessel set (so it is also a basic set). Let \( M \) be as defined in Definition 1. Then \( H_E^k(\hat{f}) \) converges to \( \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \hat{f}^k_{m_j} \cdot \chi_{2^k E} \) and \( H_E(\hat{f}) \) converges \( \sum_{k \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \hat{f}^k_{m_j} \cdot \chi_{2^k E} \) for all \( f \in L^2(\mathbb{R}) \). Here the convergence is unconditional under the norm of \( L^2(\mathbb{R}) \).

§2. THEOREMS AND PROOFS

We begin this section with the following theorem, which is a natural generalization of the characterization (Theorem 5.4, [7]) of normalized tight frame wavelet set for \( L^2(\mathbb{R}) \) to the reducing subspaces.

**Theorem 1.** Let \( X \) be a reducing subspace so that \( \hat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega \), where \( \Omega = 2\Omega \) is a measurable subset in \( \mathbb{R} \). Then a measurable set \( E \) is an NTF wavelet set for \( X \) if and only if \( E = E(\delta, 1) = E(\tau, 1) \) and \( \bigcup_{k \in \mathbb{Z}} 2^k E = \Omega \).

**Proof.** Assume that \( E \) satisfies the conditions in Theorem 1. For any \( f \in X \), we have \( f = \sum_{n \in \mathbb{Z}} f \chi_{2^n E} \) since \( \Omega = \bigcup_{n \in \mathbb{Z}} 2^n E \) and the union is a disjoint union. But by Proposition 2, \( H_E(\hat{f}) \) also converges in \( L^2(\mathbb{R}) \) to \( \sum_{n \in \mathbb{Z}} H_E^0(\hat{f}) = \sum_{n \in \mathbb{Z}} \hat{f} \chi_{2^n E} \) since \( \mu(E(\tau, m)) = \mu(E(\delta, m)) = 0 \) for all \( m \geq 2 \).

Now assume that \( E \) is an NTF wavelet set for \( X \). Since \( X \) is a reducing subspace by the assumption, \( \chi_\Omega \in X = L^2(\mathbb{R}) \cdot \chi_\Omega \), where \( \Omega = 2\Omega \). So \( E \subseteq \Omega \) and hence \( E(\tau, m) \subseteq \Omega \). Therefore, \( f = \chi_{E(\tau, m)} \in X \) and \( \hat{f} = H_E(\hat{f}) \). By Proposition 2, \( H_E^0(\hat{f}) = m \hat{f} \). So \( \|\hat{f}\|^2 = \langle H_E(\hat{f}), \hat{f} \rangle \geq \langle H_E^0(\hat{f}), \hat{f} \rangle = m\|\hat{f}\|^2 \). Therefore \( \mu(E(\tau, m)) = 0 \) for all \( m > 1 \) and \( E = E(\tau, 1) \). If \( \mu(E(\delta, k)) > 0 \) for some \( k > 1 \),
we can find a subset $F$ of $E$ such that $\mu(F) > 0$ and $2^n F \subset E$ for some $p > 0$. A contradiction can be then derived in a similar way by choosing $\hat{f} = \chi_F$. The fact that $\Omega = \bigcup_{n \in \mathbb{Z}} 2^n E$ is rather obvious.

**Theorem 2.** Let $E$ be a Bessel set. Then:

(i) If $E = E(\delta, 1)$, then $X_E$ is a reducing subspace with $\hat{X}_E = L^2(\mathbb{R})\chi_\Omega$, where $\Omega = \bigcup_{n \in \mathbb{Z}} 2^n E(\tau, 1)$.

(ii) If $E = E(\tau, 1)$, then $X_E$ is a reducing subspace with $\hat{X}_E = L^2(\mathbb{R})\chi_\Omega$, where $\Omega = \bigcup_{n \in \mathbb{Z}} 2^n E(\delta, 1)$.

(iii) There exists a maximal reducing subspace $Y \subset X_E$. Furthermore, $\hat{Y} = L^2(\mathbb{R})\chi_\Omega$, where $\Omega = \bigcup_{n \in \mathbb{Z}} 2^n (E(\delta, 1) \cap E(\tau, 1))$.

**Proof.** (i) Let $\hat{f} \in L^2(\mathbb{R}) \cdot \chi_\Omega$, and $F = E(\tau, 1)$. Since sup$\{\hat{f}\} \subset \Omega = \bigcup_{n \in \mathbb{Z}} 2^n F$, $\hat{f}_{m,j}^k = 0$ for all $m \geq 2$. Therefore, by Proposition 3 $H_E(\hat{f}) = \sum_{n \in \mathbb{Z}} \hat{f} : \chi_{2^n E} = \hat{f}$. So $f \in X_E$ and it follows that $L^2(\mathbb{R}) \cdot \chi_\Omega \subseteq X_E$.

Now assume $f \in X_E$. By Proposition 2, we have

$$\hat{f} = H_E(\hat{f}) = \sum_{n \in \mathbb{Z}} \sum_{m=1}^{M} \sum_{j=1}^{m} \hat{f}_{m,j} \chi_{2^n E}$$

where the summation converges in $L^2(\mathbb{R})$ norm and hence converges almost everywhere. Let $s \in 2^n E(\tau, m)$ be any point such that the summation converges at all its $2^n+1 \pi$ equivalent points (denoted by $s_1$, $s_2$, ..., $s_m$) in $2^n E(\tau, m)$. Then we have $\hat{f}(s_i) = \sum_{1 \leq j \leq m} \hat{f}(s_j)$ for any $1 \leq i \leq m$ by Proposition 1. This leads to $\hat{f}(s_i) = 0$ if $m > 1$. Since $s$ is arbitrary, $\hat{f} = 0$ on $2^n E(\tau, m)$ for any $n \in \mathbb{Z}$ and any $m > 1$.

Therefore, sup$\{\hat{f}\} \subset \bigcup_{n \in \mathbb{Z}} 2^n E(\tau, 1)$.

(ii) It is obvious that $L^2(\mathbb{R}) \chi_\Omega \subseteq \hat{X}_E$. Let $F = E \setminus E(\delta, 1)$, and let $\Omega_1 = \bigcup_{n \in \mathbb{Z}} 2^n E(\delta, 1)$, $\Omega_2 = \bigcup_{n \in \mathbb{Z}} 2^n F$. For any $f \in X_E$, $\|f\|^2 = \|f\chi_{\Omega_1}\|^2 + \|f\chi_{\Omega_2}\|^2$. On the other hand, since $E = E(\tau, 1)$, we have $\hat{f} = \sum_{n \in \mathbb{Z}} \hat{f} \chi_{2^n E}$ by Proposition 2. It follows that $\|\hat{f}\|^2 \geq \int_{\Omega_1} |\hat{f}|^2 ds + 2 \int_{\Omega_2} |\hat{f}|^2 ds$. Thus, $\int_{\Omega_2} |\hat{f}|^2 ds = 0$ and hence the support of $\hat{f}$ is contained in $\Omega_1$.

(iii) It is obvious that the subspace defined by $\hat{Y} = L^2(\mathbb{R}) \chi_\Omega$ is a reducing subspace. We need only to show that if $Y_1$ is also a reducing subspace contained in $X_E$, then $Y_1 \subset Y$. Let $f \in Y_1$: then $\hat{f} \in \hat{Y}_1$ and hence $|\hat{f}| \in \hat{Y}_2$. It follows that $|\hat{f}| \in \hat{X}_E$. By Proposition 1 and similar arguments in the above proofs, we get sup$\{\hat{f}\} \subset \Omega$. Hence $\hat{f} \in \hat{Y}$. Details are left to our reader.

**Theorem 3.** Let $X$ be a closed subspace of $L^2(\mathbb{R})$. The following statements are equivalent:

(i) $X$ is a reducing subspace.

(ii) $\hat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$, where $\Omega = 2\Omega$ is a Lebesque measurable set.

(iii) There exists a Bessel set $E$ which is a wavelet set for $X$.

(iv) There exists a Bessel set $E$ which is an NTF wavelet set for $X$.

(v) $X = X_E$ for a Bessel set $E$ such that $E = E(\delta, 1) = E(\tau, 1)$.

**Proof.** (i)⇒(ii) is from Proposition 4.3 in [2]. (ii)⇒(iii) is Theorem 4.4 in [2]. (iii)⇒(iv) is trivial and (v)⇒(i) is implied by Theorem 2 above. So we only need to show that (iv)⇒(v). Notice that in general we have $\langle H_E(\hat{f}), \hat{f} \rangle \geq \langle H_E^0(\hat{f}), \hat{f} \rangle$.
for any $f \in L^2(\mathbb{R})$. Let $\hat{f} = \chi_E \in \hat{X}$; we have $\mu(E) = \|\hat{f}\|^2 = (H_E(\hat{f}), \hat{f})$. On the other hand, $H_E^2(\hat{f}) = \sum_{m=1}^{M} m\chi_{E(t,m)}$ by Proposition 2. It follows that $\mu(E) \geq \sum_{m=1}^{M} \int_{\mathbb{R}} m\chi_{E(t,m)} ds = \sum_{m=1}^{M} m\mu(E(t,m))$, hence $\mu(E(t,m)) = 0$ for all $m > 1$. Similarly, one can show that $\mu(E(\delta,m)) = 0$ for all $m > 1$. $X = X_E$ since $E$ is an NTF set for $X$ as pointed out in the note after Definition 3.

§3. EXAMPLES AND DISCUSSIONS

We now conclude this paper with some examples and discussions.

1. Let $X$ be a closed reducing subspace of $D,T$. Then for any $0 < \alpha < 2\pi$, we can find an NTF wavelet set $E$ for $X$ such that $\mu(E) = \alpha$ as shown below.

Let $\tilde{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$ where $\Omega = 2\Omega$. It is shown in [2] that there exists a measurable set $S$ such that $S = S(\delta,1)$, $\bigcup_{k \in \mathbb{Z}} 2^k S = \Omega$ and $S \approx [0,2\pi]$. Let $n \in \mathbb{Z}$ be such that $\frac{\alpha}{\pi} < \min\{\alpha, 2\pi - \alpha\}$. For any $t \in [2^{n+2}, \pi)$, define $A = \bigcup_{k \in \mathbb{Z}} (S \cap (-t, t + 2\pi))$ and $B = S \cap ((-t, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, t))$. Let $C = S \setminus (A \cup B)$ and $D$ be the subset of $(\frac{\pi}{2}, \frac{\pi}{2} + \pi)$ that is 2-dilation equivalent to $A \cup B$. It can be verified that $S_t = C \cup D$ is an NTF wavelet set. Note that when $t = \frac{2\pi - \alpha}{\pi}$, the measure of $S_t$ is greater than $\alpha$, whereas when $t = \pi$, the measure of $S_t$ is less than $\frac{\alpha}{\pi}$ which is less than $\alpha$. Since the measure of $S_t$ is a continuous function of $t$, the conclusion follows.

2. Let $E$ be a wavelet set (for $L^2(\mathbb{R})$ or a reducing subspace). Let $F$ be a measurable subset of $E$. Then by Theorem 1 $F$ is a normalized tight frame wavelet set for the subspace $X_F$ with $\tilde{X}_F = L^2(\mathbb{R}) \cdot \chi_\Omega$, where $\Omega = \bigcup_{k \in \mathbb{Z}} 2^n F$. In particular, any measurable subset in $[-2\pi, -\pi) \cup [\pi, 2\pi)$ with positive measure is such a set.

3. Any basic set $E$ such that $E \neq E(\delta,1)$ or $E \neq E(\tau,1)$ is not an NTF wavelet set for $X_E$. This can be seen from Proposition 2.

4. Although Theorem 2 guarantees the existence of a maximal reducing subspace in $X_E$, it does not say when or whether $X_E$ is itself a reducing subspace. The following example shows that in general $X_E$ is not necessarily a reducing subspace.

Let $E_1 = [2\pi, 2.75\pi)$, $E_2 = [4\pi, 4.75\pi)$, $E_3 = [4.75\pi, 5\pi)$, $E_4 = [10\pi, 10.75\pi)$, $E_5 = [10.75\pi, 11\pi)$ and $E = \bigcup_j E_j$. We have $E = E(\delta,2), E(\tau,1) = 0, E(\tau,2) = E_3 \cup E_5$ and $E(\tau,3) = E_1 \cup E_2 \cup E_4$. Define $\hat{f}(s) = 2\chi_{E_1} - \chi_{E_2} - \chi_{E_4} + \chi_{E_5} - \chi_{E_1}$. $H_E(\hat{f})(s) = f(s)$ by Proposition 1 so $\hat{f}(s) \in \hat{X}_E$, but $H_E(\|f\|)(s) \neq |\hat{f}(s)|$. Thus, $X_E$ cannot be a reducing subspace. Since if it were, then $|\hat{f}(s)|$ would be in it as well.

5. Although we proved in Theorem 3 that any subspace admitting an NTF wavelet defined by a Bessel set is necessarily a reducing subspace, this is not true for general NTF wavelets. In fact, one can construct non-reducing subspaces of $L^2(\mathbb{R})$ that admit wavelets (hence NTF wavelets); see [2].

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REFERENCES


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