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# ON THE DISTRIBUTION SINGULAR VALUES OF TOEPLITZ MATRICES

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ABSTRACT. We prove a second order formula concerning distribution of singular values of Toeplitz matrices in some cases when conditions of the H. Widom Theorem are not satisfied.

## 1. INTRODUCTION AND NOTATION

In 1920 G. Szegö proved a basic result concerning the distribution of the eigenvalues $\left\{ \lambda_k^{(n)} \right\}_{k=1}^n$ of the Toeplitz matrix

$$T_n(f) = \left(\hat{f}_{i-j}\right)_{i,j=0}^{n-1} \left(\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}\, d\theta\right)$$

associated with a bounded real valued function $f$ on the interval $[-\pi, \pi]$: For any continuous function $F$ one has

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n F\left(\lambda_k^{(n)}\right)}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\left(f(\theta)\right)\, d\theta. \tag{1}$$

An analogous result holds for the singular values $s_1^{(n)} \geq s_2^{(n)} \geq \ldots \geq s_n^{(n)}$ of not necessarily selfadjoint Toeplitz matrices $T_n(f)$. The analogue of (1) is

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n F\left(s_k^{(n)}\right)}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\left(|f(\theta)|\right)\, d\theta. \tag{2}$$

Let

$$K = \left\{ f \in L^\infty(-\pi, \pi) \,:\, \sum_{k=-\infty}^{\infty} |k| |\hat{f}_k|^2 < \infty \right\},$$
$$M = \|f\|_\infty,\ m = dist\,(0, convR(f))$$

where $R(f)$ denotes the essential range of $f$ and "conv" denotes the convex hull. For $f \in K$ we let

$$||f||_K = \left( \sum_{k\in\mathbb{Z}} |k| \left| \hat{f}_k \right|^2 \right)^{\frac{1}{2}}.$$

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Then it is easy to see that each singular value of $T_n(f)$ belongs to the interval $[m, M]$ (see Lemma 1.2 in [7]).

Let $t_k^{(n)} = \left(s_k^{(n)}\right)^2$. H. Widom [7] proved a more exact formula than (1) and (2). Namely he proved

**Theorem 1.** *If $f \in K$ and $G \in C^3[m^2, M^2]$, then*

$$\begin{aligned}(3)\qquad &\lim_{n\to\infty}\left(\sum_{k=1}^{n} G\left(t_k^{(n)}\right) - \frac{n}{2\pi}\int_{-\pi}^{\pi} G\left(|f(\theta)|^2\right)\, d\theta\right)\\ &= \operatorname{tr}\left(G\left(T(\overline{f})T(f)\right) + G\left(T(f)\,T(\overline{f})\right) - 2T\left(G(|f|^2)\right)\right).\end{aligned}$$

We denote by $T(f)$ the infinite Toeplitz matrix $\left(\hat{f}_{i-j}\right)_{i,j=0}^{\infty}$ and by $H(f) = \left(\hat{f}_{i+j+1}\right)_{i,j=0}^{\infty}$ the infinite Hankel matrix associated with the symbol $f$.

If $f \in K$, it is obvious that $H(f)$ is a Hilbert-Schmidt operator. Moreover, $|H(f)|_2 \le \|f\|_K$, where $|\cdot|_2$ denotes the Hilbert-Schmidt norm of the operator $H(f)$ acting on $l^2$ of the nonnegative integers. For the little bit of the theory of trace class (i.e., nuclear) and the Hilbert-Schmidt operators that will be needed we refer the reader to [3].

It is easy to see that $T(\overline{f}) = (T(f))^*$ and hence the operators $T(\overline{f})T(f)$ and $T(f)\,T(\overline{f})$ are selfadjoint. Therefore, operators $G\left(T(\overline{f})T(f)\right)$ and $G\left(T(f)\,T(\overline{f})\right)$ are defined by the spectral theorem. If $f \ge 0$, the operator $T(f)$ is obviously nonnegative.

In the case of eigenvalues, formulae similar to (3) are established in [1] and [4] but under much more restrictive assumptions on $G$ (which is assumed to be analytic) and on the symbol $f$.

The function $G$ in Theorem 1 is given in terms of the function $F$ in relation (2) by $G(\lambda) = F\left(\sqrt{\lambda}\right)$. For $G$ to belong to $C^3$ it is not enough that $F$ belongs to $C^3$ but we also must have $F'(0) = F''(0) = F'''(0) = 0$ (in the case when $m = 0$).

It is conjectured in [7] that the condition $F \in C^3[0, M]$, $F'(0) = 0$ (in the case $m = 0$) is sufficient for the statement of Theorem 1.

Essentially, in the case $m = 0$ it is necessary to prove Theorem 1 when $F(\lambda) = \lambda^{\beta}$ and $\beta$ small enough. In this paper, we shall prove formula (3) in the case $m = 0$ and when the conditions of Theorem 1 are not satisfied.

## 2. Result

**Theorem 2.** *Let $f \in K$ and $m = 0$. If $F(\lambda) = \lambda^{\alpha}$ $(\alpha \ge 2)$, or $F \in C^6[0, M]$, $F'(0) = 0$, then the operator*

$$S = F\left(\sqrt{T(\overline{f})T(f)}\right) + F\left(\sqrt{T(f)T(\overline{f})}\right) - 2T\left(F(|f|)\right)$$

*is of the trace class and*

$$(4)\qquad \lim_{n\to\infty}\left(\sum_{k=1}^{n} F\left(s_k^{(n)}\right) - \frac{n}{2\pi}\int_{-\pi}^{\pi} F\left(|f(\theta)|\right)\, d\theta\right) = \operatorname{tr} S.$$

*Remark* 1. Theorem 2 is stated for the case $m = 0$. If $m > 0$, (4) holds according to Theorem 1. (Of course, in that case the condition $F'(0) = 0$ is superficial.)

In the proof of Theorem 2 we will use the following Lemma of Lizorkin [5]:

**Lemma 1.** *Let $\alpha \geq 1$, $\sigma > 0$. Then*

$$(i\lambda)^{\alpha} = A\exp\left(-i\frac{\pi\,\alpha\,\lambda}{2\cdot\sigma}\right)\int_{-\infty}^{\infty} e^{-i(\lambda-\sigma)\xi}\,d\varrho(\xi) \qquad (|\lambda| \leq \sigma)$$

*where $\varrho$ is a nondecreasing function of bounded variation and $A = \exp(-i\pi\alpha)$, $\varrho(\xi) = \sum_{k<\frac{\xi\sigma}{\pi}} a_k$, where $a_k$ are positive numbers related to Fourier coefficients $c_n$ of the function $(i\lambda)^{\alpha} e^{i\lambda\frac{\pi\alpha}{2\sigma}}$ by*

$$c_k = \overline{A} a_k e^{-ik\pi}.$$

Here the function $(i\lambda)^{\alpha} e^{i\lambda\frac{\pi\alpha}{2\sigma}}$ is assumed to be periodically extended from $(-\sigma,\sigma)$ to the entire real line. Our convention is $z^{\gamma} = e^{\gamma\ln z}$, $\ln z = \ln|z| + i\,arg\,z$, $0 \leq arg\,z < 2\pi$.

**Lemma 2.** *If $1 < \alpha < 2$ and $M > 0$, then there exists a nondecreasing function $\varrho_0$ of bounded variation, such that for each $\lambda \in [0, M^2]$*

$$\lambda^{\alpha} = \int_{-\infty}^{\infty} e^{-i2\pi\alpha - iM^2 t} e^{i\lambda t} d\varrho_0(t)$$

*holds and $\int\limits_{-\infty}^{\infty} t^2\,d\varrho_0(t) < \infty$.*

*Proof.* We apply Lemma 1, with $\sigma = M^2$. Since $\lambda \geq 0$, we have $(i\lambda)^{\alpha} = e^{i\frac{\pi}{2}\alpha}\cdot\lambda^{\alpha}$ and thus

$$\lambda^{\alpha} e^{-i\frac{\pi}{2}\alpha} = e^{-\pi\alpha} e^{i\lambda\frac{\pi\alpha}{2M^2}} \int_{-\infty}^{\infty} e^{i(\lambda - M^2)\xi}\,d\varrho(\xi),$$

i.e.,

$$\lambda^{\alpha} = e^{-\frac{3\pi i\alpha}{2}} \int_{-\infty}^{\infty} e^{-iM^2\xi} e^{i\lambda\left(\xi - \frac{\pi\alpha}{2M^2}\right)}\,d\varrho(\xi).$$

Substituting $\xi - \frac{\pi\alpha}{2M^2} = t$, $\varrho_0(t) \overset{def}{=} \varrho\left(t + \frac{\pi\alpha}{2M^2}\right)$ in the last formula we get

$$\lambda^{\alpha} = \int_{-\infty}^{\infty} e^{-2\pi i\alpha} e^{-iM^2 t} e^{i\lambda t}\,d\varrho_0(t).$$

The last formula holds for $\lambda \in [0, M^2]$ and $\alpha \geq 1$. Since $\varrho$ is a function of bounded variation, so is $\varrho_0$.

We will show now that for $1 < \alpha < 2$,

$$\int_{-\infty}^{\infty} t^2\,d\varrho_0(t) < \infty. \tag{5}$$

Since $\varrho_0$ is a function of bounded variation, applying the Cauchy inequality to (5) we get

$$\int_{-\infty}^{\infty} |t|\,d\varrho_0(t) < \infty.$$

Since $\varrho_0(t) = \varrho\left(t + \frac{\pi\alpha}{2M^2}\right)$, in order to prove (5) it is enough to show that

$$\int_{-\infty}^{\infty} t^2\,d\varrho(t) < \infty.$$

From the way the function $\varrho$ is defined it follows that it suffices to prove

$$\sum_{n\in\mathbb{Z}} n^2 a_n < \infty$$

where $a_n$ is the sequence of positive numbers from Lemma 1 (with $\sigma = M^2$), i.e., the convergence of the series

$$\sum_{n\in\mathbb{Z}} (-1)^n n^2 c_n \tag{6}$$

where $c_n$ are the Fourier coefficients of the function $(i\,\lambda)^{\alpha} e^{i\,\lambda\,\frac{\pi\,\alpha}{2M^2}}$ on the interval $[-M^2, M^2]$. Since

$$\begin{aligned}\int_{-M^2}^{M^2} (i\,\lambda)^{\alpha} \exp\left(\frac{i\,\lambda\pi\,\alpha}{2M^2}\right)\cdot\exp\left(-\frac{i\,\lambda n\pi}{M^2}\right) d\lambda \\ = \left(\frac{M^2}{\pi}\right)^{\alpha+1}\cdot\int_{-\pi}^{\pi}(i\,x)^{\alpha}\, e^{\frac{i\,\alpha\,x}{2}} e^{-i\,nx}\,dx,\end{aligned}$$

in order to prove the convergence of the series (6) it is enough to prove that for $1 < \alpha < 2$ the series

$$\sum_{n\in\mathbb{Z}} (-1)^n n^2 \int_{-\pi}^{\pi}(i\,x)^{\alpha}\, e^{\frac{i\,\alpha\,x}{2}} e^{-i\,nx}\,dx$$

converges.

Since $\alpha > 1$, integrating by parts twice and having in mind the definition of the function $z \longmapsto z^{\gamma}$, we conclude that the convergence of the above series will be established once we prove that the series

$$\sum_{n\in\mathbb{Z}} (-1)^n A_n$$

converges for $1 < \alpha < 2$. Here $A_n \overset{def}{=} \int_{-\pi}^{\pi}(i\,x)^{\alpha-2} e^{\frac{i\,\alpha\,x}{2}} e^{-i\,nx}\,dx$. Consider now the behavior of $A_n$ as $n \to \infty$. If $n > 0$, one gets

$$\begin{aligned}A_n = \left(n-\frac{\alpha}{2}\right)^{1-\alpha}\Bigg[-e^{\frac{i\,\pi\,\alpha}{2}}\cdot\int_0^{\pi\left(n-\frac{\alpha}{2}\right)} t^{\alpha-2}e^{-i\,t}\,dt - e^{\frac{3\pi i\,\alpha}{2}} \\ \cdot\int_0^{\pi\left(n-\frac{\alpha}{2}\right)} t^{\alpha-2}e^{i\,t}\,dt\Bigg].\end{aligned}$$

Since for $1 < \alpha < 2$ $\int_0^{\infty} t^{\alpha-2}e^{\pm it}\,dt = \Gamma(\alpha-1)e^{\pm i\frac{\pi}{2}(\alpha-1)}$ we obtain

$$\begin{aligned}A_n = \left(n-\frac{\alpha}{2}\right)^{1-\alpha}\Bigg[i\Gamma(\alpha-1)(e^{2\pi i\,\alpha}-1) + e^{\frac{i\,\pi\,\alpha}{2}}\cdot\int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2}e^{-i\,t}\,dt \\ + e^{\frac{3\pi i\,\alpha}{2}}\cdot\int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2}e^{i\,t}\,dt\Bigg].\end{aligned}$$

Integrating by parts, we get

$$\int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2}e^{-it}\,dt = (-1)^{n+1} i\left(\pi\left(n-\frac{\alpha}{2}\right)\right)^{\alpha-2}\cdot e^{i\frac{\pi\alpha}{2}} + O\left(n^{\alpha-3}\right),$$

$$\int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2}e^{it}\,dt = (-1)^{n} i\left(\pi\left(n-\frac{\alpha}{2}\right)\right)^{\alpha-2}\cdot e^{-i\frac{\pi\alpha}{2}} + O\left(n^{\alpha-3}\right), \quad n\to\infty,$$

and thus,

$$A_n = \left(n-\frac{\alpha}{2}\right)^{1-\alpha}\left[i\,\Gamma(\alpha-1)\left(e^{2\pi i\,\alpha}-1\right)+O\left(n^{\alpha-3}\right)\right].$$

Therefore, the series $\sum_{n=1}^{\infty}(-1)^n A_n$ converge. In a similar way one shows that the series

$$\sum_{n=-\infty}^{-1}(-1)^n A_n$$

also converge. □

*Remark* 2. From Lemma 2 (by integrating over $\lambda$) we obtain the representation

$$\lambda^{\alpha+1} = \int_{-\infty}^{+\infty} e^{-2\pi i\alpha - iM^2 t}\cdot\frac{e^{i\lambda t}-1}{it}\,(\alpha+1)\,d\rho_0\,(t)\,, \tag{7}$$

for $1<\alpha<2, \lambda\in\left[0,M^2\right]$.

Let $d\nu = \frac{\alpha+1}{it}d\rho_0$. If $1<\alpha<2$, then the function $\rho_0$ does not have a jump at the $t=0$, hence

$$\int_{-\infty}^{+\infty}|t|^k\,d\,|\nu|<\infty \quad \text{for } k=0,1,2,3$$

($|\nu|$ is a variation of measure $\nu$). From (7), putting $\beta=\alpha+1$, we get

$$\lambda^{\beta} = \int_{-\infty}^{+\infty} e^{-2\pi i\alpha - iM^2 t}e^{i\lambda t}d\nu\,(t) - A, \quad 2<\beta<3, \lambda\in\left[0,M^2\right], \tag{8}$$

and

$$A = \int_{-\infty}^{+\infty} e^{-2\pi i\alpha - iM^2 t}\,d\nu\,(t)\,.$$

We write $P_n$ for the projection operator, defined by

$$P_n(x_0,x_1\ldots) = (x_0,x_1,x_2,\ldots,x_{n-1},0,0,\ldots)$$

from $l^2$ to the subspace of $l^2$ on which $T_n(f)$ may be thought of as acting. We identify $T_n(f)$ with $P_nT(f)P_n$ in the obvious way. We define an operator $Q_n$ on $l^2$ by

$$Q_n(x_0,x_1,\ldots) = (x_{n-1},x_{n-2},\ldots,x_1,x_0,0,0,\ldots).$$

For $f\in L^{\infty}(-\pi,\pi)$ we define $\tilde{f}(\theta)=f(-\theta)$.

**Lemma 3.** **1**) *For any* $f, g \in L^\infty(-\pi,\pi)$ *we have*

$$T(f\cdot g) - T(f)\cdot T(g) = H(f)H(\tilde g),$$
$$T_n(f\cdot g) - T_n(f)\cdot T_n(g) = P_n H(f)H(\tilde g)P_n + Q_n H(\tilde f)H(g)Q_n.$$

**2**) *If* $f \in K$, *then we have* $\tilde f \in K$, $|f|^2 \in K$, $e^{itf} \in K$ $(\forall t\in\mathbb{R})$ *and*

$$\left\||f^2|\right\|_K \le \ \mathrm{const}\,\|f\|_K,$$
$$\left\|e^{itf}\right\|_K \le \ \mathrm{const}\cdot|t|\cdot\|f\|_K.$$

*Proof.* **1)** Routine computation. (Or see [2], Propositions 2.7 and 3.6.)

**2)** Can be proved in a same way as Proposition 1 in [6]. □

**Lemma 4.** *If* $f \in K$, $1 < \alpha < 2$, *then the operator* $\left(T\left(|f|^2\right)\right)^\alpha - T\left(|f|^{2\alpha}\right)$ *is nuclear.*

*Proof.* Integrating the identity

$$\frac{d}{ds}\left(T\left(e^{is|f|^2}\right)e^{-isT\left(|f|^2\right)}\right) = \left[T\left(e^{is|f|^2}\cdot i|f|^2\right) - T\left(e^{is|f|^2}\right)T\left(i\,|f|^2\right)\right]e^{-isT\left(|f|^2\right)}$$

on the interval $[0,t]$ and multiplying the result by $e^{itT\left(|f|^2\right)}$ on the right, we get

$$T\left(e^{it|f|^2}\right) - e^{itT\left(|f|^2\right)} = \int_0^t H\left(e^{is|f|^2}\right)\cdot H\left(i\left|\tilde f\right|^2\right)e^{i(t-s)T\left(|f|^2\right)}\,ds. \tag{9}$$

Since $f \in K$, we have $e^{is|f|^2} \in K$, $i\,|f|^2 \in K$. Applying Lemma 3, formula (9) yields

$$\left|T\left(e^{it|f|^2}\right) - e^{itT\left(|f|^2\right)}\right|_1 \le \int_0^{|t|}\left|H\left(e^{is|f|^2}\right)\cdot H\left(i|\tilde f|^2\right)\right|_1 ds$$

for all $t\in\mathbb{R}$, since the operator $e^{i(t-s)T\left(|f|^2\right)}$ is unitary. Here $|\cdot|_1$ denotes the nuclear norm of an operator. Since the operators $H\left(e^{is|f|^2}\right)$ and $H\left(i\left|\tilde f\right|^2\right)$ are Hilbert-Schmidt, their product is nuclear, and thus, according to Lemma 3 (statement 2)) we get

$$\left|H\left(e^{is|f|^2}\right)\cdot H\left(i|\tilde f|^2\right)\right|_1 \le c_0\cdot|s|\cdot\|f\|_K^2$$

($c_0$ is independent of $s$), and thus

$$\left|T\left(e^{it|f|^2}\right) - e^{itT\left(|f|^2\right)}\right|_1 \le c_0\|f\|_K^2\int_0^{|t|} s\,ds = \frac{c_0}{2}t^2\|f\|_K^2. \tag{10}$$

According to Lemma 2

$$\left(T\left(|f|^2\right)\right)^\alpha = \int_{\mathbb{R}} e^{-2\pi i\alpha - iM^2t}e^{itT\left(|f|^2\right)}\,d\varrho_0(t),$$
$$|f|^{2\alpha} = \int_{\mathbb{R}} e^{-2\pi i\alpha - iM^2t}e^{it|f|^2}\,d\varrho_0(t),$$

and thus,

$$T\left(|f|^{2\alpha}\right) = \int e^{-2\pi i\alpha - iM^2t}T\left(e^{it|f|^2}\right)\,d\varrho_0(t).$$

Therefore,

$$\left(\left(T\left(|f|^2\right)\right)^{\alpha} - T\left(|f|^{2\alpha}\right)\right) = \int_{\mathbb{R}} e^{-2\pi i\alpha - iM^2 t}\left(e^{itT\left(|f|^2\right)} - T\left(e^{it|f|^2}\right)\right) d\varrho_0(t).$$

Inequality (10) shows that the integral on the right side in the formula above, converges in nuclear norm and thus $\left(\left(T\left(|f|^2\right)\right)^{\alpha} - T\left(|f|^{2\alpha}\right)\right)$ is a nuclear operator. Moreover,

$$\left|\left(\left(T\left(|f|^2\right)\right)^{\alpha} - T\left(|f|^{2\alpha}\right)\right)\right|_1 \le \frac{c_0}{2}\|f\|_K^2 \int_{\mathbb{R}} t^2\, d\varrho_0(t) < +\infty \quad \text{(by Lemma 2)}.$$

□

**Lemma 5.** *If $1 < \alpha < 2$ and $f \in K$, then*

$$\lim_{n\to\infty} \operatorname{tr}\left(\left(T_n(|f|^2)\right)^{\alpha} - T_n(|f|^{2\alpha})\right) = 2\operatorname{tr}\left(\left(T(|f|^2)\right)^{\alpha} - T(|f|^{2\alpha})\right).$$

*Proof.* In a same way as we proved (9) we get

$$\begin{aligned} &T_n\left(e^{it|f|^2}\right) - e^{itT_n\left(|f|^2\right)} \\ &\quad= \int_0^t \left[T_n\left(e^{is|f|^2} i\,|f|^2\right) - T_n\left(e^{is|f|^2}\right) T_n\left(i\,|f|^2\right)\right] e^{i(t-s)T_n\left(|f|^2\right)}\, ds \end{aligned}$$

and thus by Lemma 3

$$\begin{aligned} T_n\left(e^{it|f|^2}\right) - e^{itT_n\left(|f|^2\right)} = \int_0^t &\left[P_n H\left(e^{is|f|^2}\right) H\left(i\,|\tilde{f}|^2\right) P_n \right.\\ &\left. + Q_n H\left(e^{is|\tilde{f}|^2}\right) H\left(i\,|f|^2\right) Q_n\right] e^{i(t-s)T_n\left(|f|^2\right)}\, ds. \end{aligned} \tag{11}$$

From Lemma 2 we obtain

$$\begin{aligned} &\left(\left(T_n\left(|f|^2\right)\right)^{\alpha} - T_n\left(|f|^{2\alpha}\right)\right) \\ &\quad= \int_{\mathbb{R}} e^{-2\pi i\alpha - iM^2 t}\left(e^{itT_n\left(|f|^2\right)} - T_n\left(e^{it|f|^2}\right)\right) d\varrho_0(t) \qquad (1 < \alpha < 2). \end{aligned} \tag{12}$$

It follows from (11) that

$$\left|T_n\left(e^{it|f|^2}\right) - e^{itT_n\left(|f|^2\right)}\right|_1 \le \text{const}\ |t|^2, \quad \forall t \in \mathbb{R}.$$

(const does not depend on $t$ and $n$) and thus, since $\int_{\mathbb{R}} |t|^2\, d\varrho_0(t) < \infty$, by the same arguments as in the proof of (14) in [7] and by the Lebesgue theorem on dominant convergence, (11) and (12) give

$$\lim_{n\to\infty} \operatorname{tr}\left(\left(T_n(|f|^2)\right)^{\alpha} - T_n(|f|^{2\alpha})\right) = 2\operatorname{tr}\left(\left(T(|f|^2)\right)^{\alpha} - T(|f|^{2\alpha})\right).$$

□

**Lemma 6.** *If $f \in K$, $1 < \alpha < 2$, then the operator $\left(T(\overline{f})T(f)\right)^{\alpha} + \left(T(f)T(\overline{f})\right)^{\alpha} - 2T\left(|f|^{2\alpha}\right)$ is nuclear and*

$$\begin{aligned} &\lim_{n\to\infty} \operatorname{tr}\left[\left(T_n(\overline{f})T_n(f)\right)^{\alpha} - T_n(|f|^{2\alpha})\right] \\ &\qquad= \operatorname{tr}\ \left[\left(T(\overline{f})T(f)\right)^{\alpha} + \left(T(f)T(\overline{f})\right)^{\alpha} - 2T(|f|^{2\alpha})\right]. \end{aligned}$$

*Proof.* By Lemma 2 we have

$$\begin{aligned}(13)\qquad &\left(T(\overline{f})T(f)\right)^{\alpha} - T_n\left(|f|^2\right)^{\alpha}\\ &= \int_{-\infty}^{\infty} e^{-2\pi i\,\alpha - i\,M^2 t}\cdot\left(e^{i\,tT(\overline{f})T(f)} - e^{i\,tT(|f|^2)}\right)\,d\varrho_0(t).\end{aligned}$$

Since $\left|e^{i\,tT(\overline{f})T(f)} - e^{i\,tT(|f|^2)}\right|_1 \le$ const $\cdot |t|$ $(t\in\mathbb{R})$, and $\int_{\mathbb{R}}|t|\,d\varrho_0(t) < +\infty$, the integral in (13) converges (in nuclear norm) and thus the operator $\left(T(\overline{f})T(f)\right)^{\alpha} - T\left(|f|^2\right)^{\alpha}$ is nuclear. In a similar way we prove that the operator $\left(T(f)T(\overline{f})\right)^{\alpha} - T_n\left(|f|^2\right)^{\alpha}$ is nuclear. Thus $\left(T(\overline{f})T(f)\right)^{\alpha} + \left(T(f)T(\overline{f})\right)^{\alpha} - 2T\left(|f|^2\right)^{\alpha}$ is also nuclear. Therefore, according to Lemma 4 the operator $\left(T(\overline{f})T(f)\right)^{\alpha} + \left(T(f)T(\overline{f})\right)^{\alpha} - 2T\left(|f|^{2\,\alpha}\right)$ is nuclear. Since $\left|e^{i\,tT_n(\overline{f})T(f)} - e^{i\,tT_n(|f|^2)}\right|_1 \le d_0\cdot|t|$ $(t\in\mathbb{R},\ d_0$ is independent of $n$ and $t)$ and $\int_{-\infty}^{\infty}|t|\,d\varrho_0(t) < +\infty$, it follows from

$$\begin{aligned}&\left(T_n(\overline{f})T_n(f)\right)^{\alpha} - T_n\left(|f|^2\right)^{\alpha}\\ &= \int_{-\infty}^{\infty} e^{-2\pi\,i\,\alpha - i\,M^2 t}\left(e^{i\,tT_n(\overline{f})T(f)} - e^{i\,tT_n(|f|^2)}\right)\,d\varrho_0(t)\end{aligned}$$

that

$$\begin{aligned}&\lim_{n\to\infty}\operatorname{tr}\ \left(\left(T_n(\overline{f})T_n(f)\right)^{\alpha} - T_n\left(|f|^2\right)^{\alpha}\right)\\ &= \int_{-\infty}^{\infty} e^{-2\pi\,i\,\alpha - i\,M^2 t}\lim_{n\to\infty}\ \operatorname{tr}\ \left(e^{i\,tT_n(\overline{f})T(f)} - e^{i\,tT_n(|f|^2)}\right)\,d\varrho_0(t).\end{aligned}$$

From relation (14) in [7] and from Lemma 2, the last equality becomes

$$\begin{aligned}(14)\qquad &\lim_{n\to\infty}\operatorname{tr}\ \left(\left(T(\overline{f})T(f)\right)^{\alpha} - T_n\left(|f|^2\right)^{\alpha}\right)\\ &= \operatorname{tr}\ \left(\left(T(\overline{f})T(f)\right)^{\alpha} + \left(T(f)T(\overline{f})\right)^{\alpha} - 2T\left(|f|^2\right)^{\alpha}\right).\end{aligned}$$

From (14) and Lemma 5, adding, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\operatorname{tr}\left[\left(T_n(\overline{f})T_n(f)\right)^{\alpha} - T_n\left(|f|^{2\,\alpha}\right)\right]\\ &= \operatorname{tr}\ \left[\left(T(\overline{f})T(f)\right)^{\alpha} + \left(T(f)T(\overline{f})\right)^{\alpha} - 2T\left(|f|^{2\,\alpha}\right)\right].\end{aligned}$$

□

*Remark* 3. By using representation (8) by the same method as the one used for proving Lemmas 4, 5, 6 one can show that:

**Lemma 7.** *If $f\in K$ and $2<\beta<3$, then the operator $T\left(|f|^2\right)^{\beta} - T\left(|f|^{2\beta}\right)$ is the trace class and the following holds*:

$$\lim_{n\longrightarrow\infty}\operatorname{tr}\left[T_n\left(|f|^2\right)^{\beta} - T_n\left(|f|^{2\beta}\right)\right] = 2\operatorname{tr}\left[T\left(|f|^2\right)^{\beta} - T\left(|f|^{2\beta}\right)\right].$$

*Also, The operator $\left(T(\overline{f})T(f)\right)^{\beta} + \left(T(f)T(\overline{f})\right)^{\beta} - 2T\left(|f|^{2\beta}\right)$ is the trace class and the following holds*:

$$\begin{aligned}&\lim_{n\to\infty}\operatorname{tr}\ \left[\left(T_n(\overline{f})T_n(f)\right)^{\beta} - T_n\left(|f|^{2\beta}\right)\right]\\ &= \operatorname{tr}\ \left[\left(T(\overline{f})T(f)\right)^{\beta} + \left(T(f)T(\overline{f})\right)^{\beta} - 2T\left(|f|^{2\beta}\right)\right].\end{aligned}$$

## 3. Proof of Theorem 2

Note that Theorem 2 holds for the functions $\lambda \longmapsto \lambda^2$, $\lambda \longmapsto \lambda^4$ and $\lambda \longmapsto \lambda^{\alpha}$ ($\alpha \geq 6$) as a consequence of Theorem 1. In other words,

$$\begin{aligned}(15)\qquad &\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(s_k^{(n)}\right)^{\alpha} - \frac{n}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^{\alpha}\,d\theta\right)\\ &= \operatorname{tr}\left[\left(T(\overline{f})T(f)\right)^{\frac{\alpha}{2}} + \left(T(f)T(\overline{f})\right)^{\frac{\alpha}{2}} - 2T\left(|f|^{\alpha}\right)\right],\end{aligned}$$

for $\alpha = 2, 4$ and $\alpha \geq 6$.

From Lemma 6 and Lemma 7 we obtain that (15) also holds if $2 < \alpha < 4$ and $4 < \alpha < 6$. Therefore, formula (4) holds if $F(\lambda) = \lambda^{\alpha}$ and $\alpha \geq 2$.

Now let $F \in C^6\left[0, M^2\right]$ and $F'(0) = 0$. Then, for the function $F_0(\lambda) = \sum_{k=2}^{6} \frac{F^{(k)}(0)}{k!}\lambda^k$ we have

$$\begin{aligned}(16)\qquad &\lim_{n\to\infty}\left(\sum_{k=1}^{n}F_0\left(s_k^{(n)}\right) - \frac{n}{2\pi}\int_{-\pi}^{\pi}F_0\left(|f(\theta)|\right)\,d\theta\right)\\ &= \operatorname{tr}\left[F_0\left(\sqrt{T(\overline{f})T(f)}\right) + F_0\left(\sqrt{T(f)T(\overline{f})}\right) - 2T\left(F_0\left(|f|\right)\right)\right].\end{aligned}$$

Let $R(\lambda) = F(\lambda) - F_0(\lambda)$. The function $\lambda \longmapsto R(\sqrt{\lambda})$ satisfies the conditions of Theorem 1 and hence

$$\begin{aligned}(17)\qquad &\lim_{n\to\infty}\left(\sum_{k=1}^{n}R\left(s_k^{(n)}\right) - \frac{n}{2\pi}\int_{-\pi}^{\pi}R\left(|f(\theta)|\right)\,d\theta\right)\\ &= \operatorname{tr}\left[R\left(\sqrt{T(\overline{f})T(f)}\right) + R\left(\sqrt{T(f)T(\overline{f})}\right) - 2T\left(R\left(|f|\right)\right)\right].\end{aligned}$$

Adding (16) and (17) one gets (4). (The operators on the right-hand side of (16) and (17) are nuclear and so is their sum, i.e., the operator $S$ is nuclear.)

*Remark* 4. The question of whether the condition $F'(0) = 0$ in Theorem 2 is necessary remains open. To answer it affirmatively it is enough to find the example of a function $f \in K$ such that $m = dist(0, convR(f)) = 0$ and

$$\begin{aligned}&\lim_{n\to\infty}\left(\sum_{k=1}^{n}s_k^{(n)} - \frac{n}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|\,d\theta\right.\\ &\qquad \left. - \operatorname{tr}\left[\sqrt{T(\overline{f})T(f)} + \sqrt{T(f)T(\overline{f})} - 2T\left(|f|\right)\right]\right) \neq 0.\end{aligned}$$

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