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ANALYTIC EXTENSION OF DIFFERENTIABLE FUNCTIONS DEFINED IN CLOSED SETS BY MEANS OF CONTINUOUS LINEAR OPERATORS

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ABSTRACT. In this paper we solve the following problem posed by Schmets and Valdivia: Under which conditions does there exist an extension operator from the space $\mathscr{E}(F)$ of the Whitney jets on a closed set $F \subset \mathbb{R}^n$ to $\mathscr{E}(\mathbb{R}^n)$ so that the extended functions are real analytic outside F?

For a closed set $F \subset \mathbb{R}^n$ we denote by $\mathscr{E}(F)$ the space of the Whitney jets on F (see e.g. [5] or [2]). Whitney's extension theorem [5] states that every Whitney jet on F can be extended to a C^∞-function on $\mathbb{R}^n$, which is real analytic outside F. An extension operator E for $\mathscr{E}(F)$ is a linear continuous operator from $\mathscr{E}(F)$ into the space $\mathscr{E}(\mathbb{R}^n)$ which is a right inverse of the restriction $f \mapsto (\partial^\alpha f|_F)_{\alpha\in\mathbb{N}_0^n}$. Tidten [4] gave a characterization for the existence of an extension operator in terms of topological properties of the space $\mathscr{E}(F)$. More references to this topic can be found in [3], [1] and [4]. In [3] Schmets and Valdivia proved that for a compact set $K \subset \mathbb{R}^n$, admitting an extension operator E_1 for $\mathscr{E}(K)$, there is an extension operator E for $\mathscr{E}(K)$ with $E(f)$ analytic in $(\mathbb{R}^n \setminus K)^\star = \{u + iv : u \in \mathbb{R}^n \setminus K, v \in \mathbb{R}^n, |v| < \operatorname{dist}(u, K)\}$ for every $f \in \mathscr{E}(K)$. In particular, the values of E are real analytic outside K. They asked the question: what happens in the case of an unbounded closed $F \subset \mathbb{R}^n$? A complete characterization is contained in:

Theorem 1. *Assume that $F \subset \mathbb{R}^n$ is closed and $\mathscr{E}(F)$ admits an extension operator. Then the following are equivalent:*

i) *$\mathscr{E}(F)$ admits an extension operator E whose values are real analytic outside F.*
ii) *$\mathscr{E}(F)$ admits an extension operator E such that $E(f)$ has an analytic extension to $(\mathbb{R}^n \setminus F)^\star$ for all $f \in \mathscr{E}(F)$.*
iii) *For every $\rho > 0$ the boundary of the union of those components of the complement of F, which have nonempty intersection with the ball $B_\rho := \{x \in \mathbb{R}^n : |x| \le \rho\}$ of radius ρ, is bounded.*

For the proof of our result we use the following result of Schmets and Valdivia [3, Theorem 4.1]. In fact, it is the main tool in their construction.

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Theorem 2 (Schmets and Valdivia)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $BC^\infty(\Omega)$ denote the space of arbitrary often differentiable functions f on Ω for which all its partial derivatives $\partial^\alpha f$, $\alpha \in \mathbb{N}_0^n$, are bounded. Then there exists a linear continuous operator $T_\Omega : BC^\infty(\Omega) \to BC^\infty(\Omega)$ with the following two properties:*

i) *$T_\Omega(f)$ admits an analytic extension to*

$$\Omega^\star := \{u + iv : u \in \Omega, v \in \mathbb{R}^n, |v| < \operatorname{dist}(u, \partial\Omega)\}$$

for all $f \in BC^\infty(\Omega)$.

ii) *For all $f \in BC^\infty(\Omega)$, all $m \in \mathbb{N}$ and all $\varepsilon > 0$ there is $K \subset \Omega$ compact such that*

$$\sup\{|\partial^\alpha f(x) - \partial^\alpha T_\Omega(f)(x)| : |\alpha| \le m, x \in \Omega \setminus K\} < \varepsilon.$$

If there is an extension operator E_1 for $\mathscr{E}(K)$, where $K \subset \mathbb{R}^n$ is compact, then we may assume that there is a compact $L \subset \mathbb{R}^n$ such that $E_1(f)$ vanishes outside L for every $f \in \mathscr{E}(K)$. If we set

$$E(f)(x) := \begin{cases} f^{(0)}(x), & x \in K, \\ T_{\mathbb{R}^n \setminus K}(E_1(f))(x), & x \in \mathbb{R}^n \setminus K, \end{cases}$$

for $f = (f^{(\alpha)})_{\alpha \in \mathbb{N}_0^n} \in \mathscr{E}(K)$ and $x \in \mathbb{R}^n$, then E is an extension operator for $\mathscr{E}(K)$ so that the extended functions are real analytic on $\mathbb{R}^n \setminus K$; see [3]. Instead of $T_{\mathbb{R}^n \setminus K}$ one could also use the operator constructed by Langenbruch [1] which is given by an explicit formula.

Now we are ready for the proof of Theorem 1.

Proof. Let Ω denote the complement of F, let $\Omega^\star$ be as in Theorem 2, and let ω_ρ denote the union of those components of Ω, which have a nonempty intersection with B_ρ. Let us show first that i) implies iii).

For this we assume that there is $\rho > 0$ such that $\partial\omega_\rho$ is unbounded. The continuity of E implies that there exist $m \in \mathbb{N}$, $m > \rho$, and $C \ge 1$ such that

$$\sup\{|E(f)(x)| : |x| \le \rho\} \le C\|f\|_{m,B_m} \tag{1}$$

for all $f \in \mathscr{E}(F)$. Here $\|\cdot\|_{m,B_m}$ denotes the m-th Whitney norm with respect to the compact set $F \cap B_m$. Choose $x_0 \in \partial\omega_\rho \setminus B_{m+1} \subset F$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$ with $\varphi(x_0) \ne 0$ and $\operatorname{supp}(\varphi) \cap B_{m+1} = \emptyset$. By (1) we have $g \equiv 0$ on B_ρ, where $g := E((\partial^\alpha \varphi|_F)_{\alpha \in \mathbb{N}_0^n})$. On the other hand, g does not vanish in a neighborhood of x_0 and therefore, because g is real analytic, $g \not\equiv 0$ in B_ρ, which is a contradiction.

That ii) implies i) is trivial, so it remains to show that iii) implies ii). For this let us assume that $n \ge 2$, the case $n = 1$ can be treated similarly. In the case $n \ge 2$, the condition in iii) implies that F is compact or ω_ρ is bounded for all $\rho > 0$. The compact case is covered by the result of Schmets and Valdivia, so we may assume that F is not compact. Choose a sequence $(R_n)_{n\in\mathbb{N}}$ of positive numbers with $R_{n+1} > R_n + 1$ and $\omega_{R_n} \subset B_{R_{n+1}}$, $n \in \mathbb{N}$. We set $C_1 := \omega_{R_1}$, $C_n := \omega_{R_n} \setminus \omega_{R_{n-1}}$, $n \ge 2$. Without loss of generality, we assume that the C_n are nonempty. Then the sets C_n are open, pairwise disjoint and their union is Ω. Moreover, for every n there is m such that B_n is contained in $F \cup \bigcup_{1 \le k \le m} C_k$. We have $|\{n \in \mathbb{N} : x \in \partial C_n\}| \le 3$ for every $x \in F$.

Let E_1 be an extension operator for $\mathscr{E}(F)$ and define the linear and continuous operators $S_n : \mathscr{E}(F) \to BC^\infty(C_n)$ by $S_n(f) := E_1(f)|_{C_n}$, $n \in \mathbb{N}$. Now choose operators $T_{C_n} : BC^\infty(C_n) \to BC^\infty(C_n)$ according to Theorem 1 and define

$$E(f)(x) := \begin{cases} f^{(0)}(x), & x \in F, \\ T_{C_n}(S_n(f))(x), & x \in C_n, n \in \mathbb{N}. \end{cases}$$

Due to the properties of the C_n we obtain a well-defined linear continuous operator $E : \mathscr{E}(F) \to \mathscr{E}(\mathbb{R}^n)$ which is an extension operator for $\mathscr{E}(F)$ with the desired analyticity property. □

Remark 3. For $n = 1$ condition iii) is always satisfied. For $n \geq 2$ condition iii) is equivalent to the following: either F is compact or all ω_ρ are bounded. This implies that either F is compact or all components of the complement are bounded. But it is easy to see that there are closed sets F such that all components of the complement are bounded, nevertheless, F does not satisfy condition iii).

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