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LOGARITHMIC CONVEXITY OF EXTENDED MEAN VALUES

FENG QI

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Abstract. In this article, the logarithmic convexity of the extended mean values are proved and an inequality of mean values is presented. As by-products, two analytic inequalities are derived. Two open problems are proposed.

1. Introduction

The so-called extended mean values $E(r,s;x,y)$ were first defined by Professor K. B. Stolarsky in [21] as follows:

$$E(r,s;x,y)=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y)\neq 0, \tag{1}$$

$$E(r,0;x,y)=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, \qquad r(x-y)\neq 0, \tag{2}$$

$$E(r,r;x,y)=\mathrm{e}^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, \qquad r(x-y)\neq 0, \tag{3}$$

$$E(0,0;x,y)=\sqrt{xy}, \qquad x\neq y, \tag{4}$$

$$E(r,s;x,x)=x, \qquad x=y.$$

Define a function g by

$$g(t)=g(t;x,y)=\begin{cases}\dfrac{(y^t-x^t)}{t}, & t\neq 0,\\ \ln y-\ln x, & t=0.\end{cases} \tag{5}$$

It is easy to see that g can be expressed in integral form as

$$g(t;x,y)=\int_x^y u^{t-1}\,\mathrm{d}u, \quad t\in\mathbb{R}, \tag{6}$$

and

$$g^{(n)}(t)=\int_x^y (\ln u)^n u^{t-1}\,\mathrm{d}u, \quad t\in\mathbb{R}. \tag{7}$$

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Therefore, the extended mean values can be represented [8, 19] in terms of g by

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{g(s;x,y)}{g(r;x,y)}\right)^{1/(s-r)}, & (r-s)(x-y)\neq 0,\\ \exp\left(\dfrac{\partial g(r;x,y)/\partial r}{g(r;x,y)}\right), & r=s,\ x-y\neq 0,\end{cases} \tag{8}$$

and

$$\ln E(r,s;x,y)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s \dfrac{\partial g(t;x,y)/\partial t}{g(t;x,y)}\,\mathrm{d}t, & (r-s)(x-y)\neq 0,\\ \dfrac{\partial g(r;x,y)/\partial r}{g(r;x,y)}, & r=s,\ x-y\neq 0.\end{cases} \tag{9}$$

Leach and Sholander [3] showed that $E(r,s;x,y)$ are increasing with both r and s, or with both x and y. The monotonicities of E have also been researched by the author and others in [1] and [13]–[19] using different ideas and simpler methods.

Leach and Sholander [4] and Páles [7] solved the problem of comparison of E; that is, they found necessary and sufficient conditions for the parameters r,s,u,v in order that

$$E(r,s;x,y)\leq E(u,v;x,y) \tag{10}$$

be satisfied for all positive x and y.

Most of two variable means are special cases of E, for example [2],

$$E(1,2;x,y)=A(x,y),\quad E(1,1;x,y)=I(x,y),\quad E(0,1;x,y)=L(x,y). \tag{11}$$

They are called the arithmetic mean, the identric mean, and the logarithmic mean, respectively.

Recently, the concepts of mean values have been generalized by the author in [9]–[12].

The main purpose of this paper is to verify the logarithmic convexity of the extended mean values $E(r,s;x,y)$. As applications, an inequality among the arithmetic mean, the identric mean and the logarithmic mean is established; two open problems are proposed. As by-products, an inequality for the exponential function is obtained.

2. Logarithmic convexity of $E(r,s;x,y)$

In order to prove our main result, the following lemma is necessary.

Lemma 1 ([19]). *If $f(t)$ is an increasing integrable function on I, then the arithmetic mean of function $f(t)$,*

$$\phi(r,s)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,\mathrm{d}t, & r\neq s,\\ f(r), & r=s,\end{cases} \tag{12}$$

is also increasing with both r and s on I.

If f is a twice-differentiable convex function, then the function $\phi(r,s)$ is also convex with both r and s on I.

Proof. Direct calculation yields

$$\frac{\partial \phi(r,s)}{\partial s} = \frac{1}{(s-r)^2}\Big[(s-r)f(s) - \int_r^s f(t)\,\mathrm{d}t\Big], \tag{13}$$

$$\frac{\partial^2 \phi(r,s)}{\partial s^2} = \frac{(s-r)^2 f'(s) - 2(s-r)f(s) + 2\int_r^s f(t)\,\mathrm{d}t}{(s-r)^3} \equiv \frac{\varphi(r,s)}{(s-r)^3}, \tag{14}$$

$$\frac{\partial \varphi(r,s)}{\partial s} = (s-r)^2 f''(s). \tag{15}$$

In the case of $f(t)$ being increasing, we have $\partial\phi(r,s)/\partial s \geq 0$, thus $\phi(r,s)$ increases in both r and s, since $\phi(r,s) = \phi(s,r)$.

In the case of $f''(t) \geq 0$, $\varphi(r,s)$ increases with s. Since $\varphi(r,r) = 0$, we have $\partial^2\phi(r,s)/\partial s^2 \geq 0$. Therefore $\phi(r,s)$ is convex with respect to either r or s, since $\phi(r,s) = \phi(s,r)$. This completes the proof. □

By formula (9) and the above Lemma, we can see that, in order to prove the logarithmic convexity of the extended mean values $E(r,s;x,y)$, it suffices to verify the convexity of function

$$g'(t)/g(t) \triangleq g'_t(t;x,y)/g(t;x,y) \triangleq \big(\partial g(t;x,y)/\partial t\big)/g(t;x,y)$$

with respect to t, where $g(t) = g(t;x,y)$ is defined by (5) or (6).

Straightforward computation results in

$$\left(\frac{g'(t)}{g(t)}\right)' = \frac{g''(t)g(t) - [g'(t)]^2}{g^2(t)}, \tag{16}$$

$$\left(\frac{g'(t)}{g(t)}\right)'' = \frac{g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3}{g^3(t)}. \tag{17}$$

For $y > x = 1$, expanding $g(t;1,y)$ into series at $t_0 = 0$ with respect to t directly gives us

$$\begin{aligned} g(t;1,y) &= \sum_{i=0}^{\infty} \frac{(\ln y)^{i+1}}{(i+1)!} t^i, \\ g'_t(t;1,y) &= \sum_{i=0}^{\infty} \frac{(i+1)(\ln y)^{i+2}}{(i+2)!} t^i, \\ g''_t(t;1,y) &= \sum_{i=0}^{\infty} \frac{(i+1)(i+2)(\ln y)^{i+3}}{(i+3)!} t^i, \\ g'''_t(t;1,y) &= \sum_{i=0}^{\infty} \frac{(i+1)(i+2)(i+3)(\ln y)^{i+4}}{(i+4)!} t^i. \end{aligned} \tag{18}$$

From the four fundamental operations of arithmetic and suitable properties of series, we have

$$g^2(t;1,y) = 2\sum_{k=0}^{\infty} \frac{(2^{k+1}-1)(\ln y)^{k+2}}{(k+2)!} t^k, \tag{19}$$

$$g(t;1,y)g'_t(t;1,y) = \sum_{k=0}^{\infty} \frac{(k+1)(2^{k+2}-1)(\ln y)^{k+3}}{(k+3)!} t^k, \tag{20}$$

$$g(t;1,y)g_t''(t;1,y)+[g_t'(t;1,y)]^2=\sum_{k=0}^{\infty}\frac{(k+1)(k+2)(2^{k+3}-1)(\ln y)^{k+4}}{(k+4)!}t^k. \tag{21}$$

By standard arguments for series, we can get two combinatorial identities:

$$\sum_{i=0}^{k}(i+1)\binom{k+3}{i+2}=(k+1)(2^{k+2}-1), \tag{22}$$

$$\sum_{i=0}^{k}(i+1)(k-i+1)\binom{k+4}{i+2}=2(k+3)(1+k\cdot 2^{k+1}). \tag{23}$$

By further computation, the following expansions are obtained:

$$\begin{aligned}
&g^2(t;1,y)g_t'''(t;1,y)\equiv\sum_{k=0}^{\infty}\alpha_k(\ln y)^{k+6}t^k\\
&=\sum_{k=0}^{\infty}\frac{2}{(k+6)!}\sum_{i=0}^{k}(i+1)(i+2)(i+3)(2^{k-i+1}-1)\binom{k+6}{i+4}(\ln y)^{k+6}t^k,
\end{aligned} \tag{24}$$

$$\begin{aligned}
&g(t;1,y)g_t'(t;1,y)g_t''(t;1,y)\equiv\sum_{k=0}^{\infty}\beta_k(\ln y)^{k+6}t^k\\
&=\sum_{k=0}^{\infty}\frac{1}{(k+6)!}\sum_{i=0}^{k}(i+1)(i+2)(k-i+1)(2^{k-i+2}-1)\binom{k+6}{i+3}\big(\ln y\big)^{k+6}t^k,
\end{aligned} \tag{25}$$

$$\begin{aligned}
&g_t'(t;1,y)\big([g_t'(t;1,y)]^2+g(t;1,y)g_t''(t;1,y)\big)\equiv\sum_{k=0}^{\infty}\gamma_k(\ln y)^{k+6}t^k\\
&=\sum_{k=0}^{\infty}\frac{1}{(k+6)!}\sum_{i=0}^{k}(i+1)(i+2)(k-i+1)(2^{i+3}-1)\binom{k+6}{i+4}\big(\ln y\big)^{k+6}t^k.
\end{aligned} \tag{26}$$

Proposition 1. *For an arbitrary nonnegative integer k, we have*

$$\begin{aligned}
&2\sum_{i=0}^{k}(i+1)(i+2)\big\{(i+3)2^{k-i+1}+(k-i+1)2^{i+3}-(k+4)\big\}\binom{k+6}{i+4}\\
&\qquad\le 5\sum_{i=0}^{k}(i+1)(i+2)(k-i+1)(2^{k-i+2}-1)\binom{k+6}{i+3}.
\end{aligned} \tag{27}$$

Proof. The inequality (27) is equivalent to

$$\begin{aligned}
&\sum_{j=0}^{k+6}(j-3)(j-2)\big[(k-j+5)2^j+(j-1)2^{k-j+6}-2(k+4)\big]\binom{k+6}{j}\\
&\qquad+(k^2+12k+38)(2^{k+6}+k+3)\\
&\qquad\le 5\sum_{j=0}^{k+6}(j-2)(j-1)(k-j+4)(2^{k-j+5}-1)\binom{k+6}{j}.
\end{aligned} \tag{28}$$

Using expansions-into-series of $(1+2x)^n$ and its derivatives, we get

$$\begin{aligned} \sum_{i=0}^{n} 2^i \binom{n}{i} &= 3^n, \\ \sum_{i=0}^{n} i2^i \binom{n}{i} &= 2n3^{n-1}, \\ \sum_{i=0}^{n} i(i-1)2^i \binom{n}{i} &= 4n(n-1)3^{n-2}, \\ \sum_{i=0}^{n} i(i-1)(i-2)2^i \binom{n}{i} &= 8n(n-1)(n-2)3^{n-3}. \end{aligned} \tag{29}$$

Similarly, using expansions-into-series of $(1+x/2)^n$ and its derivatives, we have

$$\begin{aligned} \sum_{j=0}^{n} \frac{1}{2^j} \binom{n}{j} &= \left(\frac{3}{2}\right)^n, \\ \sum_{j=0}^{n} \frac{j}{2^j} \binom{n}{j} &= \frac{n}{2}\left(\frac{3}{2}\right)^{n-1}, \\ \sum_{j=0}^{n} \frac{j(j-1)}{2^j} \binom{n}{j} &= \frac{n(n-1)}{4}\left(\frac{3}{2}\right)^{n-2}, \\ \sum_{j=0}^{n} \frac{j(j-1)(j-2)}{2^j} \binom{n}{j} &= \frac{n(n-1)(n-2)}{8}\left(\frac{3}{2}\right)^{n-3}. \end{aligned} \tag{30}$$

The following formulae are also well-known:

$$\begin{aligned} \sum_{j=0}^{n} \binom{n}{j} &= 2^n, \\ \sum_{j=0}^{n} j\binom{n}{j} &= n2^{n-1}, \\ \sum_{j=0}^{n} j(j-1)\binom{n}{j} &= n(n-1)2^{n-2}, \\ \sum_{j=0}^{n} j(j-1)(j-2)\binom{n}{j} &= n(n-1)(n-2)2^{n-3}. \end{aligned} \tag{31}$$

Substitution of (29), (30) and (31) into (28) and simplification give us

$$(k^3+15k^2+74k+114)+(k^3+15k^2+74k+168)2^{k+3}-2\times 3^{k+6} \le 0. \tag{32}$$

From Taylor's expansion

$$a^x = \sum_{i=0}^{\infty} \frac{(\ln a)^i}{i!} x^i, \tag{33}$$

the inequality (32) reduces to

$$
\begin{aligned}
&1458 + (666 + 1344\ln 2)k + 8\big[84(\ln 2)^2 + 74\ln 2 + 15\big]k^2 + k^3 \\
&+ 8\sum_{i=3}^{\infty}\left[\frac{168(\ln 2)^3}{i(i-1)(i-2)} + \frac{74(\ln 2)^2}{(i-1)(i-2)} + \frac{15\ln 2}{i-2} + 1\right]\frac{(\ln 2)^{i-3}}{(i-3)!}\cdot k^i \\
&\le 1458\sum_{i=0}^{\infty}\frac{(\ln 3)^i}{i!}\cdot k^i.
\end{aligned} \tag{34}
$$

To prove inequality (34), by equating the coefficients of k^i in (34), it is sufficient to verify that

$$
4\big[168(\ln 2)^3 + 74(\ln 2)^2 i + 15i(i-1)\ln 2 + i(i-1)(i-2)\big] \le 729(\ln 2)^3(\log_2 3)^i \tag{35}
$$

holds for $i \ge 4$.

Let

$$
\begin{aligned}
\psi(x) = {}& 4\big\{x^3 + (15\ln 2 - 3)x^2 + \big[2 - 15\ln 2 + 74(\ln 2)^2\big]x + 168(\ln 2)^3\big\} \\
&- 729(\ln 2)^3(\log_2 3)^x
\end{aligned} \tag{36}
$$

for $x \ge 4$. By standard argument, we have

$$
\begin{aligned}
\psi'(x) &= 4\big\{3x^2 + 2(15\ln 2 - 3)x + 2 - 15\ln 2 + 74(\ln 2)^2\big\} \\
&\quad - 729(\ln 2)^3\ln(\log_2 3)(\log_2 3)^x, \\
\psi''(x) &= 4\big\{6x + 2(15\ln 2 - 3)\big\} - 729(\ln 2)^3[\ln(\log_2 3)]^2(\log_2 3)^x, \\
\psi'''(x) &= 24 - 729(\ln 2)^3[\ln(\log_2 3)]^3(\log_2 3)^x, \\
\psi^{(4)}(x) &= -729(\ln 2)^3(\log_2 3)^x(\ln(\log_3 2))^4 < 0.
\end{aligned}
$$

Direct computation yields

$$
\begin{aligned}
\psi'''(4) &= 24 - 729(\ln 2)^3[\ln(\log_2 3)]^3(\log_2 3)^4 < -125, \\
\psi''(4) &= 4(18 + 30\ln 2) - 729(\ln 2)^3[\ln(\log_2 3)]^2(\log_2 3)^4 < -169, \\
\psi'(4) &= 4\big\{26 + 105\ln 2 + 74(\ln 2)^2\big\} - 729(\ln 2)^3\ln(\log_2 3)(\log_2 3)^4 < -168, \\
\psi(4) &= 4\big\{24 + 180\ln 2 + 296(\ln 2)^2 + 168(\ln 2)^3\big\} - 729(\ln 2)^3(\log_2 3)^4 < -144.
\end{aligned}
$$

Since the function $\psi'''(x)$ is decreasing, then $\psi'''(x) < 0$ for $x \ge 4$, and $\psi''(x) < 0$, $\psi'(x) < 0$, and then $\psi(x) < 0$ for $x \ge 4$. Inequality (35) follows. This completes the proof. □

Remark 1. In fact, inequality $\psi(x) < 0$ holds for $x \ge 0$.

Corollary 1. *For any nonnegative number $x \ge 0$, we have*

$$
\frac{1 + 3^{x+3}}{1 + 2^{x+3}} \ge \frac{x^3 + 15x^2 + 74x + 168}{54}. \tag{37}
$$

Proposition 2. *If $y > x = 1$, then, for $t \ge 0$,*

$$
g^2(t;1,y)g_t'''(t;1,y) - 3g(t;1,y)g_t'(t;1,y)g_t''(t;1,y) + 2[g_t'(t;1,y)]^3 \le 0. \tag{38}
$$

Proof. It is clear that

$$
\begin{aligned}
&g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2\big[g'(t)\big]^3 \\
&= g^2(t)g'''(t) - 5\big(g(t)g'(t)\big)g''(t) + 2g'(t)\big[g(t)g''(t) + \big(g'(t)\big)^2\big] \\
&= \sum_{k=0}^{\infty}(\alpha_k - 5\beta_k + 2\gamma_k)(\ln y)^{k+6}t^k.
\end{aligned} \tag{39}
$$

Furthermore, Proposition 1 implies

$$\alpha_k - 5\beta_k + 2\gamma_k \le 0, \quad k \ge 0. \tag{40}$$

The proof of Proposition 2 is completed. □

Theorem 1. *For all fixed $x, y > 0$ and $s \in [0, +\infty)$ (or $r \in [0, +\infty)$, respectively), the extended mean values $E(r, s; x, y)$ are logarithmically concave in r (or in s, respectively) on $[0, +\infty)$. For all fixed $x, y > 0$ and $s \in (-\infty, 0]$ (or $r \in (-\infty, 0]$, respectively), the extended mean values $E(r, s; x, y)$ are logarithmically convex in r (or in s, respectively) on $(-\infty, 0]$.*

Proof. The combination of Proposition 2 with equality (17) proves that $\frac{g_t'(t;1,y)}{g(t;1,y)}$ is concave on $[0, +\infty)$ with t for $y > x = 1$. Therefore, from Lemma 1, it follows that the extended mean values $E(r, s; 1, y)$ are logarithmically concave on $[0, +\infty)$ with respect to either r or s for $y > x = 1$.

By standard arguments, we obtain

$$E(r, s; x, y) = xE(r, s; 1, \frac{y}{x}), \tag{41}$$

$$E(-r, -s; x, y) = \frac{xy}{E(r, s; x, y)}. \tag{42}$$

Hence, $E(r, s; x, y)$ are logarithmically concave on $[0, +\infty)$ with either r or s, respectively, and logarithmically convex on $(-\infty, 0]$ in either r or s, respectively. The proof of Theorem 1 is completed. □

Corollary 2. *The logarithmic means $L(r; x, y) = \big(L(x^r, y^r)\big)^{1/r}$ and the extended logarithmic means $S_r(x, y) = E(r, 1; x, y)$ are logarithmically concave on $[0, +\infty)$ with respect to r.*

Corollary 3. *For any $y > x > 0$, if $t \ge 0$, then*

$$g^2(t; x, y)g_t'''(t; x, y) - 3g(t; x, y)g_t'(t; x, y)g_t''(t; x, y) + 2[g_t'(t; x, y)]^3 \le 0. \tag{43}$$

If $t \le 0$, the inequality (43) reverses.

As concrete applications of the logarithmic convexity of $E(r, s; x, y)$, an inequality of mean values is given as follows.

Theorem 2. *Let $A(x, y)$, $L(x, y)$ and $I(x, y)$ denote the arithmetic mean, the logarithmic mean, and the identric mean of two variables x and y. Then, for $x \ne y$, we have*

$$I(x, y) > \frac{L(x, y) + A(x, y)}{2}. \tag{44}$$

Proof. P. Montel [5, p. 19] verified that a positive function $f(x)$ is logarithmically convex if and only if $x \longmapsto \mathrm{e}^{ax} f(x)$ is a convex function for all real values of a. Thus, applied this conclusion to the function $\mathrm{e}^{ar} E(r,s;x,y)$, for any given $a \in \mathbb{R}$ and $x, y > 0$, $x \neq y$, we get

$$I(x,y) \geq \frac{e^{-a} L(x,y) + e^{a} A(x,y)}{2} \geq \sqrt{A(x,y)L(x,y)}. \tag{45}$$

The proof is complete. □

3. Miscellanea

A function $f(t)$ is said to be absolutely monotonic on (a,b) if it has derivatives of all orders and $f^{(k)}(t) \geq 0$, $t \in (a,b)$, $k \in \mathbb{N}$.

A function $f(t)$ is said to be completely monotonic on (a,b) if it has derivatives of all orders and $(-1)^k f^{(k)}(t) \geq 0$, $t \in (a,b)$, $k \in \mathbb{N}$.

The famous Bernstein-Widder theorem [22] states that a function $f(x)$, $x \in (0,+\infty)$, is absolutely monotone if and only if there exists a bounded and nondecreasing function $\sigma(t)$ such that the integral

$$f(x) = \int_0^{+\infty} e^{xt} \,\mathrm{d}\sigma(t) \tag{46}$$

converges for $0 \leq x < +\infty$, and a function $f(x)$, $x \in (0,+\infty)$, is completely monotone if and only if there exists a bounded and nondecreasing function $\eta(t)$ such that the integral

$$f(x) = \int_0^{+\infty} e^{-xt} \,\mathrm{d}\eta(t) \tag{47}$$

converges for $0 \leq x < +\infty$.

Proposition 3. *Suppose $F(t) = \int_a^b p(u) f^t(u)\,\mathrm{d}u$, $t \in \mathbb{R}$, $p(u) \not\equiv 0$, is a nonnegative and continuous function, and $f(u)$ a positive and continuous function on a given interval $[a,b]$. Then*

$$F^{(n)}(t) = \int_a^b p(u) f^t(u) \big[\ln f(u)\big]^n \,\mathrm{d}u. \tag{48}$$

If $f(u) \geq 1$, $F(t)$ is absolutely monotone on $(-\infty,+\infty)$; if $0 < f(u) < 1$, then $F(t)$ is completely monotone on $(-\infty,+\infty)$. Moreover, $F(t)$ is absolutely convex on $(-\infty,+\infty)$.

Proof. This is obvious. □

Corollary 4 ([17, 18])**.** *The function $g(t;x,y)$ is absolutely and regularly monotonic on $(-\infty,+\infty)$ for $y > x > 1$, or on $(0,+\infty)$ for $y > x^{-1} > 1$, completely and regularly monotonic on $(-\infty,+\infty)$ for $0 < x < y < 1$, or on $(-\infty,0)$ for $1 < y < x^{-1}$. Furthermore, $g(x)$ is absolutely convex on $(-\infty,+\infty)$.*

The generalized weighted mean values $M_{p,f}(r,s;x,y)$, with two parameters r and s, are defined in [9, 10] by

$$M_{p,f}(r,s;x,y)=\left(\frac{\int_x^y p(u)f^s(u)\,\mathrm{d}u}{\int_x^y p(u)f^r(u)\,\mathrm{d}u}\right)^{1/(s-r)}, \qquad (r-s)(x-y)\neq 0, \tag{49}$$

$$M_{p,f}(r,r;x,y)=\exp\left(\frac{\int_x^y p(u)f^r(u)\ln f(u)\,\mathrm{d}u}{\int_x^y p(u)f^r(u)\,\mathrm{d}u}\right), \qquad x-y\neq 0, \tag{50}$$

$$M_{p,f}(r,s;x,x)=f(x),$$

where $x,y,r,s\in\mathbb{R}$, $p(u)\not\equiv 0$ is a nonnegative and integrable function and $f(u)$ a positive and integrable function on the interval between x and y.

It is clear that $E(r,s;x,y)$ is a special case of $M_{p,f}(r-1,s-1;x,y)$ applied to $p(u)\equiv 1$, $f(u)=u$, and $M^{[r]}(f;p;x,y)=M_{p,f}(r,0;x,y)$.

The basic properties and monotonicities of $M_{p,f}(r,s;x,y)$ were studied in [9, 15, 16, 20].

At last, we propose the following two open problems.

Open Problem 1. If $f(x)$ is an absolutely or completely monotonic function on the interval $(-\infty,+\infty)$, then the following inequality holds for $0\leq x<+\infty$ or reverses for $-\infty<x\leq 0$:

$$f^2(x)f'''(x)-3f(x)f'(x)f''(x)+2\left[f'(x)\right]^3\leq 0. \tag{51}$$

Open Problem 2. Suppose $p(u)$ is nonnegative and continuous, and $f(u)$ positive and continuous on a given interval $[a,b]$. If $f(u)\geq 1$ or $0<f(u)<1$, then the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are logarithmically concave on $(0,+\infty)$ with respect to either r or s, respectively, or logarithmically convex on $(-\infty,0)$ with respect to either r or s, respectively.

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Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, People's Republic of China

E-mail address: qifeng@jzit.edu.cn

URL: http://rgmia.vu.edu.au/qi.html