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DIVISION ALGEBRAS OVER C_2- AND C_3-FIELDS

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ABSTRACT. Using elementary methods we prove a theorem of Rost, Serre, and Tignol that any division algebra of degree 4 over a C_3-field containing $\sqrt{-1}$ is cyclic. Our methods also show any division algebra of degree 8 over a C_2-field containing $\sqrt[4]{-1}$ is cyclic.

INTRODUCTION

Throughout the discussion A will always denote a central simple algebra over a field F which is fixed. We write D for the underlying division algebra. We want to study the structure of A in terms of the assumption that F is a C_m-field for some m; we recall that this means every form of degree d in $n > d^m$ variables has a nontrivial zero. C_m-fields were discovered by Tsen [9] and rediscovered by Lang [6]. Greenberg [5] contains an excellent treatment. The Tsen-Lang theorem shows the Brauer group over a C_1-field is trivial. Examples of C_m-fields include all fields of transcendence degree m over an algebraically closed field. More generally, by [5, Theorem 3.6], if F is a C_m-field and K/F is a field extension of transcendence degree t, then K is a C_{m+t}-field.

Since the general structure theory [2] shows A is a tensor product of algebras of prime power index, one may assume index(A) is a power of a prime number p. The motivation of this research was the following question which I heard from Zinovy Reichstein:

Question A. Over the purely transcendental field $F = \mathbb{C}(x, y)$ or $F = \mathbb{C}(x, y, z)$, is the tensor product of two cyclic algebras of index p necessarily cyclic? As Reichtstein pointed out, this is true for $p = 2$ over arbitrary C_2-fields containing $\sqrt{-1}$, seen by solving the equations

$$\operatorname{tr}(a) = tr(a^2) = \operatorname{tr}(a^3) = 0.$$

This is the first question one would ask in trying to find a noncyclic algebra over a field of low transcendence degree over an algebraically closed field. Note that over the transcendental field extension $\mathbb{C}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of $\mathbb{C}$ one has the noncyclic

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algebra which is the tensor product of two symbol algebras $(\lambda_1, \lambda_2) \otimes (\lambda_3, \lambda_4)$, so if we want a cyclicity theorem we must study C_m-fields for $m \leq 3$.

Note that Albert [1] has produced a noncyclic division algebra of degree 4 and exponent 2 over $F = \mathbb{R}(x, y)$; any such algebra must be the tensor product of two quaternion algebras.

We prove cyclicity of all division algebras of degree 4 over C_3-fields containing $\sqrt{-1}$, and of division algebras of degree 8 over C_2-fields containing $\sqrt[4]{-1}$. The former result follows as an immediate consequence of a lovely new theorem of Rost-Serre-Tignol [7, Theorem 1], but our proof is elementary and direct, yielding an explicit element whose fourth power is central, even without the presence of $\sqrt{-1}$. For the reader's convenience, the statement of the Rost-Serre-Tignol Theorem is:

Let E be a central simple F-algebra of degree 4, with $-1 \in F^{\times 2}$. Let n_Q be the norm form of the quaternion algebra Q Brauer-equivalent to $E \otimes_F E$. The trace form $q_E : E \to F$, defined by $q_E(x) = \mathrm{Trd}_E(x^2)$, satisfies the following equation in the Witt ring $W F$:

$$q_E = n_Q + q_4$$

for some (uniquely determined) 4-fold Pfister form q_4. The form q_4 is hyperbolic if and only if E is cyclic, i.e. E is split by some Galois $(\mathbb{Z}/4\mathbb{Z})$-algebra over F.

§1. Division algebras of degree 4 over C_3-fields

Theorem 1.1. *Over a C_3-field of characteristic $\neq 2$, if $\deg(D) = 4$, then there is an element c such that $c^4 \in F$ but $c^2 \notin F$. In particular if $\sqrt{-1} \in F$, then D is cyclic.*

Proof. Take an element a such that $a^2 \in F$ but $a \notin F$. (We know that D contains such elements by [2, Theorem 11.9].) Now let $V = \{b : ab = -ba\}$, easily seen to be a subspace of dimension $\frac{4^2}{2} = 8$. (One way of seeing this is to note that $V \supseteq \{ad - da : d \in D\}$.) Let $V' = V + Fa$, which has dimension 9. For any $c = b + \gamma a \in V'$, we have

$$c^2 = b^2 + \gamma^2 a^2 + b\gamma a + \gamma ab = b^2 + \gamma^2 a^2.$$

Recalling $a^2 \in F$, we see that $c^2 \in F[b^2]$. But $F[b^2] \subset F[b]$ for $b \neq 0$, since $b \notin F[b^2]$ (because b^2 commutes with a), so c^2 is quadratic over F. The quadratic form given by $\mathrm{tr}(c^2) = 0$ has a nontrivial solution over a C_3-field since $\dim V' = 9 > 2^3 = 8$. But such c satisfies

$$c^4 = (c^2)^2 \in F$$

although $c^2 \notin F$. □

Remark. If $\sqrt{-1} \in F$, then any such c of Theorem 1.1 can be recovered via the proof. Indeed, take c' such that $c'c = \sqrt{-1}\, cc'$, and let $a = (c')^2$. Then $ac = -ca$, so we could take $b = c$ and $\gamma = 0$.

Remark 1.2. One can refine this result by means of the Amitsur-Saltman construction [3] of an arbitrary central simple algebra of degree 4 in terms of a maximal subfield K Galois over the center F. We can write $K = F(a_1, a_2)$ where $a_i^2 \in F$, and one has $z_i \in K$ such that $z_i a_i z_i^{-1} = -a_i$ and $z_i a_{2-i} = a_{2-i} z_i$, for $i = 1, 2$.

Suppose F has a C_2-subfield F_0 such that $a_i^2 \in F_0$ and $z_1^2 \in F_0(a_1, a_2)$. Then letting $K_0 = F_0(a_1, a_2)$ and

$$V'' = F_0 a_1 + K_0 z_1,$$

which has dimension 5 over F_0, one sees V'' has an element $v \neq 0$ such that $\operatorname{tr} v^2 = 0$. But $v^2 \in F_0(a_2)$. (Indeed, for any $k \in K_0$ one has $(kz_1)^2 \in K_0$ is fixed under conjugation by z_1, so is in $F_0(a_2)$.) Consequently

$$v^2 \in F_0 a_2$$

and $v^4 \in F_0$. The more precise form for this v will be used in later work.

§2. Division algebras of degree 4 over C_3-fields

Along the same lines, we can study division algebras of degree 8 over C_2-fields. Artin-Harris and Artin-Tate [4, Theorem 6.2 and Appendix] proved over a C_2-field that $\operatorname{index}(A) = \exp(A)$ for any central simple algebra A such that $\operatorname{index}(A)$ is a power of 2 or 3. Since their theorem was proved before the major structure theorems of Merkurjev and Suslin, their statement was somewhat weaker, but examining [4, Theorem 6.2, Step I, and Appendix] one can piece together the components of their arguments for a straightforward proof over an arbitrary C_2-field F. Let us sketch this proof for $\operatorname{index}(A)$ a power of 2:

(1) The tensor product of m quaternion algebras over F has index 2. Indeed, this is true for $m = 2$ by a degree counting argument which shows that any two quaternion algebras over a C_2-field have a common (quadratic) subfield; the assertion for arbitrary m follows by induction.

(2) The case $\exp(A) = 2$ follows by (1) by Merkurjev's theorem that every algebra of exponent 2 is similar to a tensor product of quaternion algebras. Note that for $\operatorname{index}(A) \leq 8$, one could substitute the much more basic theorems of Albert and Tignol [8].

(3) The case of arbitrary exponent 2^t is achieved by applying induction to $A \otimes A$, which has exponent 2^{t-1} and therefore has a splitting field L of dimension 2^{t-1}. Then $A \otimes L$ has exponent 2, so one concludes with (2) and Albert's index reduction formula [2, Theorem 4.20].

Theorem 2.1. *Suppose F is a C_2-field of characteristic $\neq 2$ containing $\sqrt{-1}$. If $\deg(D) = 8$, then D contains an element c such that $c^8 \in F$ but $c^4 \notin F$. In particular if $\sqrt[4]{-1} \in F$, then D is cyclic.*

Proof. Since $\exp(D) = \operatorname{index}(D) = 8$, we have $\operatorname{index} D^{\otimes 2} = 4$. Hence, by Theorem 1.1 there is an element $a \in D^{\otimes 2}$ such that $a^4 \in F$ but $a^2 \notin F$. Hence $L = F[a]$ splits $D^{\otimes 2}$, so $D \otimes_F L$ has exponent 2, and thus is a quaternion algebra. By Albert's index reduction formula [2, Theorem 4.20], L is isomorphic to a subfield of D. Thus we may assume $a \in D$. Now, writing $i = \sqrt{-1}$, let $V = \{b : ab = iba\}$, easily seen to be a subspace of dimension $\frac{8^2}{4} = 16$. Let $V' = V + Fa$, which has dimension 17. For any $c = b + \gamma a \in V'$, we have

$$c^2 = b^2 + \gamma^2 a^2 + b\gamma a + \gamma ab = b^2 + \gamma^2 a^2 + (1 + i)\gamma ba,$$

so

$$c^4 = (c^2)^2 = b^4 + \gamma^4 a^4 + (1 + i)^2 \gamma^2 (ba)^2 + 2\gamma^2 a^2 b^2 = b^4 + \gamma^4 a^4.$$

Since $a^4 \in F$ we see that $c^4 \in F[b^4]$. But $F[b^4] \subset F[b^2] \subset F[b]$ when $b \neq 0$, so c^4 is quadratic over F. The quartic form given by $\operatorname{tr}(c^4) = 0$ has a nontrivial solution over a C_2-field since $\dim V' = 17 > 2^4 = 16$. But such c satisfies

$$c^8 = (c^4)^2 \in F$$

although $c^4 \notin F$. $\square$

Remark 2.2. If $\sqrt[4]{-1} \in F$, then any such c of Theorem 2.1 can be recovered via the proof. Indeed, take c' such that $c'c = \sqrt[4]{-1}\, cc'$, and let $a = (c')^2$. Then $ac = ica$, so we could take $b = c$ and $\gamma = 0$.

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