

## PRIMARY DECOMPOSITION: COMPATIBILITY, INDEPENDENCE AND LINEAR GROWTH

YONGWEI YAO

(Communicated by Wolmer V. Vasconcelos)

**ABSTRACT.** For finitely generated modules  $N \subsetneq M$  over a Noetherian ring  $R$ , we study the following properties about primary decomposition: (1) The Compatibility property, which says that if  $\text{Ass}(M/N) = \{P_1, P_2, \dots, P_s\}$  and  $Q_i$  is a  $P_i$ -primary component of  $N \subsetneq M$  for each  $i = 1, 2, \dots, s$ , then  $N = Q_1 \cap Q_2 \cap \dots \cap Q_s$ ; (2) For a given subset  $X = \{P_1, P_2, \dots, P_r\} \subseteq \text{Ass}(M/N)$ ,  $X$  is an open subset of  $\text{Ass}(M/N)$  if and only if the intersections  $Q_1 \cap Q_2 \cap \dots \cap Q_r = Q'_1 \cap Q'_2 \cap \dots \cap Q'_r$  for all possible  $P_i$ -primary components  $Q_i$  and  $Q'_i$  of  $N \subsetneq M$ ; (3) A new proof of the ‘Linear Growth’ property, which says that for any fixed ideals  $I_1, I_2, \dots, I_t$  of  $R$  there exists a  $k \in \mathbb{N}$  such that for any  $n_1, n_2, \dots, n_t \in \mathbb{N}$  there exists a primary decomposition of  $I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M \subsetneq M$  such that every  $P$ -primary component  $Q$  of that primary decomposition contains  $P^{k(n_1+n_2+\dots+n_t)}M$ .

### 0. INTRODUCTION

Throughout this paper  $R$  is a Noetherian ring and  $M \neq 0$  is a finitely generated  $R$ -module unless explicitly stated otherwise. Let  $N \subsetneq M$  be a proper  $R$ -submodule of  $M$ . By primary decomposition  $N = Q_1 \cap Q_2 \cap \dots \cap Q_s$  of  $N$  in  $M$ , we always mean an irredundant and minimal primary decomposition, where  $Q_i$  is a  $P_i$ -primary submodule of  $M$ , i.e.  $\text{Ass}(M/Q_i) = \{P_i\}$ , for each  $i = 1, 2, \dots, s$ , unless explicitly mentioned otherwise. Then  $\text{Ass}(M/N) = \{P_1, P_2, \dots, P_s\}$  and we say that  $Q_i$  is a  $P_i$ -primary component of  $N$  in  $M$ . As a subset of  $\text{Spec}(R)$  with the Zariski topology,  $\text{Ass}(M/N)$  inherits a topology structure. For an ideal  $I$  in  $R$ , we use  $(N :_M I^\infty)$  to denote  $\bigcup_i (N :_M I^i)$ .

*Notation 0.1.* Let  $N \subsetneq M$  be finitely generated  $R$ -modules. We use  $\Lambda_P(N \subsetneq M)$ , or  $\Lambda_P$  if the  $R$ -modules  $N \subsetneq M$  are clear from the context, to denote the set of all possible  $P$ -primary components of  $N$  in  $M$  for every  $P \in \text{Ass}(M/N)$ .

We know that if  $P \in \text{Ass}(M/N)$  is an embedded prime ideal, then  $\Lambda_P(N \subsetneq M)$  contains more than one element. (Also see the passage following Theorem 2.2 and the reference to [HRS].) Suppose that  $N = Q_1 \cap Q_2 \cap \dots \cap Q_s$  is a primary decomposition of  $N \subsetneq M$  such that  $Q_i \in \Lambda_{P_i}$  for  $i = 1, 2, \dots, s$ . Then if we choose a  $P_i$ -primary submodule  $Q'_i$  of  $M$  such that  $N \subseteq Q'_i \subseteq Q_i$  for each  $i = 1, 2, \dots, s$ , we get a primary decomposition  $N = Q'_1 \cap Q'_2 \cap \dots \cap Q'_s$  of  $N \subsetneq M$ . For example

---

Received by the editors October 5, 2000 and, in revised form, January 12, 2001.

2000 *Mathematics Subject Classification.* Primary 13E05; Secondary 13C99, 13H99.

*Key words and phrases.* Primary decomposition, Linear Growth, Artin-Rees number.

we may choose  $Q'_i = \ker(M \rightarrow (M/(P_i^{n_i}M + N))_{P_i})$  for all  $n_i \gg 0$  to get primary decompositions  $N = \bigcap_{1 \leq i \leq s} \ker(M \rightarrow (M/(P_i^{n_i}M + N))_{P_i})$  for all  $n_i \gg 0$ . But given an arbitrary  $Q''_i \in \Lambda_{P_i}$  for each  $i = 1, 2, \dots, s$ , we do not know *a priori* if  $N = Q''_1 \cap Q''_2 \cap \dots \cap Q''_s$ . This compatibility question is answered positively in Theorem 1.1:

**Theorem 1.1** (Compatibility). *Let  $N \subsetneq M$  be finitely generated  $R$ -modules and  $\text{Ass}(M/N) = \{P_1, P_2, \dots, P_s\}$ . Suppose that for each  $i = 1, 2, \dots, s$ ,  $Q_i$  is a  $P_i$ -primary component of  $N$  in  $M$ , i.e.  $Q_i \in \Lambda_{P_i}$ . Then  $N = Q_1 \cap Q_2 \cap \dots \cap Q_s$ , which is necessarily an irredundant and minimal primary decomposition.*

**Definition 0.2.** Let  $N \subsetneq M$  be finitely generated  $R$ -modules and  $X$  a subset of  $\text{Ass}(M/N)$ , say  $X = \{P_1, P_2, \dots, P_r\} \subseteq \text{Ass}(M/N) = \{P_1, P_2, \dots, P_r, P_{r+1}, \dots, P_s\}$ . We say that the primary decompositions of  $N$  in  $M$  are independent over  $X$ , or  $X$ -independent, if for any two primary decompositions, say,  $N = Q_1 \cap Q_2 \cap \dots \cap Q_s = Q'_1 \cap Q'_2 \cap \dots \cap Q'_s$ , of  $N \subset M$  such that  $\{Q_i, Q'_i\} \subseteq \Lambda_{P_i}(N \subset M)$  for  $i = 1, 2, \dots, s$ , we have  $Q_1 \cap Q_2 \cap \dots \cap Q_r = Q'_1 \cap Q'_2 \cap \dots \cap Q'_r$ . In this case, we denote the invariant intersection by  $Q_X(N \subset M)$ , or  $Q_X$  if  $N \subset M$  is clear from the context.

It is well known that primary decompositions are independent over open subsets of  $\text{Ass}(M/N)$ . (See Observations 0.3 below.) Actually it turns out that independence property characterizes open subsets of  $\text{Ass}(M/N)$ :

**Theorem 2.2.** *Let  $N \subsetneq M$  be finitely generated  $R$ -modules and  $X \subseteq \text{Ass}(M/N)$  be a subset of  $\text{Ass}(M/N)$ . Then the primary decompositions of  $N$  in  $M$  are independent over  $X$  if and only if  $X$  is an open subset of  $\text{Ass}(M/N)$ .*

In Section 3 we use *Artin-Rees numbers* to prove the following:

**Theorem 3.3.** *Let  $R$  be a Noetherian ring,  $M$  a finitely generated  $R$ -module and  $I_1, I_2, \dots, I_t$  ideals of  $R$ . Then there exists a  $k \in \mathbb{N}$  such that for all  $n_1, n_2, \dots, n_t \in \mathbb{N}$  and for all ideals  $J \subset R$ ,*

$$(J^{k|\underline{n}|}M + I_1^{n_1}I_2^{n_2} \dots I_t^{n_t}M) \cap (I_1^{n_1}I_2^{n_2} \dots I_t^{n_t}M :_M J^\infty) = I_1^{n_1}I_2^{n_2} \dots I_t^{n_t}M,$$

where  $|\underline{n}| := n_1 + n_2 + \dots + n_t$ .

As a corollary of Theorem 3.3, we have a new proof of the ‘Linear Growth’ property, which was first proved by I. Swanson [Sw] and then by R. Y. Sharp using different methods and in a more general situation [Sh2]:

**Corollary 3.4** (Linear Growth; [Sw] and [Sh2]). *Let  $R$  be a Noetherian ring,  $M$  a finitely generated  $R$ -module and  $I_1, I_2, \dots, I_t$  ideals of  $R$ . Then there exists a  $k \in \mathbb{N}$  such that for any  $n_1, n_2, \dots, n_t \in \mathbb{N}$ , there exists a primary decomposition of  $I_1^{n_1}I_2^{n_2} \dots I_t^{n_t}M \subseteq M$*

$$I_1^{n_1}I_2^{n_2} \dots I_t^{n_t}M = Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \dots \cap Q_{\underline{n}_{r_{\underline{n}}}},$$

where the  $Q_i$ ’s are  $P_i$ -primary components of the primary decomposition such that  $P_i^{k|\underline{n}|}M \subseteq Q_i$  for all  $i = 1, 2, \dots, r_{\underline{n}}$ , where  $\underline{n} = (n_1, n_2, \dots, n_t)$  and  $|\underline{n}| = n_1 + n_2 + \dots + n_t$ .

Before ending this introductory section, we make the following well-known observations, which is to the effect of saying that primary decompositions are independent over open subsets.

*Observations on independence* 0.3. Suppose  $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$  is a primary decomposition of  $N$  in a finitely generated  $R$ -module  $M$  such that  $Q_i$  is  $P_i$ -primary for each  $i = 1, 2, \dots, s$ .

- (1) For any ideal  $I \subseteq R$ , the intersection  $\bigcap_{I \not\subseteq P_i} Q_i = (N :_M I^\infty)$  is independent of the particular primary decomposition of  $N$  in  $M$  (cf. D. Eisenbud [Ei], page 101, Proposition 3.13). This means that the primary decompositions of  $N \subsetneq M$  are independent over  $X = \{P \in \text{Ass}(M/N) \mid I \not\subseteq P\}$  and  $Q_X = (N :_M I^\infty)$ .
- (2) Alternatively, for any multiplicatively closed set  $W \subset R$ , the intersection  $\bigcap_{P_i \cap W = \emptyset} Q_i = \ker(M \rightarrow (M/N)_W)$  is independent of the particular primary decomposition (cf. D. Eisenbud [Ei], page 113, Exercise 3.12). That is to say that the primary decompositions of  $N \subsetneq M$  are independent over  $Y = \{P \in \text{Ass}(M/N) \mid P \cap W = \emptyset\}$  and  $Q_Y = \ker(M \rightarrow (M/N)_W)$ .

1. COMPATIBILITY

The main theorem in this section is to show that all the primary components of  $R$ -modules  $N \subsetneq M$  are totally compatible in forming the primary decompositions of  $N \subsetneq M$ .

**Theorem 1.1** (Compatibility). *Let  $N \subsetneq M$  be finitely generated  $R$ -modules and  $\text{Ass}(M/N) = \{P_1, P_2, \dots, P_s\}$ . Suppose that for each  $i = 1, 2, \dots, s$ ,  $Q_i$  is a  $P_i$ -primary component of  $N$  in  $M$ , i.e.  $Q_i \in \Lambda_{P_i}(N \subsetneq M)$ . Then  $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ , which is necessarily an irredundant and minimal primary decomposition.*

*Proof.* We induct on  $s$ , the cardinality of  $\text{Ass}(M/N)$ .

If  $s = 1$ , then  $N = Q_1$  and the claim is trivially true.

Suppose  $s \geq 2$ . By rearranging the order of  $P_1, P_2, \dots, P_s$ , we may assume that  $P_s$  is a maximal prime ideal in  $\text{Ass}(M/N)$ . Since  $Q_i \in \Lambda_{P_i}$  for  $i = 1, 2, \dots, s$ , we can find  $s$  specific primary decompositions

$$N = Q_{(i,1)} \cap Q_{(i,2)} \cap \cdots \cap Q_{(i,i)} \cap \cdots \cap Q_{(i,s)} \quad \text{for } i = 1, 2, \dots, s,$$

where  $Q_{(i,j)} \in \Lambda_{P_j}$  and  $Q_{(i,i)} = Q_i$  for all  $i, j = 1, 2, \dots, s$ . Let  $W = R \setminus \bigcup_{1 \leq i \leq s-1} P_i$ . By Observation 0.3(2) and our assumption on  $P_s$ , we know that the primary decompositions of  $N \subsetneq M$  are independent over  $X = \{P \in \text{Ass}(M/N) \mid P \cap W = \emptyset\} = \{P_1, P_2, \dots, P_{s-1}\}$  with  $Q_X = \ker(M \rightarrow (M/N)_W)$ . That is to say,

$$Q_X = \ker(M \rightarrow (M/N)_W) = Q_{(i,1)} \cap Q_{(i,2)} \cap \cdots \cap Q_{(i,s-1)} \quad \text{for } i = 1, 2, \dots, s,$$

are all primary decompositions of  $Q_X \subsetneq M$  and in particular  $Q_i = Q_{(i,i)} \in \Lambda_{P_i}(Q_X \subsetneq M)$  for  $i = 1, 2, \dots, s - 1$ . Since the cardinality of  $\text{Ass}(M/Q_X)$  is  $s - 1$ , we use the induction hypothesis to see that

$$Q_X = Q_1 \cap Q_2 \cap \cdots \cap Q_{s-1}.$$

But we already know that  $Q_X = Q_{(s,1)} \cap Q_{(s,2)} \cap \cdots \cap Q_{(s,s-1)}$  by the  $X$ -independence of primary decompositions of  $N \subsetneq M$ . Hence we have

$$\begin{aligned} N &= Q_{(s,1)} \cap Q_{(s,2)} \cap \cdots \cap Q_{(s,s-1)} \cap Q_{(s,s)} \\ &= Q_X \cap Q_s \\ &= Q_1 \cap Q_2 \cap \cdots \cap Q_{s-1} \cap Q_s. \end{aligned}$$

□

*Remark 1.2.* In [Bo, Chapter IV], the notion of primary decomposition is generalized to not necessarily finitely generated modules over not necessarily Noetherian rings. Let  $R$  be a (not necessarily Noetherian) ring and  $M$  be a (not necessarily finitely generated)  $R$ -module. A prime ideal  $P$  of  $R$  is said to be *weakly associated* with  $M$  if there exists an  $x \in M$  such that  $P$  is minimal over the ideal  $\text{Ann}(x)$  and we denote by  $\text{Ass}_f(M)$  the set of prime ideals weakly associated with  $M$  (cf. [Bo, page 289, Chapter IV, § 1, Exercise 17]). We say that an element  $r \in R$  is nearly nilpotent on  $M$  if for any  $x \in M$ , there exists an  $n(x) \in \mathbb{N}$ , such that  $r^{n(x)}x = 0$  (cf. [Bo, page 267, Chapter IV, § 1.4, Definition 2]). Then for any  $R$ -submodule  $N$  of  $M$ , we define  $r_M(N) := \{r \in R \mid r \text{ is nearly nilpotent on } M/N\}$  (cf. [Bo, page 292, Chapter IV, § 2, Exercise 11]). An  $R$ -submodule  $Q$  of  $M$  is said to be  $P$ -primary in  $M$  if  $\text{Ass}_f(M/Q) = \{P\}$ , which is equivalent to the statement that every  $r \in R$  is either a non-zero-divisor or nearly nilpotent on  $M/Q$ , and in this case we have  $r_M(Q) = P$  (cf. [Bo, page 292, Chapter IV, § 2, Exercise 12(a)]). Then we say that an  $R$ -submodule  $N$  has a primary decomposition in  $M$  if there exist  $P_i$ -primary submodules  $Q_i \subset M$ ,  $i = 1, 2, \dots, s$ , such that  $N = Q_1 \cap Q_2 \cap \dots \cap Q_s$  (cf. [Bo, page 294, Chapter IV, § 2, Exercise 20]). Again we always assume primary decompositions to be irredundant and minimal (i.e. reduced) if they exist. If  $N$  has primary decompositions in  $M$ , then Observation 0.3(2) still holds (replace  $\text{Ass}(M/N)$  by  $\text{Ass}_f(M/N)$ ). Therefore the proof of compatibility, i.e. Theorem 1.1, also applies to the case where  $N \subsetneq M$  are not necessarily finitely generated  $R$ -modules over a not necessarily Noetherian ring  $R$  as long as the primary decompositions exist.

## 2. INDEPENDENCE OVER OPEN SUBSETS OF $\text{Ass}(M/N)$

Because of the compatibility property, i.e. Theorem 1.1, we have an equivalent statement to the definition of  $X$ -independence.

**Lemma 2.1.** *Let  $N \subsetneq M$  be finitely generated  $R$ -modules and  $X = \{P_1, P_2, \dots, P_r\} \subseteq \text{Ass}(M/N) = \{P_1, P_2, \dots, P_r, P_{r+1}, \dots, P_s\}$ . Then the following are equivalent:*

- (1) *The primary decompositions of  $N$  in  $M$  are independent over  $X$ .*
- (2) *For any  $Q_i$  and  $Q'_i$  in  $\Lambda_{P_i}$ , where  $i = 1, 2, \dots, r$ , the equality  $Q_1 \cap Q_2 \cap \dots \cap Q_r = Q'_1 \cap Q'_2 \cap \dots \cap Q'_r$  holds.*

It turns out that the independence observed in Observations 0.3 actually exhausts all the possibilities.

**Theorem 2.2.** *Let  $N \subsetneq M$  be finitely generated  $R$ -modules and  $X \subseteq \text{Ass}(M/N)$  be a subset of  $\text{Ass}(M/N)$ . Then the primary decompositions of  $N$  in  $M$  are independent over  $X$  if and only if  $X$  is an open subset of  $\text{Ass}(M/N)$ .*

*Proof.* Without loss of generality we assume  $N = 0$ .

The “if” part is just Observation 0.3(1). To prove the “only if” part, it suffices to show  $X$  is stable under specialization since  $\text{Ass}(M/N) = \text{Ass}(M)$  is finite. Let  $P$  be an arbitrary prime ideal in  $X \subseteq \text{Ass}(M/N)$ . All we need to show is that for any  $P' \in \text{Ass}(M)$  such that  $P' \subset P$ , we have  $P' \in X$ .

Say  $X = \{P = P_1, P_2, \dots, P_t, P_{t+1}, \dots, P_r\}$  such that  $P_i \subseteq P$  for  $i = 1, 2, \dots, t$  and  $P_i \not\subseteq P$  for  $i = t + 1, \dots, r$ . Let

$$X_P := X \cap \text{Ass}(M_P) = \{P_P = (P_1)_P, (P_2)_P, \dots, (P_t)_P\}.$$

We first show that the primary decompositions of  $0 \subsetneq M_P$  are independent over  $X_P$ : For any  $L_i \in \Lambda_{(P_i)_P}(0 \subsetneq M_P)$ ,  $i = 1, 2, \dots, t$ , let  $Q_i$  be the full pre-image

of  $L_i$  under the map  $M \rightarrow M_P$ . Then choose  $Q_i \in \Lambda_{P_i}(0 \subsetneq M)$  for  $i = t + 1, \dots, r$ . Then it is easy to see that  $(Q_1 \cap Q_2 \cap \dots \cap Q_r)_P = L_1 \cap L_2 \cap \dots \cap L_t$ . Then the  $X$ -independence assumption implies that the primary decompositions of  $0 \subsetneq M_P$  are independent over  $X_P = X \cap \text{Ass}(M_P)$ .

Hence by replacing  $M$  with  $M_P$  we may assume that  $(R, P)$  is local with the maximal ideal  $P$  and  $P \in X = \{P = P_1, P_2, \dots, P_t\} \subseteq \text{Ass}(M)$ . In this case to prove that  $X$  is stable under specialization is simply to prove that  $X = \text{Ass}(M)$ . For each  $i = 1, 2, \dots, t$ , choose a  $P_i$ -primary component  $Q_i$  of  $0 \subsetneq M$ . There exists a  $k \in \mathbb{N}$  such that  $P^k M \subseteq Q_1$  and therefore  $P^n M \in \Lambda_P$  for all  $n \geq k$ . Set  $L = Q_2 \cap Q_3 \cap \dots \cap Q_t$ . Then by Lemma 2.1 the assumption that the primary decompositions of  $0$  in  $M$  are independent over  $X$  simply means that  $Q_1 \cap L = P^n M \cap L$  for all  $n \geq k$ , which implies  $Q_1 \cap L = 0$  by Krull Intersection Theorem. This forces  $0 = Q_1 \cap Q_2 \cap \dots \cap Q_t$  to be a primary decomposition of  $0$  in  $M$ . In particular it means that  $\text{Ass}(M) = \{P = P_1, P_2, \dots, P_t\} = X$ .  $\square$

In particular, if  $P \in \text{Ass}(M/N)$  is not minimal over  $\text{Ann}(M/N)$ , then the  $P$ -primary components of  $N$  in  $M$  are not unique. In fact, in [HRS], W. Heinzer, L. J. Ratliff, Jr. and K. Shah showed that if  $P \in \text{Ass}(M/N)$  is an embedded prime ideal, then there are infinitely many maximal  $P$ -primary components of  $N$  in  $M$  with respect to containment. See [HRS] and their following papers for more information about the embedded primary components.

### 3. ‘LINEAR GROWTH’ PROPERTY

In this section we give a new proof of the ‘Linear Growth’ property using *Artin-Rees numbers* and compatibility. The ‘Linear Growth’ property was first proved by I. Swanson [Sw] and then by R. Y. Sharp using different methods and in a more general situation [Sh2].

We first give a definition of *Artin-Rees numbers*,  $\text{AR}(J, N \subseteq M)$ , of a pair of finitely generated  $R$ -modules  $N \subseteq M$  with respect to an ideal  $J$  of  $R$ . These numbers have been studied in [Hu], where a set of ideals is considered instead of one single ideal.

**Definition 3.1.** Let  $N \subseteq M$  be finitely generated  $R$ -modules over a Noetherian ring  $R$  and  $J$  an ideal of  $R$ . We define  $\text{AR}(J, N \subseteq M) := \min\{k \mid J^n M \cap N \subseteq J^{n-k} N \text{ for all } n \geq k\}$ .

*Remark 3.2.* If  $K \subseteq L \subseteq M$ , then

$$\text{AR}(J, K \subseteq M) \leq \text{AR}(J, K \subseteq L) + \text{AR}(J, L \subseteq M).$$

If  $J^n M \subseteq N$  for some  $n$ , then  $\text{AR}(J, N \subseteq M) \leq n$ .

**Theorem 3.3.** Let  $R$  be a Noetherian ring,  $M$  a finitely generated  $R$ -module and  $I_1, I_2, \dots, I_t$  ideals of  $R$ . Then there exists a  $k \in \mathbb{N}$  such that for all  $n_1, n_2, \dots, n_t \in \mathbb{N}$  and for all ideals  $J \subset R$ ,

$$\begin{aligned} I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M &\supseteq J^{k|\underline{n}|} M \cap (I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M :_M J^\infty), \quad \text{i.e.} \\ I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M &= (J^{k|\underline{n}|} M + I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M) \cap (I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M :_M J^\infty), \end{aligned}$$

where  $|\underline{n}| := n_1 + n_2 + \dots + n_t$ .

*Proof.* It is enough to prove the theorem for

$$\begin{aligned} \mathcal{R} &= R[I_1T_1, I_2T_2, \dots, I_tT_t, T_1^{-1}, T_2^{-1}, \dots, T_t^{-1}], \\ \mathcal{M} &= \bigoplus_{n_1, n_2, \dots, n_t \in \mathbb{Z}} I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M T_1^{n_1} T_2^{n_2} \dots T_t^{n_t}, \\ \mathcal{I}_i &= T_i^{-1} \mathcal{R} \quad \text{for each } i = 1, 2, \dots, t, \quad \text{and} \\ \mathcal{J} &= \mathcal{J}\mathcal{R}. \end{aligned}$$

That is because if we contract the result for  $\mathcal{R}$  back to  $R$ , we get the desired result. Hence without loss of generality we assume  $I_i = (x_i)$  is generated by an  $M$ -regular element  $x_i \in R$  for each  $i = 1, 2, \dots, t$ . The same technique is also used in [Sw] and [Sh2].

And it also suffices to prove the theorem for one fixed ideal  $J$ . The reason is that for every  $J$  in  $R$ , we have

$$J \subseteq J' := \bigcap_{\substack{P \in Y \\ J \subseteq P}} P, \quad \text{where } Y = \bigcup_{(n_1, n_2, \dots, n_t) \in \mathbb{Z}^t} \text{Ass}(M/I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M),$$

and, furthermore, there are only finitely many such  $J'$  to deal with since the set  $Y = \bigcup_{(n_1, n_2, \dots, n_t)} \text{Ass}(M/I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M)$  is finite (cf. [Mc, page 125, Lemma 13.1]).

For each  $i = 1, 2, \dots, t$ , let  $N_i = x_i M :_M J^\infty \subseteq M$ ,  $k'_i = \text{AR}(J, N_i \subseteq M)$  and  $k''_i$  be such that  $J^{k''_i} N_i \subseteq x_i M$ . Then  $\text{AR}(J, x_i M \subseteq N_i) \leq k''_i$ .

Let  $k' = \max\{k'_i \mid 1 \leq i \leq t\}$ ,  $k'' = \max\{k''_i \mid 1 \leq i \leq t\}$  and  $k = k' + k''$ . It is easy to see by Remark 3.2 that  $\text{AR}(J, x_i M \subseteq M) \leq k'_i + k''_i \leq k$  for all  $i = 1, \dots, t$ . Since each  $x_i$  is regular on  $M$ , we have  $\text{AR}(J, x_1^{m_1} x_2^{m_2} \dots x_{i-1}^{m_{i-1}} x_i^{m_i+1} x_{i+1}^{m_{i+1}} \dots x_t^{m_t} M \subseteq x_1^{m_1} x_2^{m_2} \dots x_t^{m_t} M) = \text{AR}(J, x_i M \subseteq M) \leq k$  because of the  $R$ -linear isomorphism  $M \cong x_1^{m_1} x_2^{m_2} \dots x_t^{m_t} M$  induced by multiplication by  $x_1^{m_1} x_2^{m_2} \dots x_t^{m_t}$ . Therefore we have  $\text{AR}(J, x_1^{n_1} x_2^{n_2} \dots x_t^{n_t} M \subseteq M) \leq k(n_1 + n_2 + \dots + n_t) = k|\underline{n}|$  by the same Remark 3.2 applied to the filtration

$$x_1^{n_1} x_2^{n_2} \dots x_t^{n_t} M \subseteq x_1^{n_1-1} x_2^{n_2} \dots x_t^{n_t} M \subseteq \dots \subseteq x_t^2 M \subseteq x_t M \subseteq M$$

of  $x_1^{n_1} x_2^{n_2} \dots x_t^{n_t} M \subseteq M$  so that each quotient is isomorphic to  $M/x_i M$  for some  $i = 1, 2, \dots, t$ .

We prove the theorem by induction on  $|\underline{n}| = n_1 + n_2 + \dots + n_t$ . If  $|\underline{n}| = 0$ , the claim is trivially true.

Now suppose  $|\underline{n}| \geq 1$ . By symmetry we assume  $n_1 \geq 1$ . Notice, by the induction hypothesis, that

$$\begin{aligned} & J^{k|\underline{n}|} M \cap (x_1^{n_1} x_2^{n_2} \dots x_t^{n_t} M :_M J^\infty) \\ (*) \quad & \subseteq J^{k(|\underline{n}|-1)} M \cap (x_1^{n_1-1} x_2^{n_2} \dots x_t^{n_t} M :_M J^\infty) \\ & \subseteq x_1^{n_1-1} x_2^{n_2} \dots x_t^{n_t} M. \end{aligned}$$

Therefore, using the definition of integers  $k, k', k''$  and the fact that

$$\begin{aligned} \text{AR} \left( J, (x_1^{n_1} x_2^{n_2} \dots x_t^{n_t} M :_{x_1^{n_1-1} x_2^{n_2} \dots x_t^{n_t} M} J^\infty) \subseteq x_1^{n_1-1} x_2^{n_2} \dots x_t^{n_t} M \right) \\ &= \text{AR}(J, x_1 M :_M J^\infty \subseteq M) \quad \text{and} \\ (x_1^{n_1} x_2^{n_2} \dots x_t^{n_t} M :_{x_1^{n_1-1} x_2^{n_2} \dots x_t^{n_t} M} J^\infty) / x_1^{n_1} x_2^{n_2} \dots x_t^{n_t} M \\ &\cong (x_1 M :_M J^\infty) / x_1 M, \end{aligned}$$

we have

$$\begin{aligned}
 & J^{k|\underline{n}|}M \cap (x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} M :_M J^\infty) \\
 &= (x_1^{n_1-1} x_2^{n_2} \cdots x_t^{n_t} M) \cap J^{k|\underline{n}|}M \cap (x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} M :_M J^\infty) \quad \text{by } (*) \\
 &= (x_1^{n_1-1} x_2^{n_2} \cdots x_t^{n_t} M) \cap J^{k|\underline{n}|}M \cap (x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} M :_{x_1^{n_1-1} x_2^{n_2} \cdots x_t^{n_t} M} J^\infty) \\
 &= (J^{k|\underline{n}|}M \cap (x_1^{n_1-1} x_2^{n_2} \cdots x_t^{n_t} M)) \cap (x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} M :_{x_1^{n_1-1} x_2^{n_2} \cdots x_t^{n_t} M} J^\infty) \\
 &\subseteq (J^k(x_1^{n_1-1} x_2^{n_2} \cdots x_t^{n_t} M)) \cap (x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} M :_{x_1^{n_1-1} x_2^{n_2} \cdots x_t^{n_t} M} J^\infty) \\
 &\subseteq J^{k''}(x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} M :_{x_1^{n_1-1} x_2^{n_2} \cdots x_t^{n_t} M} J^\infty) \\
 &\subseteq x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} M.
 \end{aligned}$$

□

**Corollary 3.4** (Linear Growth; [Sw] and [Sh2]). *Let  $R$  be a Noetherian ring,  $M$  a finitely generated  $R$ -module and  $I_1, I_2, \dots, I_t$  ideals of  $R$ . Then there exists a  $k \in \mathbb{N}$  such that for any  $n_1, n_2, \dots, n_t \in \mathbb{N}$ , there exists a primary decomposition of  $I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M \subseteq M$ ,*

$$I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M = Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \cdots \cap Q_{\underline{n}_s},$$

where the  $Q_i$ 's are  $P_i$ -primary components of the primary decomposition such that  $P_i^{k|\underline{n}|}M \subseteq Q_i$  for all  $i = 1, 2, \dots, s$ , where  $\underline{n} = (n_1, n_2, \dots, n_t)$  and  $|\underline{n}| = n_1 + n_2 + \cdots + n_t$ .

*Proof.* Let  $k$  be as in Theorem 3.3. By Theorem 1.1 (Compatibility), it suffices to show that for each  $\underline{n} \in \mathbb{N}^t$  and each  $P \in \text{Ass}(M/I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M)$ , there is a  $P$ -primary component  $Q$  of  $I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M \subseteq M$  such that  $P^{k|\underline{n}|}M \subseteq Q$ . So we fix  $\underline{n}$  and  $P \in \text{Ass}(M/I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M)$ . Let

$$\begin{aligned}
 (P^{k|\underline{n}|}M + I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M) &= Q_1 \cap Q_2 \cap \cdots \cap Q_r \quad \text{and} \\
 (I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M :_M P^\infty) &= Q_{r+1} \cap Q_{r+2} \cap \cdots \cap Q_s
 \end{aligned}$$

be irredundant and minimal primary decompositions of the corresponding submodules of  $M$ , where  $Q_i$  is a  $P_i$ -primary submodule of  $M$  for each  $i = 1, 2, \dots, s$ . As  $P \notin \text{Ass}(M/(I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M :_M P^\infty))$ , we may assume that  $P_1 = P$ . By Theorem 3.3,  $(P^{k|\underline{n}|}M + I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M) \cap (I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M :_M P^\infty) = I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M$ . Hence

$$I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M = Q_1 \cap Q_2 \cap \cdots \cap Q_r \cap Q_{r+1} \cap Q_{r+2} \cap \cdots \cap Q_s.$$

Although the above intersection may not necessarily be irredundant and minimal, we know that  $Q_1$  is a  $P_1 = P$ -primary component of  $I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M \subseteq M$  since  $P \in \text{Ass}(M/I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M)$  and  $Q_1$  is the only  $P$ -primary submodule in the above intersection. Evidently  $P^{k|\underline{n}|}M \subseteq Q_1$ . □

Actually Theorem 3.3 can be stated in a more general situation: The filtration  $\{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M \mid (n_1, n_2, \dots, n_t) \in \mathbb{Z}^t\}$  may be replaced by a ‘multi-graded’ filtration  $\{M_{(n_1, n_2, \dots, n_t)} \mid (n_1, n_2, \dots, n_t) \in \mathbb{Z}^t\}$  of  $M$  such that

$$\mathcal{M} = \bigoplus_{(n_1, n_2, \dots, n_t) \in \mathbb{Z}^t} M_{(n_1, n_2, \dots, n_t)} T_1^{n_1} T_2^{n_2} \cdots T_t^{n_t}$$

naturally forms a multi-graded Noetherian module over a multi-graded sub-ring  $\mathcal{R}$  in  $R[T_1, T_2, \dots, T_t, T_1^{-1}, T_2^{-1}, \dots, T_t^{-1}]$  with the usual grading such that  $T_1^{-1}$ ,

$T_2^{-1}, \dots, T_t^{-1}$  are all contained in  $\mathcal{R}$  and the  $(0, 0, \dots, 0)$  part of  $\mathcal{R}$  is  $R$ . We call such a filtration ‘Noetherian’. To simplify notation, we use  $\underline{n}$  to denote  $(n_1, n_2, \dots, n_t)$  and use  $|\underline{n}|$  to denote  $n_1 + n_2 + \dots + n_t$ . And  $\mathbb{N}^t := \{(n_1, n_2, \dots, n_t) \mid n_i \geq 0, i = 1, 2, \dots, t\}$ .

The next theorem and its corollary look apparently more general than Theorem 3.3 and Corollary 3.4, although in essence they are the same.

**Theorem 3.5.** *Let  $R$  be a Noetherian ring,  $M$  a finitely generated  $R$ -module and  $\{M_{(n_1 n_2 \dots n_t)} \mid (n_1, n_2, \dots, n_t) \in \mathbb{Z}^t\}$  a Noetherian filtration of  $M$ . Then:*

- (1) *There exists a  $k \in \mathbb{N}$  such that for all  $\underline{m} \in \mathbb{Z}^t$ , for all  $\underline{n} \in \mathbb{N}^t$  and for all ideals  $J \subset R$ ,  $J^{k|\underline{n}|} M_{\underline{m}} \cap (M_{\underline{m}+\underline{n}} :_{M_{\underline{m}}} J^\infty) \subseteq M_{\underline{m}+\underline{n}}$ , i.e.  $(J^{k|\underline{n}|} M_{\underline{m}} + M_{\underline{m}+\underline{n}}) \cap (M_{\underline{m}+\underline{n}} :_{M_{\underline{m}}} J^\infty) = M_{\underline{m}+\underline{n}}$ .*
- (2) *The set  $\bigcup_{\underline{m} \in \mathbb{Z}^t, \underline{n} \in \mathbb{N}^t} \text{Ass}(M_{\underline{m}}/M_{\underline{m}+\underline{n}})$  is finite.*

*Proof.* The proof of (1) may be carried out in almost the same way as the proof of Theorem 3.3. But here we choose to use Theorem 3.3 and provide a sketch of the proof: Simply apply Theorem 3.3 to the Noetherian  $\mathcal{R}$  module  $\mathcal{M}$  and ideals  $\mathcal{I}_i = T_i^{-1}\mathcal{R}$  and then restrict the results to each of the homogeneous pieces. Theorem 3.3 gives results for all the ideals of  $\mathcal{R}$ , but here we are only interested in the ideals  $J\mathcal{R}$ , the ideals extended from ideals  $J \subset R$ .

To prove (2), we notice that the set

$$\bigcup_{\underline{n} \in \mathbb{N}^t} \text{Ass}_{\mathcal{R}}(\mathcal{M}/T_1^{-n_1} T_2^{-n_2} \dots T_t^{-n_t} \mathcal{M})$$

is finite. Then (2) follows by contracting to each of the homogeneous pieces. □

**Corollary 3.6.** *Let  $R$  be a Noetherian ring,  $M$  a finitely generated  $R$ -module and  $\{M_{(n_1 n_2 \dots n_t)} \mid (n_1, n_2, \dots, n_t) \in \mathbb{Z}^t\}$  a Noetherian filtration of  $M$ . Then there exists a  $k \in \mathbb{N}$  such that for any  $\underline{m} \in \mathbb{Z}^t$ ,  $\underline{n} \in \mathbb{N}^t$  and  $P \in \text{Ass}(M_{\underline{m}}/M_{\underline{m}+\underline{n}})$ , there exists a  $Q \in \Lambda_P(M_{\underline{m}+\underline{n}} \subseteq M_{\underline{m}})$  such that  $P^{k|\underline{n}|} M_{\underline{m}} \subseteq Q$ .*

**Example 3.7** (Compare with [Sh1]). Assume that  $R$  is Nagata (e.g.  $R$  is excellent) and  $M$  is a finitely generated  $R$ -module and  $I_1, I_2, \dots, I_t$  ideals of  $R$ . Then we have a multi-graded filtration  $\{\overline{I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M} \mid \underline{n} \in \mathbb{Z}^t\}$ . In order to see if the filtration satisfies the Linear Growth property, we may mod out the nil-radical and hence assume that  $R$  is reduced. Then it is straightforward to see that the associated graded module is finite over  $\mathcal{R} = R[I_1 T_1, I_2 T_2, \dots, I_t T_t, T_1^{-1}, T_2^{-1}, \dots, T_t^{-1}]$ . Hence the filtration satisfies the Linear Growth property. Similarly we can show the Linear Growth property of the filtration  $\{\overline{I_1^{n_1} \cdot I_2^{n_2} \dots I_t^{n_t} M} \mid \underline{n} \in \mathbb{Z}^t\}$  provided  $R$  is reduced and Nagata.

In [Sh1] R. Y. Sharp proved the Linear Growth property of the filtration  $\{\overline{I^n} \mid n \in \mathbb{Z}\}$  of Noetherian ring  $R$  without the Nagata assumption. The argument there also works for the filtration  $\{\overline{I_1^{n_1} I_2^{n_2} \dots I_t^{n_t}} \mid \underline{n} \in \mathbb{Z}^t\}$  of any Noetherian ring  $R$ . That is because the set  $\bigcup_{\underline{n} \in \mathbb{Z}^t} \text{Ass}(R/\overline{I_1^{n_1} I_2^{n_2} \dots I_t^{n_t}})$  is finite (cf. [Ra]) and hence we can localize and then complete. In fact, if we know in advance the set

$$\bigcup_{\underline{n} \in \mathbb{Z}^t} \text{Ass}(M/\overline{I_1^{n_1} I_2^{n_2} \dots I_t^{n_t} M})$$

is finite for a finitely generated faithful  $R$ -module  $M$ , we can localize, complete and then contract the result of Example 3.7 for  $\hat{M}$  back to  $M$  to deduce that the



filtration  $\{\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M} \mid \underline{n} \in \mathbb{Z}^t\}$  satisfies the Linear Growth property. We need  $M$  to be faithful so that the process of contraction works.

**Example 3.8.** Assume  $R$  is Nagata and has characteristic  $p$ , where  $p$  is a prime number and  $M$  is a finitely generated  $R$ -module. Then for any ideal  $I$  in  $R$ , tight closure of  $I$ , denoted by  $I^*$ , is defined [HH]. It is shown that  $\sqrt{0} \subseteq I^* \subseteq \overline{I}$  for any ideal  $I$  in  $R$  [HH]. By the same argument as in Example 3.7 we can deduce that the filtration  $\{(I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t})^* M \mid \underline{n} \in \mathbb{Z}^t\}$  is Noetherian and hence has the Linear Growth property. If, furthermore,  $R$  is reduced, then the filtration  $\{I_1^{n_1*} I_2^{n_2*} \cdots I_t^{n_t*} M \mid \underline{n} \in \mathbb{Z}^t\}$  satisfies the Linear Growth property.

In [Ra] it is shown that  $\text{Ass}(R/\overline{I^n})$  is non-decreasing and eventually stabilizes for any ideal  $I$  in a Noetherian ring  $R$ . For any finitely generated  $R$ -module  $M$ , a result of [Br] says that  $\text{Ass}(M/I^n M)$  also stabilizes for large  $n$ . If  $R$  is Nagata and of characteristic  $p > 0$ , then it follows from Example 3.8 and Theorem 3.5 that the set  $\bigcup_{\underline{n} \in \mathbb{Z}^t} \text{Ass}(M/(I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t})^* M)$  is finite. In the case of  $t = 1$ , we would like to study the stability of  $\text{Ass}(M/(I^{n*} M))$ . Since  $\bigoplus_{n \in \mathbb{Z}} I^{n*} M T^n$  is finite over  $R[IT, T^{-1}]$  (see Example 3.8), we know the filtration  $\{I^{n*} M \mid n \in \mathbb{N}\}$  of  $M$  is eventually stable, i.e.  $I^{n+1*} M = I I^{n*} M$  for all large  $n$ . Hence the argument in [Br] can be applied to show that  $\text{Ass}(M/I^{n*} M)$  stabilizes for large  $n$ .

#### ACKNOWLEDGMENT

I would like to thank Craig Huneke for the helpful conversations and valuable inspiration, which made this paper possible.

#### REFERENCES

- [Bo] N. Bourbaki, *Commutative algebra (Elements of mathematics)*, Translated from the French, Addison-Wesley Publishing Co. Hermann, Paris, 1972. MR **50**:12997
- [Br] M. Brodmann, *Asymptotic stability of  $\text{Ass}(M/I^n M)$* , Proc. Amer. Math. Soc. **74** (1979), no. 1, 16–18. MR **80c**:13012
- [Ei] D. Eisenbud, *Commutative Algebra. With a view towards algebraic geometry*, Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995. MR **97a**:13001
- [HH] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116. MR **91g**:13010
- [Hu] C. Huneke, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992), 203–223. MR **93b**:13027
- [HRS] W. Heinzer, L. J. Ratliff, Jr. and K. Shah, *On the embedded primary components of ideals. I*, J. Algebra **167** (1994) no. 3, 724–744. MR **95f**:13023
- [Mc] S. McAdam, *Primes associated to an ideal*, Contemporary Mathematics, 102, American Mathematical Society, Providence, RI, 1989, ISBN: 0-8218-5108-X. MR **90m**:13004
- [Ra] L. J. Ratliff, Jr., *On asymptotic prime divisors*, Pacific J. Math. **111** (1984) no. 2, 395–413. MR **85c**:13003
- [Sh1] R. Y. Sharp, *Linear growth of primary decompositions of integral closures*, J. Algebra **207** (1998) no. 1, 276–284. MR **99g**:13008
- [Sh2] R. Y. Sharp, *Injective modules and linear growth of primary decompositions*, Proc. Amer. Math. Soc. **128** 2000 no. 3, 717–722. MR **2000e**:13004
- [Sw] I. Swanson, *Powers of Ideals: Primary Decompositions, Artin-Rees Lemma and Regularity*, Math Ann. **307** (1997) no. 2, 299–313. MR **97j**:13005

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045  
*E-mail address:* yyao@math.ukans.edu