PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 6, Pages 1717–1723
S 0002-9939(01)06291-8
Article electronically published on October 17, 2001

INDEX OF B-FREDHOLM OPERATORS AND GENERALIZATION OF A WEYL THEOREM

M. BERKANI

(Communicated by Joseph A. Ball)

ABSTRACT. The aim of this paper is to show that if S and T are commuting B-Fredholm operators acting on a Banach space X, then ST is a B-Fredholm operator and $ind(ST) = ind(S) + ind(T)$, where ind means the index. Moreover if T is a B-Fredholm operator and F is a finite rank operator, then $T + F$ is a B-Fredholm operator and $ind(T + F) = ind(T)$. We also show that if 0 is isolated in the spectrum of T, then T is a B-Fredholm operator of index 0 if and only if T is Drazin invertible. In the case of a normal bounded linear operator T acting on a Hilbert space H, we obtain a generalization of a classical Weyl theorem.

1. INTRODUCTION

This paper is a continuation of our previous works [2], [3], [4], [5]. We consider a Banach space X and $L(X)$ the Banach algebra of bounded linear operators acting on X. For $T \in L(X)$ we will denote by $N(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $R(T)$ the range of T and by $\beta(T)$ its defect. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. A semi-Fredholm operator is an upper or a lower semi-Fredholm operator. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator and the index of T is defined by $ind(T) = \alpha(T) - \beta(T)$.

Now for a bounded linear operator T and for each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is a Fredholm (resp. semi-Fredholm) operator, then T is called a B-Fredholm operator (resp. a semi-B-Fredholm) operator. In this case and from [2, Proposition 2.1] T_m is a Fredholm operator and $ind(T_m) = ind(T_n)$ for each $m \geq n$. This enables us to define the index of a B-Fredholm operator T as the index of the Fredholm operator T_n where n is any integer such that $R(T^n)$ is closed and T_n is a Fredholm operator. Let $BF(X)$ be the class of all B-Fredholm operators. In [2] the author has studied this class of operators and has proved [2, Theorem 2.7] that an operator $T \in L(X)$ is a B-Fredholm operator if and only if $T = T_0 \oplus T_1$, where T_0 is a Fredholm operator and T_1 is a nilpotent one.

The aim of this paper is to study the properties of the index of B-Fredholm operators and to derive a generalization of a classical Weyl theorem.

Received by the editors December 5, 2000.
1991 *Mathematics Subject Classification.* Primary 47A53, 47A55.

It appears that the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. Let A be an algebra with a unit. Following [14] we say that an element x of A is Drazin invertible of degree k if there is an element b of A such that

$$x^k bx = x^k, bxb = b, xb = bx. \tag{1}$$

Recall that the concept of Drazin invertibility was originally considered by M. P. Drazin in [6] where elements satisfying relation (1) are called pseudo-invertible elements. The Drazin spectrum is defined by $\sigma_D(a) = \{\lambda \in C : a - \lambda I$ is not Drazin invertible$\}$ for every $a \in A$. In the case of a Banach algebra A and from [5, Theorem 2.3] we know that the Drazin spectrum satisfies the spectral mapping theorem.

In the case of a bounded linear operator T acting on a Banach space X, it is well known that T is Drazin invertible if and only if it has a finite ascent and descent (Definition 2.1), which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is an invertible operator and T_1 is a nilpotent one. (See [14, Proposition 6] and [12, Corollary 2.2].) In [5, Theorem 3.4] we have shown that a bounded linear operator T acting on a Banach space X is a B-Fredholm operator if and only if its projection in the algebra $L(X)/F_0(X)$ is Drazin invertible, where $F_0(X)$ is the ideal of finite rank operators in the algebra $L(X)$. This characterization of B-Fredholm operators shows easily that the class of B-Fredholm operators is stable under finite rank perturbation and the product of two commuting B-Fredholm operators is a B-Fredholm operator [5, Corollary 3.5].

After giving some preliminaries in the second section, we prove in the third section that if S, T are two commuting B-Fredholm operators, then the product ST is a B-Fredholm operator and $ind(ST) = ind(S) + ind(T)$. Moreover if T is a B-Fredholm operator and F is a finite rank operator, then $T + F$ is a B-Fredholm operator and $ind(T + F) = ind(T)$. Those two results give a positive answer to two open questions of [5]. We also show that if 0 is isolated in the spectrum of T, then T is a B-Fredholm operator of index 0 if and only if T is Drazin invertible. Then we define B-Weyl operators and the B-Weyl spectrum as follows:

Definition 1.1. Let $T \in L(X)$. Then T is called a B-Weyl operator if it is a B-Fredholm operator of index 0.

The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbf{C} : T - \lambda I$ is not a B-Weyl operator$\}$.

In Theorem 4.3 we show that for $T \in L(X)$ we have

$$\sigma_{BW}(T) = \bigcap_{F \in F_0(X)} \sigma_D(T + F),$$

and in the case a normal operator T acting on a Hilbert space H, we show that

$$\sigma_{BW}(T) = \sigma(T) \backslash E(T),$$

where $E(T)$ is the set of isolated eigenvalues of T, which gives a generalization of the classical Weyl theorem. Recall that the classical Weyl theorem [15] asserts that if T is a normal operator acting on a Hilbert space H, then the Weyl spectrum $\sigma_W(T)$ is exactly the set of all points in $\sigma(T)$ except the isolated eigenvalues of finite multiplicity, that is,

$$\sigma_W(T) = \sigma(T) \backslash \Pi_{00}(T),$$

where $\Pi_{00}(T)$ is the set of isolated eigenvalues of finite multiplicity and $\sigma_W(T)$ is the Weyl spectrum of T, that is, $\sigma_W(T) = \{\lambda \in \mathbf{C}$ such that $T - \lambda I$ is not a Fredholm operator of index $0\}$. It is known from [8, Theorem 6.5.2] that

$$\sigma_W(T) = \bigcap_{F \in F_0(X)} \sigma(T + F).$$

Henceforth, if M and N are two vector spaces, the notation $M \simeq N$ will mean that M and N are isomorphic. We also define the infimum of the empty set to be ∞.

2. Preliminaries

Definition 2.1 ([3])**.** Let $T \in L(X)$ and let $n \in \mathbf{N}$.

i) The sequence $(c_n(T))$ is defined by $c_n(T) = dim R(T^n)/R(T^{n+1})$, and the descent of T is defined by $\delta(T) = inf\{n : c_n(T) = 0\} = inf\{n : R(T^n) = R(T^{n+1})\}$.

ii) The sequence $(c'_n(T))$ is defined by $c'_n(T) = dim N(T^{n+1})/N(T^n)$, and the ascent $a(T)$ of T is defined by

$$a(T) = inf\{n : c'_n(T) = 0\} = inf\{n : N(T^n) = N(T^{n+1})\}.$$

iii) The sequence $(k_n(T))$ is defined by

$$k_n(T) = dim[(R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T))].$$

Definition 2.2 ([11])**.** Let $T \in L(X)$ and let $\Delta(T) = \{n \in \mathbf{N} : \quad \forall m \in \mathbf{N} \ m \geq n \Rightarrow (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T))\}$. Then the degree of stable iteration $dis(T)$ of T is defined as $dis(T) = inf \Delta(T)$.

Definition 2.3 ([7])**.** Let $T \in L(X)$ and let $d \in \mathbf{N}$. Then T has a uniform descent for $n \geq d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for all $n \geq d$, in other words, if $k_n(T) = 0 \ (n \geq d)$.

If in addition $R(T) + N(T^d)$is closed, then T is said to have a topological uniform descent for $n \geq d$.

Theorem 2.4 ([7, Theorem 4.7])**.** *Suppose that T is a bounded operator with topological uniform descent for $n \geq d$ on the Banach space X, $n, d \in \mathbf{N}$, and that V is a bounded operator that commutes with T. If $V - T$ is sufficiently small and invertible, then:*

(a) *V has closed range and $k_p(V) = 0$ for each integer $p \geq 0$.*

(b) *$c_p(V) = c_d(T)$ for each integer $p \geq 0$.*

(c) *$c'_p(V) = c'_d(T)$ for each integer $p \geq 0$.*

Remark A. As it has already been observed in [2] it is immediately seen that a B-Fredholm operator is an operator of topological uniform descent. Using this fact and the properties of operators with topological uniform descent, we have the following properties of the index:

i) If $S, T \in BF(X), ST = TS$ and $|| \, T - S \, ||$ is small, then $ind(T) = ind(S)$. (See [7, Theorem 4.6].)

ii) If $S, T \in BF(X), ST = TS$ and $T - S$ is compact, then $ind(T) = ind(S)$. (See [7, Theorem 5.8].)

iii) If $T \in BF(X), ST = TS, || \, T - S \, ||$ is small and $T - S$ is invertible, then S is a Fredholm operator and $ind(S) = ind(T)$. (See [7, Theorem 4.7].) In particular, if T is a B-Fredholm operator and n is an integer large enough, then $T - \frac{1}{n}I$ is a Fredholm operator and $ind(T - \frac{1}{n}I) = ind(T)$.

3. Index of B-Fredholm operators

If T is a bounded linear operator T such that both of $\alpha(T)$ and $\beta(T)$ are finite, then the range $R(T)$ of T is closed and T is a Fredholm operator. In the following theorem, we prove a similar result for B-Fredholm operators giving a simple characterization of this class of operators.

Theorem 3.1. *Let $T \in L(X)$. Then T is a B-Fredholm operator if and only if there exists an integer $n \in \mathbf{N}$ such that $\alpha(T_n)$ and $\beta(T_n)$ are finite.*

Proof. Suppose that T is a B-Fredholm operator and let $d = dis(T)$. Then from [2, Proposition 2.6] we know that $R(T^d) \cap N(T)$ is of finite dimension and $R(T)+N(T^d)$ is of finite codimension. So $\alpha(T_d)$ and $\beta(T_d)$ are both finite.

Conversely suppose that $T \in L(X)$ and there exist $n \in \mathbf{N}$ such that $\alpha(T_n)$ and $\beta(T_n)$ are both finite. Then $R(T)+N(T^n)$ is of finite codimension and $N(T)\cap R(T^n)$ is of finite dimension. Since $N(T) \cap R(T^n)$ is of finite dimension, the sequence $(N(T) \cap R((T)^p))_p$ is a stationary sequence for p large enough. This shows that $d = dis(T) \in \mathbf{N}$, $R(T^d) \cap N(T)$ is of finite dimension and $R(T) + N(T^d)$ is of finite codimension. From [9, Lemma 3.1] we have $\frac{N(T^{d+1})}{N(T^d)} \simeq N(T) \cap R(T^d)$, and from [9, Lemma 3.2] we have $\frac{R(T^d)}{R(T^{d+1})} \simeq \frac{X}{R(T)+N(T^d)}$. It then follows that $c_d(T) < \infty$ and $c'_d(T) < \infty$. Since the sequences $(c_p(T))_p$ and $(c'_p(T))_p$ are stationary sequences then for $p \geq d$, we have $c_p(T) < \infty$ and $c'_p(T) < \infty$. Moreover by [9, Lemma 3.1] we have $\frac{N(T^{2p})}{N(T^p)} \simeq N(T^p) \cap R(T^p)$, and from [9, Lemma 3.2] we have $\frac{R(T^p)}{R(T^{2p})} \simeq \frac{X}{R(T^p)+N(T^p)}$. Therefore $N(T^p) \cap R(T^p)$ is of finite dimension and $R(T^p) + N(T^p)$ is of finite codimension. In particular the two sets are closed. Using the Neubauer lemma [11, Proposition 2.1.1] it follows that $R(T^p)$ is closed. Hence T_d is a Fredholm operator and so $T \in BF(X)$. □

Theorem 3.2. *Let S, T be two commuting B-Fredholm operators. Then the product ST is a B-Fredholm operator and $ind(ST) = ind(S) + ind(T)$.*

Proof. From [5, Corollary 3.5] we know that ST is a B-Fredholm operator. Moreover there exists an integer N_0 such that for any integer $n \geq N_0$, the operators $T - \frac{1}{n}I$ and $S - \frac{1}{n}I$ are both Fredholm operators, $ind(T - \frac{1}{n}I) = ind(T)$ and $ind(S - \frac{1}{n}I) = ind(S)$. Moreover for $n \geq N_0$ the difference $ST - (S - \frac{1}{n}I)(T - \frac{1}{n}I) = \frac{1}{n}(S + T - \frac{1}{n}I)$ is of small norm if the integer n is chosen large enough. Since ST and $(S - \frac{1}{n}I)(T - \frac{1}{n}I)$ are B-Fredholm operators, then by the Remark A we have $ind(ST) = ind((S - \frac{1}{n}I)(T - \frac{1}{n}I))$. Since $S - \frac{1}{n}I$ and $T - \frac{1}{n}I$ are both Fredholm operators, then
$ind((S - \frac{1}{n}I)(T - \frac{1}{n}I)) = ind(S - \frac{1}{n}I) + ind(T - \frac{1}{n}I)$. Since $ind(S - \frac{1}{n}I) = ind(S)$ and $ind(T - \frac{1}{n}I) = ind(T)$, then $ind(ST) = ind(S) + ind(T)$. □

Proposition 3.3. *Let $T \in L(X)$ be a B-Fredholm operator and let F be a finite rank operator. Then $T + F$ is a B-Fredholm operator and $ind(T + F) = ind(T)$.*

Proof. From [3, Corollary 3.10], it follows that $T + F$ is a B-Fredholm operator. Moreover there exists an integer N_0 such that for any integer $n \geq N_0$, $T - \frac{1}{n}I$ and $T + F - \frac{1}{n}I$ are Fredholm operators, $ind(T - \frac{1}{n}I) = ind(T)$ and $ind(T + F - \frac{1}{n}I) = ind(T + F)$. Since F is a finite rank operator and $T - \frac{1}{n}I$ is a Fredholm operator, by the usual properties of the index we have $ind(T + F - \frac{1}{n}I)) = ind(T - \frac{1}{n}I)$. So $ind(T + F) = ind(T)$. □

Remark B. 1) If K is a compact operator such that $R(K^n)$ is not closed for every positive integer n, then K is not a B-Fredholm operator. So if F is a finite rank operator, then F is a B-Fredholm operator, but $K+F$ is not a B-Fredholm operator, otherwise $K = K+F-F$ would be a B-Fredholm operator. Hence the class of B-Fredholm operators is not stable under compact perturbation.

2) Let $T \in L(X)$. It is easily seen that T is a B-Fredholm operator if only if T^* is a B-Fredholm operator. Moreover in this case $ind(T^*) = -ind(T)$.

4. B-Freholm operators of index 0

Lemma 4.1. *Let $T \in L(X)$. Then T is a B-Fredholm operator of index 0 if and only if $T = T_0 \oplus T_1$, where T_0 is a Fredholm operator of index 0 and T_1 is a nilpotent operator.*

Proof. If T is a B-Fredholm operator of index 0, then $X = X_0 \oplus X_1$, where X_0, X_1 are closed subspaces of X, $T_0 = T_{|X_0}$ is a Fredholm operator and $T_1 = T_{|X_1}$ is a nilpotent operator. Moreover we have $ind(T) = ind(T_n)$ for n large enough. Since T_1 is a nilpotent operator, then for n large enough we have $R(T^n) = R(T_0^n)$ and $T_n = (T_0)_n$. From [2, Proposition 1] we have $ind(T_0) = ind((T_0)_n)) = ind(T_n) = ind(T) = 0$.

Conversely if $X = X_0 \oplus X_1$, $T_0 = T_{|X_0}$ is a Fredholm operator of index 0 and $T_1 = T_{|X_1}$ is a nilpotent operator, then by the same arguments, T is a B-Fredholm operator of index 0. □

Theorem 4.2. *Let $T \in L(X)$ be such that 0 is isolated in the spectrum $\sigma(T)$ of T. Then T is a B-Fredholm operator of index 0 if and only if T is Drazin invertible.*

Proof. If T is a B-Fredholm operator of index 0, then $X = X_0 \oplus X_1$ such that $T_0 = T_{|X_0}$ is a Fredholm operator of index 0 and $T_1 = T_{|X_1}$ is a nilpotent operator. If T_0 is invertible, then T is Drazin invertible. If T_0 is not invertible, as 0 is isolated in the spectrum of T, then it is also isolated in the spectrum of T_0. Since T_0 is a Fredholm operator of index 0, it follows from [1, Proposition 2] that $T_0 = T_{00} \oplus T_{01}$, where T_{00} is invertible and T_{01} is a nilpotent operator. So $T = T_{00} \oplus T_{01} \oplus T_1$, with T_{00} invertible and $T_{01} \oplus T_1$ nilpotent. This shows that T is Drazin invertible.

Conversely if T is Drazin invertible, then T is of finite ascent and descent. It follows from [12, Theorem 1.2] that there is an integer p such that $a(T) = d(T) = p$ and $X = R(T^p) \oplus N(T^p)$. Let $X_0 = R(T^p)$ and $X_1 = N(T^p)$. Since $T_0 = T_{|X_0}$ is an invertible operator and $T_1 = T_{|X_1}$ is a nilpotent operator, from the precedent proposition it follows that T is a B-Fredholm operator of index 0. □

Theorem 4.3. *Let $T \in L(X)$. Then $\sigma_{BW}(T) = \bigcap_{F \in F_0(X)} \sigma_D(T+F)$.*

Proof. Let $\lambda \notin \sigma_{BW}(T)$. Then $T - \lambda I$ is a B-Fredholm operator of index 0. From Lemma 4.1, we have $T - \lambda I = T_0 \oplus T_1$, where T_0 is a Fredholm operator of index 0 and T_1 is a nilpotent operator. From [8, Theorem 6.5.2] there exists a finite rank operator S_0 such that $T_0 + S_0$ is invertible. Set $S = S_0 \oplus 0$; then S is of finite rank operator and $(T - \lambda I) + S = T_0 + S_0 \oplus T_1$ is Drazin invertible. Hence $\lambda \notin \bigcap_{F \in F_0(X)} \sigma_D(T+F)$.

Conversely if $\lambda \notin \bigcap_{F \in F_0(X)} \sigma_D(T+F)$, then there is a finite rank operator F such that $(T - \lambda I) + F$ is Drazin invertible. From Proposition 3.3, $(T - \lambda I) = (T - \lambda I) + F - F$ is a B-Fredholm operator and $ind(T - \lambda I) = ind((T - \lambda I) + F) = 0$, and $\lambda \notin \sigma_{BW}(T)$. □

From this theorem, we immediately obtain the following characterization of B-Weyl operators.

Corollary 4.4. *Let $T \in L(X)$. Then T is a B-Weyl operator if and only if $T = S + F$, where S is Drazin invertible operator and F is a finite rank operator.*

It is known from [13, Theorem 7.7] that if λ is isolated in the spectrum $\sigma(T)$ of a normal operator T acting on a Hilbert space H, then $T - \lambda I$ is Drazin invertible. By the following theorem we give for such an operator, a generalization of a classical Weyl theorem [15].

Theorem 4.5. *Let $T \in L(H)$ be a normal operator. Then $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, where $E(T)$ is the set of isolated eigenvalues of T.*

Proof. If $\lambda \notin \sigma_{BW}(T)$ and $\lambda \in \sigma(T)$, then $T - \lambda I$ is a B-Fredholm operator of index 0. Hence there exists an integer n such that $R((T - \lambda I)^n)$ is closed. Since $(T - \lambda I)^n$ is a normal operator, then from [13, Theorem VI.3.6],

$$H = R((T - \lambda I)^n) \oplus N((T - \lambda I)^n).$$

As $T - \lambda I$ is a normal operator, then from [13, Theorem VI.3.7], $N((T - \lambda I)^n) = N((T - \lambda I))$. Hence $R((T - \lambda I)) = R((T - \lambda I)^n)$ and $H = R((T - \lambda I)) \oplus N((T - \lambda I))$. Since $\lambda \in \sigma(T)$, then $N(T - \lambda I) \neq 0$. It follows that λ is an isolated eigenvalue of T.

Conversely if $\lambda \in E(T)$, then from [10, Theorem 7.1] we have $X = X_0 \oplus X_1$, where X_0, X_1 are closed subspaces of X, $T_0 = (T - \lambda I)_{|X_0}$ is an invertible operator and $T_1 = (T - \lambda I)_{|X_1}$ is a quasi-nilpotent operator. Since T is a normal operator, then T_1 is also a normal operator. As T_1 is quasi-nilpotent, it is a nilpotent operator. Therefore $T - \lambda I$ is Drazin invertible. From Theorem 2.2 it is a B-Fredholm operator of index 0. □

References

[1] B. A. Barnes, *Riesz Points and Weyl's Theorem.* Integr. Equ. Oper. Theory 34 (1999) 187-196. MR **2000d:**47006

[2] M. Berkani, *On a class of quasi-Fredholm operators.* Integr. Equ. Oper. Theory 34 (1999), 244-249. MR **2000d:**47023

[3] M. Berkani, *Restriction of an operator to the range of its powers.*. Studia Mathematica, Vol 140(2), (2000), 163-175. MR **2001g:**47021

[4] M. Berkani, M. Sarih, *On semi-B-Fredholm operators.* To appear in Glasgow Mathematical Journal.

[5] M. Berkani, M. Sarih, *An Atkinson-type theorem for B-Fredholm operators*, To appear in Studia Mathematica.

[6] M. P. Drazin, *Pseudoinverse in associative rings and semigroups.* Amer. Math. Monthly 65 (1958), 506-514. MR **20:**5217

[7] S. Grabiner, *Uniform ascent and descent of bounded operators;* J. Math. Soc. Japan 34, No. 2 (1982), 317-337. MR **84a:**47003

[8] R. Harte, *Invertibility and singularity for bounded linear operators;* Marcel Dekker. New York, Basel, 1988. MR **90a:**15019

[9] M. Kaashoek, *Ascent, Descent, Nullity and Defect, a Note on a Paper by A.E. Taylor;* Math. Annalen 172, 105-115 (1967). MR **36:**5719

[10] J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J. 38 (1996), 367-381. MR **98b:**46065

[11] J. P. Labrousse, *Les opérateurs quasi-Fredholm: une généralisation des opérateurs semi-Fredholm;* Rend. Circ. Math. Palermo (2), 29 (1980), 161-258. MR **83c:**47022

[12] D. C. Lay, *Spectral analysis using ascent, descent, nullity and defect;* Math. Ann. 184, 197-214 (1970).

[13] D. C. Lay, A. E. Taylor, *Introduction to Functional Analysis,* Second edition, Wiley, New York, 1980. MR **81b:**46001

[14] S. Roch, B. Silbermann, *Continuity of generalized inverses in Banach algebras.* Studia Mathematica 136 (3), (1999), 197-227. MR **2000m:**46101

[15] H. Weyl, *Über beschränkte quadratische Formen, deren Differenz vollstetig ist.* Rend. Circ. Mat. Palermo 27 (1909), 373-392.

Département de Mathématiques, Faculté des Sciences, Université Mohammed I, Oujda, Maroc

E-mail address: berkani@sciences.univ-oujda.ac.ma