EQUIVALENCE OF DOMAINS WITH ISOMORPHIC 
SEMIGROUPS OF ENDMORPHISMS

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ABSTRACT. For two bounded domains $\Omega_1$, $\Omega_2$ in $\mathbb{C}$ whose semigroups of analytic endomorphisms $E(\Omega_1)$, $E(\Omega_2)$ are isomorphic with an isomorphism $\varphi : E(\Omega_1) \to E(\Omega_2)$, Eremenko proved in 1993 that there exists a conformal or anticonformal map $\psi : \Omega_1 \to \Omega_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(\Omega_1)$.

In the present paper we prove an analogue of this result for the case of bounded domains in $\mathbb{C}^n$.

1. INTRODUCTION

A classical theorem of L. Bers says that every $\mathbb{C}$-algebra isomorphism $H(A) \to H(B)$ of algebras of holomorphic functions in domains $A$ and $B$ in the complex plane has the form $f \mapsto f \circ \theta$, where $\theta : B \to A$ is a conformal isomorphism, or $f \mapsto \overline{f} \circ \theta$ with anticonformal $\theta$. In particular, the algebras $H(A)$ and $H(B)$ are isomorphic if and only if the domains $A$ and $B$ are conformally equivalent. H. Iss’sa [9] obtained a similar theorem for fields of meromorphic functions on Stein spaces. A good reference for these results is [5].

Likewise, the question of recovering a topological space from the algebraic structure of its semigroup of continuous self-maps has been extensively studied [12].

In 1990, L. Rubel asked whether similar results hold for semigroups (under composition) $E(D)$ of holomorphic endomorphisms of a domain $D$. A. Hinkkanen constructed examples [6] which show that even non-homeomorphic domains in $\mathbb{C}$ can have isomorphic semigroups of endomorphisms. The reason is that the semigroup of endomorphisms of a domain can be too small to characterize this domain.

However, in 1993, A. Eremenko [4] proved that for two Riemann surfaces $D_1$, $D_2$, which admit bounded nonconstant holomorphic functions, and such that the semigroups of analytic endomorphisms $E(D_1)$ and $E(D_2)$ are isomorphic with an isomorphism $\varphi : E(D_1) \to E(D_2)$, there exists a conformal or anticonformal map $\psi : D_1 \to D_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(D_1)$. In the present paper we investigate the analogue of this result for the case of bounded domains in $\mathbb{C}^n$. The theorems of Bers and Iss’sa, mentioned above, do not extend to arbitrary domains in $\mathbb{C}^n$. 

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For a bounded domain \( \Omega \) in \( \mathbb{C}^n \) we denote by \( E(\Omega) \) the semigroup of analytic endomorphisms of \( \Omega \) under composition. In what follows, we say that a map is (anti-) biholomorphic, if it is biholomorphic or antibiholomorphic. We prove the following theorem.

**Theorem 1.** Let \( \Omega_1, \Omega_2 \) be bounded domains in \( \mathbb{C}^n, \mathbb{C}^m \) respectively, and suppose that there exists \( \varphi : E(\Omega_1) \rightarrow E(\Omega_2) \), an isomorphism of semigroups. Then \( n = m \) and there exists an (anti-) biholomorphic map \( \psi : \Omega_1 \rightarrow \Omega_2 \) such that

\[
\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1).
\]

(1)

The existence of a homeomorphism \( \psi \) satisfying (1) follows from simple general considerations (Section 2). The hard part is proving that \( \psi \) is (anti-) biholomorphic. In dimension 1 this is done by linearization of holomorphic germs of \( f \in E(\Omega) \) near an attracting fixed point. In several dimensions such linearization theory exists ([1], pp. 192–194), but it is too complicated (many germs with an attracting fixed point are non-linearizable, even formally). In Sections 3, 4 we show how to localize the problem. In Sections 5, 6 we describe, using only the semigroup structure, a large enough class of linearizable germs. Linearization of these germs permits us to reduce the problem to a matrix functional equation, which is solved in Section 7. In Section 8 we complete the proof that \( \psi \) is (anti-) biholomorphic.

Theorem 1 can be slightly generalized, namely one may assume that \( \varphi \) is an epimorphism. In Section 9 we prove the following theorem.

**Theorem 2.** If \( \varphi : E(\Omega_1) \rightarrow E(\Omega_2) \) is an epimorphism between semigroups, where \( \Omega_1, \Omega_2 \) are bounded domains in \( \mathbb{C}^n, \mathbb{C}^m \) respectively, then \( \varphi \) is an isomorphism.

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2. **Topology**

For a bounded domain \( \Omega \) in \( \mathbb{C}^n \) we denote by \( C(\Omega) \) the subsemigroup of \( E(\Omega) \) consisting of constant maps. An endomorphism \( c_z \) is constant if it sends \( \Omega \) to a point \( z \in \Omega \). The subset \( C(\Omega) \subset E(\Omega) \) can be described using only the semigroup structure as follows:

\[
c \in C(\Omega) \text{ iff } \forall (f \in E(\Omega)), \quad (c \circ f = c).
\]

(2)

It is clear that we have a bijection between constant endomorphisms of \( \Omega \) and points of this domain as a set: to each \( z \in \Omega \) corresponds a unique \( c_z \in C(\Omega) \) and vice versa, so we can identify the two. Under this identification, a subset of \( \Omega \) corresponds to a subsemigroup of \( C(\Omega) \).

Having defined points of a domain in terms of its semigroup structure of analytic endomorphisms, we can construct a map \( \psi \) between \( \Omega_1 \) and \( \Omega_2 \) as follows:

\[
\psi(z) = w \text{ iff } \varphi c_z = c_w.
\]

(3)

So defined, \( \psi \) satisfies (1). Indeed, let \( f \in E(\Omega_1), f(z) = \zeta \). This is equivalent to

\[
f \circ c_z = c_\zeta.
\]

(4)

Applying \( \varphi \) to both sides of (4), we have

\[
\varphi f \circ c_{\psi(z)} = c_{\psi(\zeta)}.
\]

(5)
But [5] is equivalent to $\varphi f(\psi(z)) = \psi(\zeta) = \psi(f(z))$, which is [11].

We describe the topology of a domain $\Omega$ using its injective endomorphisms. A map $f \in E(\Omega)$ is injective if and only if

$$\forall (c' \in C(\Omega)) \forall (c'' \in C(\Omega)), \quad ((f \circ c' = f \circ c'') \Rightarrow (c' = c'')).$$  

We denote the class of injective endomorphisms of $\Omega$ by $E_i(\Omega)$. For every $f \in E_i(\Omega)$, $f_i(\Omega)$ is open [2]. The family $\{f(\Omega), \ f \in E_i(\Omega)\}$ of subsets of $\Omega$ forms a base of topology, because every $z \in \Omega$ has a neighborhood $f(\Omega)$, where $f(\zeta) = z + \lambda(\zeta - z)$, $f$ belongs to $E_i(\Omega)$ for every $\lambda$ such that $|\lambda|$ is small.

To summarize, we described subsets of $\Omega$ and the topology on it using only the semigroup structure of $E(\Omega)$. Since this is so, the semigroup structure also defines the notions of an open set, closed set, compact set, closure of a set.

Now we can easily prove continuity of the map $\psi$ constructed above. Indeed, let $g(\Omega_2)$, $g \in E_i(\Omega_2)$, be a set from the base of topology of $\Omega_2$. We take $f = \varphi^{-1}g$. Then $f \in E_i(\Omega_1)$ and $\psi^{-1}(g(\Omega_2)) = f(\Omega_1)$, which proves that $\psi$ is continuous. Since $\varphi$ is an isomorphism, the same argument works to prove that $\psi^{-1}$ is also continuous, and thus $\psi$ is a homeomorphism.

Therefore the domains $\Omega_1$, $\Omega_2$ are homeomorphic, and hence [5] they have the same dimension, i.e. $n = m$.

3. Localization

We need the following lemma.

**Lemma 1.** Suppose $H$ is a semigroup with identity, and $f$ an element of $H$ with the following two properties:

(i) $hf = fh$, for every $h$ in $H$;

(ii) $h_1f = h_2f$ implies $h_1 = h_2$, for every $h_1$ and $h_2$ in $H$.

Then there exist a semigroup $S_f$ and a monomorphism $i : H \to S_f$ such that $i(f)$ is invertible in $S_f$ and commutes with all elements of $S_f$. Moreover, the semigroup $S_f$ satisfies the following universal property: for every semigroup $S_1$ with a monomorphism $i_1 : H \to S_1$ such that $i_1(f)$ is invertible in $S_1$ and commutes with all elements of $S_1$, there exists a unique monomorphism $i_1 : S_f \to S_1$ such that $i_1 = i \circ i_i$.

**Remark 1.** Uniqueness of $i_1$ implies that the semigroup $S_f$ with the universal property is unique up to an isomorphism.

**Proof.** We construct $S_f$ as follows. First we consider formal expressions of the form $hf^k$, where $h \in H$ and $k$ is an integer (it may be positive, negative or zero). Then we define a multiplication on this set: $h_1f^{k_1} \star h_2f^{k_2} = h_1h_2f^{k_1+k_2}$. Next we consider a relation on the set of formal expressions: $h_1f^{k_1} \sim h_2f^{k_2}$ if $k_1 \leq k_2$ and $h_1 = h_2f^{k_2-k_1}$ in $H$, or $k_2 \leq k_1$ and $h_2 = h_1f^{k_1-k_2}$ in $H$. It is easy to verify that this is an equivalence relation and it is compatible with the operation $\star$; that is, $x \sim y$, $u \sim v$ implies $x \star u \sim y \star v$.

Lastly, let $S_f$ be the set of equivalence classes with the binary operation induced by $\star$. For $S_f$ to be a semigroup, we need to show that the binary operation $\star$ is associative. Let $h_1f^{k_1} \sim h'_1f^{k'_1}, h_2f^{k_2} \sim h'_2f^{k'_2}$ and $h_3f^{k_3} \sim h'_3f^{k'_3}$. We need to show that $(h_1f^{k_1} \star h_2f^{k_2}) \star h_3f^{k_3} \sim h'_1f^{k'_1} \star (h'_2f^{k'_2} \star h'_3f^{k'_3})$. By the definition of the operation $\star$, the last equivalence is the same as $h_1h_2h_3f^{k_1+k_2+k_3} \sim h'_1h'_2h'_3f^{k'_1+k'_2+k'_3}$. Assuming that $k_1 + k_2 + k_3 \leq k'_1 + k'_2 + k'_3$, we have essentially one possibility to
Consider (the others are either similar or trivial): \(k_1 \leq k_1', k_2 \leq k_2', k_3' \leq k_3\). In this case \(h_1 h_2 h_3 f^{k_3-k_3'} = h_1' h_2' h_3' f^{k_1'-k_1+k_2'-k_2}\). Now we can use the cancellation property (ii) to get the desired equivalence.

The semigroup \(H\) is embedded into \(S_f\) via \(i : h \mapsto [hf^0]\). The element \(i(f) = [id_f]\), where id is the identity in \(H\), is invertible in \(S_f\) with the inverse \([id_f]^{-1}\). Clearly, \([id_f]\) commutes with all elements of \(S_f\).

Now, suppose that \(S_1, i_1 : H \to S_1\) is a semigroup and a monomorphism, such that \(i_1(f)\) is invertible in \(S_1\) and commutes with all elements of \(S_1\). Then we define

\[
i_1([hf^k]) = i_1(h)(i_1(f))^k.
\]

This definition does not depend on a representative of \([hf^k]\). Indeed, suppose \(h_1 f^{k_1} \sim h_2 f^{k_2}\) and assume \(k_1 \leq k_2\). Then \(h_1 = h_2 f^{k_2-k_1}\), and thus \(i_1(h_1) = i_1(h_2) i_1(f)^{k_2-k_1}\). Hence \(i_1(h_1) i_1(f)^{k_1} = i_1(h_2) i_1(f)^{k_2}\).

So defined, \(i_1\) is a homomorphism:

\[
i_1([h_1 f^{k_1}] [h_2 f^{k_2}]) = i_1([h_1 h_2 f^{k_1+k_2}]) = i_1(h_1 h_2) i_1(f)^{k_1+k_2} = i_1(h_1) i_1(h_2) i_1(f)^{k_1} i_1(f)^{k_2} = i_1(h_1) i_1(f)^{k_1} i_1(h_2) i_1(f)^{k_2} = i_1([h_1 f^{k_1}] [h_2 f^{k_2}]).
\]

The relation \(\hat{i}_1 \circ i = i_1\) holds, since \(\hat{i}_1([hf^0]) = i_1(h)\) for all \(h \in H\).

Uniqueness of \(i_1\) is clear. Lemma 1 is proved.

4. Extension of \(\varphi\)

Following [4], we say that for a bounded domain \(\Omega\) an element \(f \in E(\Omega)\) is good at \(z \in \Omega\), denoted by \(f \in G_z(\Omega)\), if

1. \(z\) is a unique fixed point of \(f\);
2. \(f(\Omega)\) has compact closure in \(\Omega\);
3. \(f\) is injective in \(\Omega\).

Property 3 of a good element was already stated in terms of the semigroup structure of \(\Omega\). Since the topology on \(\Omega\) was described using only the semigroup structure, Property 2 can also be stated in these terms. Property 1 can be expressed in terms of the semigroup structure as

\[
(f \circ c_z = c_z) \land ((f \circ c_\zeta = c_\zeta) \Rightarrow (c_\zeta = c_z)).
\]

Since \(f\) is an endomorphism of a domain, all eigenvalues \(\lambda\) of its linear part at \(z\) satisfy \(|\lambda| \leq 1\). Moreover, \(|\lambda| < 1\) because the closure of \(f(\Omega)\) is a compact set in \(\Omega\). The injectivity of \(f\) implies [2] that it is biholomorphic onto \(f(\Omega)\) and the Jacobian determinant of \(f\) does not vanish at any point of \(\Omega\).

It is clear that for every \(z \in \Omega\) a good element \(f\) at \(z\) exists. For example, we can take \(f(\zeta) = z + \lambda(\zeta - z)\) with sufficiently small \(|\lambda|\).

Consider a good element \(f \in G_z(\Omega)\) and its commutant \(H_f(\Omega)\) in \(E(\Omega)\):

\[
H_f(\Omega) = \{ h \in E(\Omega) : hf = fh \}.
\]

Clearly \(H_f(\Omega)\) is a subsemigroup of \(E(\Omega)\). The element \(f\), being good (hence injective), satisfies the cancellation property (ii) of Lemma 1 in \(H_f(\Omega)\). Thus, by Lemma [1] we have the extension \(S_f\) of \(H_f(\Omega)\) in which \(f\) is invertible and commutes with all elements of \(S_f\). In the case of analytic endomorphisms we can embed \(H_f(\Omega)\) into the subsemigroup of \(A_z\), the semigroup of germs of analytic mappings at \(z\) under composition, consisting of elements that commute with the germ of \(f\).
and containing the germ of $f^{-1}$. We use the universal property of Lemma \ref{lem:universal} to conclude that $S_f$ is isomorphic to a subsemigroup of $A_0$. We identify $S_f$ with this semigroup, i.e., we consider elements of $S_f$ as germs of analytic mappings at $z$.

In proving that $\psi$ is (anti-) biholomorphic we need to show that it is so in a neighborhood of every point of $\Omega$. Since an (anti-) biholomorphic type of a domain is preserved by translations in $\mathbb{C}^n$, it is enough to show that $\psi$ is (anti-) biholomorphic in a neighborhood of $0 \in \mathbb{C}^n$, assuming that $\Omega$ and $\Omega_2$ contain $0$ and $\psi(0) = 0$.

Let $\varphi : E(\Omega_1) \to E(\Omega_2)$ be an isomorphism of the semigroups, $f$ a good element, $f \in G_0(\Omega_1)$, and $H_f(\Omega_1)$ the commutant of $f$. Then clearly $H_\varphi(\Omega_2) = \varphi(H_f(\Omega_1))$ is the commutant of $g = \varphi f$. By Lemma \ref{lem:universal} we have the extensions $S_f$, $S_g$ of $H_f(\Omega_1)$ and $H_\varphi(\Omega_2)$ respectively, and by the universal property of this lemma the isomorphism $\varphi$ extends to an isomorphism

$$\Phi : S_f \to S_g.$$  

5. System of projections and linearization

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. We say that a good element $f \in G_0(\Omega)$ is very good at $0$, and write $f \in VG_0(\Omega)$, if the corresponding semigroup $S_f \subset A_0$ constructed in Section \ref{sec:linearization} contains a system of elements, which we call a system of projections, $\{p_i\}_{i=1}^n$ with the following properties:

(a) $\forall (i = 1, \ldots, n), \ (p_i \neq 0)$;  
(b) $\forall (i = 1, \ldots, n), \ (p_i^2 = p_i)$;  
(c) $\forall (i, j = 1, \ldots, n, i \neq j), \ (p_ip_j = 0)$.

There does exist a very good element, since we can take $f$ to be a homothetic transformation at $0$ with sufficiently small coefficient, $p_i$ a projection on the $i$’th coordinate of the standard coordinate system. Clearly, $p_if = fp_i$ and there exists $k$ such that $p_if^k \in E(\Omega)$, and hence $p_i \in S_f$. From now on, we fix a very good good element $f \in VG_0(\Omega)$, associated semigroups $H_f(\Omega)$, $S_f$ and a system of projections $\{p_i\}$.

We introduce another subsemigroup of $E(\Omega)$:

$$P_f(\Omega) = \{h \in G_0(\Omega) \cap H_f(\Omega), \ hp_i = p_ih, \ i = 1, \ldots, n\},$$

where the commutativity relations are in $S_f \subset A_0$. Notice that $P_f(\Omega) \neq \emptyset$ since $f$ belongs to it.

**Lemma 2.** For every $h \in P_f(\Omega)$ there exists a biholomorphic germ $\theta_h$ at $0 \in \mathbb{C}^n$ such that $\theta_h \Lambda = \Lambda \theta_h$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is an invertible diagonal matrix which is similar to $dh(0)$ in $GL(n, \mathbb{C})$.

**Proof.** The relations $p_i \neq 0$, $p_i^2 = p_i$, $p_ip_j = 0$, $i \neq j$, imply that for $P_i = dp_i(0)$, the linear part of $p_i$ at $0$, we have $P_i \neq 0$, $P_i^2 = P_i$, $P_iP_j = 0$, $i \neq j$. Since the matrices $P_i$ commute, there exists $\Lambda$ a matrix $A \in GL(n, \mathbb{C})$ such that $P_i^t = AP_iA^{-1} = \Delta_i = \text{diag}(0, \ldots, 1, \ldots, 0)$, where the only non-zero entry appears in the $i$’th place.

Since $p_i^2 = p_i$, $i = 1, \ldots, n$, we can use the argument given in \ref{sec:linearization} to linearize $p_i$, i.e., there exists a biholomorphic germ $\xi_i$ at $0$ such that $\xi_i p_i = P_i \xi_i$, $d\xi_i(0) = \text{id}$, $i = 1, \ldots, n$. The map $\xi_i$ is constructed in \ref{sec:linearization} as follows:

$$\xi_i = \text{id} + (2P_i - \text{id})(p_i - P_i), \ i = 1, \ldots, n.$$
If we take $\xi'_i = A\xi_i$, we have $\xi'_i p_i = P'_i \xi'_i$. For simplicity of notations, we assume that $\xi_i$ itself conjugates $p_i$ to a diagonal matrix, that is, $P_i = P'_i$ (in this case $P_i$ is not necessarily $d\xi_i(0)$, but rather $Ad\xi_i(0)A^{-1}$; $d\xi_i(0) = A$). For every $i = 1, \ldots, n$ we have $h_i P_i = P_i h_i$, where $h_i = \xi_i h_i^{-1}$. Let $H_i = dh_i(0)$. Then $H_i P_i = P_i H_i$, and hence in the $i$'th row and the $i$'th column the matrix $H_i$ has only one non-zero entry, $\lambda_i$, which is located at their intersection. Thus $\lambda_i$ has to be an eigenvalue of $H_i$, and hence of the linear part of $h$. In particular, $0 < |\lambda_i| < 1$.

Let $I_i : \mathbb{C} \to \mathbb{C}^n$ be the embedding $z \mapsto (0, \ldots, z, \ldots, 0)$, where the only non-zero entry is $z$, which is in the $i$'th place; and $\pi_i : \mathbb{C}^n \to \mathbb{C}$, a projection $(z_1, \ldots, z_n) \mapsto z_i$, corresponding to the $i$'th axis. For every $i = 1, \ldots, n$, the map $\pi_i h_i I_i$ sends a neighborhood of 0 in $\mathbb{C}$ into $\mathbb{C}$, and its derivative at 0, $\lambda_i$, is an eigenvalue of $h$. Hence ([3], p. 31) $\pi_i h_i I_i$ is linearized by the unique solution $\eta_{h,i}$ of the Schröder equation

$$\eta(\pi_i h_i I_i) = \lambda_i \eta_i, \quad \eta(0) = 0, \quad \eta'(0) = 1. \tag{6}$$

Since $P_i I_i = I_i$, $\pi_i P_i I_i = id_C$, we can rewrite (6) as

$$\eta_{h,i} \pi_i h_i P_i I_i = \lambda_i \eta_{h,i} \pi_i P_i I_i, \quad \text{or} \quad \eta_{h,i} \pi_i h_i P_i = \lambda_i \eta_{h,i} \pi_i P_i. \tag{7}$$

But $h_i P_i = P_i h_i$, and so

$$\eta_{h,i} \pi_i p_i h_i = \lambda_i \eta_{h,i} \pi_i p_i. \tag{8}$$

We denote

$$\theta_{h,i} = \eta_{h,i} \pi_i p_i, \tag{9}$$

a map from a neighborhood of $0 \in \mathbb{C}^n$ into $\mathbb{C}$. Then (8) becomes $\theta_{h,i} h = \lambda_i \theta_{h,i}$.

Now we define

$$\theta_h = (\theta_{h,1}, \ldots, \theta_{h,n})$$

which is a germ of an analytic map at 0. This germ linearizes $h$:

$$\theta_h h = (\theta_{h,1} h, \ldots, \theta_{h,n} h) = (\lambda_1 \theta_{h,1}, \ldots, \lambda_n \theta_{h,n}) = \Lambda \theta_h,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is an invertible diagonal matrix, which has eigenvalues of $dh(0)$ on its diagonal.

The germ $\theta_h$ is biholomorphic. Indeed,

$$\theta_{h,i} = \eta_{h,i} \pi_i \xi_i p_i = \eta_{h,i} \pi_i P_i \xi_i, \quad i = 1, \ldots, n.$$ 

Using the chain rule, we see that $d\theta_h(0) = A$, where $A$ is an invertible diagonal matrix that diagonalizes $P$. We conclude that $\theta_h$ is biholomorphic. Lemma 2 is proved. \hfill \Box

6. Simultaneous Linearization

Using Lemma 2 we can linearize elements of $P_f(\Omega)$. Namely, for every $h \in P_f(\Omega)$ there exists $\theta_h$ (constructed in Section 5) such that $\theta_h h = \Lambda_h \theta_h$, where $\Lambda_h$ is an invertible diagonal matrix. In particular, we can linearize $f$:

$$\theta_f f = \Lambda_f \theta_f,$$

where the germ $\theta_f$ is biholomorphic at 0, and $\Lambda_f$ is an invertible diagonal matrix.
Lemma 3. For every \( h \in P_f(\Omega) \) we have \( \theta_h = \theta_f \).

Proof. Let us consider the germ

\[
\theta = \Lambda_f^{-1}\theta_h f,
\]

which is clearly biholomorphic. We have

\[
\theta h = \Lambda_f^{-1}\theta_h f h = \Lambda_f^{-1}\theta_h h f = \Lambda_f^{-1}\Lambda_h \theta_h f = \Lambda_h \Lambda_f^{-1}\theta_h f = \Lambda_h \theta.
\]

Using (10), we write the equation \( \theta h = \Lambda_h \theta \) in the coordinate form:

\[
(1/\lambda_{f,i}) \theta_{h,i} f h = (\lambda_{h,i}/\lambda_{f,i}) \theta_{h,i} f, \quad i = 1, \ldots, n.
\]

By (9) and the definition of \( \xi_i \),

\[
(1/\lambda_{f,i}) \eta_{h,i} \pi_i P_i f_i h_i = (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i P_i f_i, \quad i = 1, \ldots, n,
\]

where \( f_i = \xi_i f_i \xi_i^{-1} \). Using the commutativity relations \( f_i P_i = P_i f_i, \ h_i P_i = P_i h_i, \) which hold since \( \{p_i\} \subset S_f, \ h \in P_f(\Omega) \), we get

\[
(1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i P_i = (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i f_i P_i, \quad \text{or}
\]

\[
(1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i I_i = (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i, \quad i = 1, \ldots, n.
\]

This is the same as

\[
((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)(\pi_i h_i I_i) = \lambda_{h,i}((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i), \quad i = 1, \ldots, n,
\]

since \( h_i \) locally preserves the \( i \)th coordinate axis \( (h_i P_i = P_i h_i) \). It is easily seen that

\[
((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)(0) = 0,
\]

\[
((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)'(0) = 1.
\]

A normalized solution to a Schröder equation is unique, though; thus we have

\[
\eta_{h,i} \pi_i f_i I_i = \lambda_{f,i} \eta_{h,i}, \quad \eta_{h,i}(0) = 0, \quad \eta_{h,i}'(0) = 1.
\]

Using the uniqueness argument again, we obtain \( \eta_{h,i} = \eta_{f,i} \), and hence \( \theta_h = \theta_f \).

The lemma is proved. \( \square \)

According to Lemma 3 the single biholomorphic germ \( \theta_f \) conjugates the subsemigroup \( P_f(\Omega) \) to some subsemigroup \( D_f \) of invertible diagonal matrices in \( D_n \), the set of all \( n \times n \) diagonal matrices with entries in \( \mathbb{C} \). We show that \( D_f \) contains all invertible diagonal matrices with sufficiently small entries. To do this, first we extend \( \theta_f \) to an analytic map on the whole domain \( \Omega \) using the formula

\[
\theta_f = \Lambda_f^{-1}\theta f l,
\]

where \( l \) is chosen so large that \( \text{Cl}\{f'(\Omega)\} \) is contained in a neighborhood of 0 where \( \theta_f \) is originally defined and biholomorphic; the symbol \( \text{Cl} \) denotes closure. From the procedure of extending \( \theta_f \) to \( \Omega \) we see that it is one-to-one and bounded in the whole domain.

Now, let \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) be a matrix such that \( \text{Cl}\{\Lambda \theta_f(\Omega)\} \subset W \), where \( W \) is a neighborhood of 0 in \( \mathbb{C}^n \) for which \( \text{Cl}\{\theta_f^{-1} W\} \subset \Omega \). Such a matrix \( \Lambda \) exists since \( \theta_f \) is bounded in \( \Omega \). Consider \( h = \theta_f^{-1} \Lambda \theta_f \), which belongs to \( G_0(\Omega) \). The map \( h \) commutes with \( f \) and all \( p_i \)’s. Indeed, using the formula \( \theta_f f \theta_f^{-1} = \Lambda_f \), we conclude that \( h f = f h \) is equivalent to \( \Lambda \Lambda_f = \Lambda_f \Lambda \), which is a true relation since both matrices \( \Lambda \) and \( \Lambda_f \) are diagonal. The relations \( h p_i = p_i h, \quad i = 1, \ldots, n, \) are
verified similarly, using the formula $\theta_f p \theta_f^{-1} = P$, which follows from the definition of $\theta_f$.

7. Solving a Matrix Equation

We proved that for an element $f \in VG_0(\Omega)$ there exists a biholomorphic germ $\theta_f$ conjugating the semigroup $P_f(\Omega)$ to a subsemigroup $D_f \subset D_n$, which contains all invertible diagonal matrices with sufficiently small entries.

Let $f \in VG_0(\Omega_1)$ and $g = \varphi f$. Then $g \in VG_0(\Omega_2)$, and there is an isomorphism

$$\Phi: \mathcal{S}_f \to \mathcal{S}_g.$$ 

For the mappings $f$ and $g$ we have

$$\theta_f f = \Lambda_f \theta_f, \quad \theta_g g = M_g \theta_g,$$

where $\Lambda_f$, $M_g$ are invertible diagonal matrices. 

Let us consider the germ $L = \theta_g \psi \theta_f^{-1}$. This germ conjugates the semigroups $D_f$, $D_g$:

$$L \Lambda L^{-1} = \theta_g \psi \theta_f^{-1} \Lambda \theta_f \psi^{-1} \theta_g^{-1}$$

$$= \theta_g \psi h \psi^{-1} \theta_g^{-1} = \theta_g j \theta_g^{-1} = M,$$

where $h \in P_f$, $\theta_f h = \Lambda \theta_f$; $j = \varphi h$, $\theta_g j = M \theta_g$.

Define $R(\Lambda) = L \Lambda L^{-1}$. Then $R: D_f \to D_g$,

$$R(\Lambda_1 \Lambda_2) = R(\Lambda_1) R(\Lambda_2), \quad \Lambda_1, \Lambda_2 \in D_f.$$

In what follows, we will identify $D_n$ with the multiplicative semigroup $\mathbb{C}^n$ ($D_n \cong \mathbb{C}^n$) in the obvious way and consider a topology on $D_n$ induced by the standard topology on $\mathbb{C}^n$.

We are going to extend $R$ to an isomorphism of $D_n$. First, we denote by $\overline{D}_f$, $\overline{D}_g$ the closures of $D_f$, $D_g$ in $D_n$, and for $\Lambda \in \overline{D}_f$ we set

$$R(\Lambda) = \lim R(\Lambda_k), \quad \Lambda_k \to \Lambda, \quad \Lambda_k \in D_f.$$

This limit exists and does not depend on the sequence $\{\Lambda_k\}$, which follows from the fact that $\psi \pm 1, \theta_f \pm 1, \theta_g \pm 1$ are continuous. The map $R$ is an isomorphism of topological semigroups $\overline{D}_f$ and $\overline{D}_g$ (the inverse of $R$ has a similar representation).

Next, we extend the map $R$ to $D_n$ as

$$R(\Gamma) = R(\Gamma \Lambda) R(\Lambda)^{-1}, \quad \Gamma \in D_n,$$

where $\Lambda \in D_f$ is chosen so that $\Gamma \Lambda \in \overline{D}_f$. This definition does not depend on the choice of $\Lambda$. Indeed, since all matrices in question are diagonal (hence commute), the relation $R(\Gamma \Lambda_1) R(\Lambda_1)^{-1} = R(\Gamma \Lambda_2) R(\Lambda_2)^{-1}$ is equivalent to $R(\Gamma \Lambda_1) R(\Lambda_2) = R(\Gamma \Lambda_2) R(\Lambda_1)$, which holds.

The extended map $R$ is clearly an isomorphism of $D_n$ onto itself. Thus we have

$$R(\Lambda' \Lambda'') = R(\Lambda') R(\Lambda''), \quad \Lambda', \Lambda'' \in D_n.$$ (11)

Injectivity of $R$ and (11) imply that $R(\Delta_i) = \Delta_j$ for all $i$, where $j = j(i)$ depends on $i$; $j(i)$ is a permutation on $\{1, \ldots, n\}$ (we recall that $\Delta_i = \text{diag}(0, \ldots, 1, \ldots, 0)$).

This is because $\{\Delta_i\}_{i=1}^n$ is the only system in $D_n$ with the following relations: $\Delta_i \neq 0$, $\Delta_i^2 = \Delta_i$, $\Delta_i \Delta_j = 0$, $i \neq j$. 

Since all matrices $\Lambda$ and their images $R(\Lambda)$ are diagonal, we can consider the matrix equation \((11)\) as \(n\) scalar equations:
\[
(12) \quad r_j(\lambda_1^{m}, \ldots, \lambda_n^{m}) = r_j(\lambda_1^{n}, \ldots, \lambda_n^{n}), \quad j = 1, \ldots, n,
\]
where \(r_j\) are components of \(R\). If we rewrite the equation \(R(\Delta, \Lambda) = \Delta, R(\Lambda)\) in the coordinate form, we see that \(r_j(\lambda_1, \ldots, \lambda_n) = r_j(0, \ldots, \lambda_i, \ldots, 0) = q_j(\lambda_i);\) that is, each \(r_j\) depends on only one of the \(\lambda_i\)'s. For each \(j\) the corresponding equation in \((12)\) in terms of the \(q_j\)'s becomes
\[
q_j(\lambda_i^{n}) = q_j(\lambda_i^{m}) q_j(\lambda_i^{n}).
\]
This equation has \((2), \text{p. 130}\) either the constant solution \(q_j(\lambda_i) = 1\), or
\[
q_j(\lambda_i) = \lambda_i^{\alpha_i} r_i^{\beta_i}, \quad \alpha_i, \beta_i \in \mathbb{C}, \quad \alpha_i - \beta_i = \pm 1.
\]

Going back to the function \(L\), we have
\[
L(\lambda_1, \ldots, \lambda_n) = \text{diag}(\lambda_1^{\alpha_1}, \ldots, \lambda_n^{\alpha_n}),
\]
where \(i(j)\) is the inverse permutation to \(j(i)\).

Let us choose and fix \((\mu_1, \ldots, \mu_n)\) such that \((1/\mu_1, \ldots, 1/\mu_n)\) belongs to a neighborhood \(W_0\) of \(0 \in \mathbb{C}^n\) where \(L\) is defined, and let \(W_1\) be a neighborhood of \(0 \in \mathbb{C}^n\) such that \((\mu_1 z_1, \ldots, \mu_n z_n) \in W_0, \text{whenever } (z_1, \ldots, z_n) \in W_1\). Then from \((13)\) we have
\[
L(z_1, \ldots, z_n) = L(\mu_1 z_1, \ldots, \mu_n z_n)(1/\mu_1, \ldots, 1/\mu_n)
\]
\[
= \text{diag}((\mu_i(1) z_i(1))^{\alpha_i} (\mu_i(1) z_i(1))^{\beta_i}, \ldots, (\mu_i(n) z_i(n))^{\alpha_i} (\mu_i(n) z_i(n))^{\beta_i})
\]
\[
\times L(1/\mu_1, \ldots, 1/\mu_n) = B(z_1^{\alpha_1} z_1^{\beta_1}, \ldots, z_n^{\alpha_n} z_n^{\beta_n}),
\]
where \(B\) is a constant matrix. The last formula is the explicit expression for \(L\).

8. Proving that \(\psi\) is (anti-) biholomorphic

To prove that \(\psi\) is (anti-) biholomorphic is the same as to prove that \(L\) is (anti-) biholomorphic, because the relation \(L = \theta_{\psi} \circ \psi \circ \theta_{\psi}^{-1}\) holds. We showed that
\[
L(z_1, \ldots, z_n) = B(z_1^{\alpha_1} z_1^{\beta_1}, \ldots, z_n^{\alpha_n} z_n^{\beta_n}), \quad \alpha_i - \beta_i = \pm 1, \quad i = 1, \ldots, n,
\]
in a neighborhood \(W_1\) of \(0\). From the representation \((13)\) we see that \(L\) is \(\mathbb{R}\)-differentiable and non-degenerate in \(W_1 \setminus \bigcup_{k=1}^{n} \{ (z_1, \ldots, z_n) : z_k = 0 \}\). Since this is true for every point in the domain \(\Omega_1\), the map \(\psi\) is \(\mathbb{R}\)-differentiable and non-degenerate everywhere, with the possible exception of an analytic set. Let us remove this set from \(\Omega_1\), as well as its image under \(\psi\) from \(\Omega_2\). We call the domains obtained in this way \(\Omega'\), \(\Omega''\). Now the map \(\psi : \Omega' \to \Omega''\) is \(\mathbb{R}\)-differentiable and non-degenerate everywhere. It is clear that if we prove that \(\psi\) is (anti-) biholomorphic between \(\Omega'\), \(\Omega''\), then it is (anti-) biholomorphic between \(\Omega_1\), \(\Omega_2\) due to a standard continuation argument \((11)\). So we can think that \(\psi\) is \(\mathbb{R}\)-differentiable and non-degenerate in \(\Omega_1\) itself. The map \(L\) thus has to be \(\mathbb{R}\)-differentiable and non-degenerate at 0. However, this is the case if and only if \(\alpha_i + \beta_i = 1, \quad i = 1, \ldots, n\). Together with the equation \(\alpha_i - \beta_i = \pm 1\) it gives us that either \(\alpha_i = 1, \beta_i = 0\), or \(\alpha_i = 0, \beta_i = 1\).
It remains to show that either $\alpha_i = 1$ and $\beta_i = 0$, or $\alpha_i = 0$ and $\beta_i = 1$, simultaneously for all $i$. Suppose, by way of contradiction, that we have $L(z_1, \ldots, z_n) = B(\ldots, z_i, \ldots, \overline{z}_j, \ldots)$. Then

$$L^{-1}(w_1, \ldots, w_n) = (\ldots, l_i(w_1, \ldots, w_n), \ldots, l_j(\overline{w}_1, \ldots, \overline{w}_n), \ldots),$$

where $l_i, l_j$ are linear analytic functions. Let us look at an endomorphism $f_0$ of $\Omega_1$ in the form

$$f_0 = \theta_f^{-1} \lambda(\ldots, \theta_{f_i}, \theta_{f_j}, \ldots, \theta_{f_n}) \theta_f,$$

where $\theta_{f_i}, \theta_{f_j}$ is in the $i$'th place and $\theta_{f_n}$ in the $j$'th place; $|\lambda|$ is sufficiently small. Using (1) and the definition of $L$, we have

$$\theta_g \varphi f_0 \theta_g^{-1} = \theta_g \psi f_0 \psi^{-1} \theta_g^{-1} = L \theta_f f_0 \theta_f^{-1} L^{-1}.$$

So,

$$\theta_g \varphi f_0 \theta_g^{-1}(w_1, \ldots, w_n) = B'(\ldots, l_i(w_1, \ldots, w_n), l_j(\overline{w}_1, \ldots, \overline{w}_n), \ldots, l_j(w_1, \ldots, w_n), \ldots)$$

for some constant matrix $B'$. This map, and hence $\varphi f_0$, is not analytic, though, in a neighborhood of 0, which is a contradiction. Thus $L$, and hence $\psi$, is either analytic or antianalytic in a neighborhood of 0.

Theorem 1 is proved completely.

9. Proof of Theorem 2

Since $\varphi$ is an epimorphism, it takes constant endomorphisms of $\Omega_1$ to constant endomorphisms of $\Omega_2$, which follows from (2). Thus we can define a map $\psi : \Omega_1 \to \Omega_2$ as in (3). Following the same steps as in verifying (1), we obtain

$$\varphi f \circ \psi = \psi \circ f, \text{ for all } f \in E(\Omega_1).$$

We will show that (15) implies bijectivity of $\psi$. The map $\psi$ is surjective. Indeed, let $w \in \Omega_2$, and let $c_w$ be the corresponding constant endomorphism. Since $\varphi$ is an epimorphism, there exists $f \in E(\Omega_1)$ such that $\varphi f = c_w$. If we plug this $f$ into (15), we get

$$\psi f(z) = w$$

for all $z \in \Omega_1$. Thus $\psi$ is surjective.

To prove that $\psi$ is injective, we show that for every $w \in \Omega_2$, the full preimage of $w$ under $\psi$, $\psi^{-1}(w)$, consists of one point.

Assume for contradiction that $S_w = \psi^{-1}(w)$ consists of more than one point for some $w \in \Omega_2$. The set $S_w$ cannot be all of $\Omega_1$, since $\psi$ is surjective. For $z_0 \in \partial S_w \cap \Omega_1$ we can find $z_1 \in S_w$ and $\zeta \notin S_w$ which are arbitrarily close to $z_0$. Let $z_2$ be a fixed point of $S_w$ different from $z_1$. Consider a homothetic transformation $h$ such that $h(z_1) = z_2$, $h(z_2) = \zeta$. Since the domain $\Omega_1$ is bounded, we can choose points $z_1$ and $\zeta$ sufficiently close to each other so that $h$ belongs to $E(\Omega_1)$. Applying (15) to $h$, we obtain

$$\varphi h(w) = \varphi h \circ \psi(z_1) = \psi \circ h(z_1) = \psi(z_1) = w;$$

$$\varphi h(w) = \varphi h \circ \psi(z_2) = \psi \circ h(z_2) = \psi(\zeta) \neq w.$$ 

The contradiction shows injectivity of $\psi$. Thus we have proved that $\psi$ is bijective.
According to (15) we have
\[ \varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1), \]
which implies that \( \varphi \) is an isomorphism.

Theorem 2 is proved.

REFERENCES