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EQUIVALENCE OF DOMAINS WITH ISOMORPHIC SEMIGROUPS OF ENDOMORPHISMS

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ABSTRACT. For two bounded domains Ω_1, Ω_2 in $\mathbb{C}$ whose semigroups of analytic endomorphisms $E(\Omega_1)$, $E(\Omega_2)$ are isomorphic with an isomorphism $\varphi: E(\Omega_1) \to E(\Omega_2)$, Eremenko proved in 1993 that there exists a conformal or anticonformal map $\psi: \Omega_1 \to \Omega_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(\Omega_1)$.

In the present paper we prove an analogue of this result for the case of bounded domains in $\mathbb{C}^n$.

1. Introduction

A classical theorem of L. Bers says that every $\mathbb{C}$-algebra isomorphism $H(A) \to H(B)$ of algebras of holomorphic functions in domains A and B in the complex plane has the form $f \mapsto f \circ \theta$, where $\theta: B \to A$ is a conformal isomorphism, or $f \mapsto \overline{f} \circ \theta$ with anticonformal θ. In particular, the algebras $H(A)$ and $H(B)$ are isomorphic if and only if the domains A and B are conformally equivalent. H. Iss'sa [9] obtained a similar theorem for fields of meromorphic functions on Stein spaces. A good reference for these results is [5].

Likewise, the question of recovering a topological space from the algebraic structure of its semigroup of continuous self-maps has been extensively studied [12].

In 1990, L. Rubel asked whether similar results hold for semigroups (under composition) $E(D)$ of holomorphic endomorphisms of a domain D. A. Hinkkanen constructed examples [6] which show that even non-homeomorphic domains in $\mathbb{C}$ can have isomorphic semigroups of endomorphisms. The reason is that the semigroup of endomorphisms of a domain can be too small to characterize this domain.

However, in 1993, A. Eremenko [4] proved that for two Riemann surfaces D_1, D_2, which admit bounded nonconstant holomorphic functions, and such that the semigroups of analytic endomorphisms $E(D_1)$ and $E(D_2)$ are isomorphic with an isomorphism $\varphi: E(D_1) \to E(D_2)$, there exists a conformal or anticonformal map $\psi: D_1 \to D_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(D_1)$. In the present paper we investigate the analogue of this result for the case of bounded domains in $\mathbb{C}^n$. The theorems of Bers and Iss'sa, mentioned above, do not extend to arbitrary domains in $\mathbb{C}^n$.

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For a bounded domain Ω in $\mathbb{C}^n$ we denote by $E(\Omega)$ the semigroup of analytic endomorphisms of Ω under composition. In what follows, we say that a map is *(anti-) biholomorphic*, if it is biholomorphic or antibiholomorphic. We prove the following theorem.

Theorem 1. *Let $\Omega_1,\ \Omega_2$ be bounded domains in $\mathbb{C}^n,\ \mathbb{C}^m$ respectively, and suppose that there exists $\varphi:\ E(\Omega_1) \to E(\Omega_2)$, an isomorphism of semigroups. Then $n = m$ and there exists an (anti-) biholomorphic map $\psi:\ \Omega_1 \to \Omega_2$ such that*

$$\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1). \tag{1}$$

The existence of a homeomorphism ψ satisfying (1) follows from simple general considerations (Section 2). The hard part is proving that ψ is (anti-) biholomorphic. In dimension 1 this is done by linearization of holomorphic germs of $f \in E(\Omega)$ near an attracting fixed point. In several dimensions such linearization theory exists ([1], pp. 192–194), but it is too complicated (many germs with an attracting fixed point are non-linearizable, even formally). In Sections 3, 4 we show how to localize the problem. In Sections 5, 6 we describe, using only the semigroup structure, a large enough class of linearizable germs. Linearization of these germs permits us to reduce the problem to a matrix functional equation, which is solved in Section 7. In Section 8 we complete the proof that ψ is (anti-) biholomorphic.

Theorem 1 can be slightly generalized, namely one may assume that φ is an epimorphism. In Section 9 we prove the following theorem.

Theorem 2. *If $\varphi:\ E(\Omega_1) \to E(\Omega_2)$ is an epimorphism between semigroups, where $\Omega_1,\ \Omega_2$ are bounded domains in $\mathbb{C}^n,\ \mathbb{C}^m$ respectively, then φ is an isomorphism.*

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2. Topology

For a bounded domain Ω in $\mathbb{C}^n$ we denote by $C(\Omega)$ the subsemigroup of $E(\Omega)$ consisting of constant maps. An endomorphism c_z is constant if it sends Ω to a point $z \in \Omega$. The subset $C(\Omega) \subset E(\Omega)$ can be described using only the semigroup structure as follows:

$$c \in C(\Omega) \text{ iff } \forall (f \in E(\Omega)), \quad (c \circ f = c). \tag{2}$$

It is clear that we have a bijection between constant endomorphisms of Ω and points of this domain as a set: to each z corresponds a unique $c_z \in C(\Omega)$ and vice versa, so we can identify the two. Under this identification, a subset of Ω corresponds to a subsemigroup of $C(\Omega)$.

Having defined points of a domain in terms of its semigroup structure of analytic endomorphisms, we can construct a map ψ between Ω_1 and Ω_2 as follows:

$$\psi(z) = w \text{ iff } \varphi c_z = c_w. \tag{3}$$

So defined, ψ satisfies (1). Indeed, let $f \in E(\Omega_1)$, $f(z) = \zeta$. This is equivalent to

$$f \circ c_z = c_\zeta. \tag{4}$$

Applying φ to both sides of (4), we have

$$\varphi f \circ c_{\psi(z)} = c_{\psi(\zeta)}. \tag{5}$$

But (5) is equivalent to $\varphi f(\psi(z)) = \psi(\zeta) = \psi(f(z))$, which is (1).

We describe the topology of a domain Ω using its injective endomorphisms. A map $f \in E(\Omega)$ is injective if and only if

$$\forall (c' \in C(\Omega)) \, \forall (c'' \in C(\Omega)), \quad ((f \circ c' = f \circ c'') \Rightarrow (c' = c'')).$$

We denote the class of injective endomorphisms of Ω by $E_i(\Omega)$. For every $f \in E_i(\Omega)$, $f_i(\Omega)$ is open [2]. The family $\{f(\Omega), \; f \in E_i(\Omega)\}$ of subsets of Ω forms a base of topology, because every $z \in \Omega$ has a neighborhood $f(\Omega)$, where $f(\zeta) = z + \lambda(\zeta - z)$, f belongs to $E_i(\Omega)$ for every λ such that $|\lambda|$ is small.

To summarize, we described subsets of Ω and the topology on it using only the semigroup structure of $E(\Omega)$. Since this is so, the semigroup structure also defines the notions of an open set, closed set, compact set, closure of a set.

Now we can easily prove continuity of the map ψ constructed above. Indeed, let $g(\Omega_2)$, $g \in E_i(\Omega_2)$, be a set from the base of topology of Ω_2. We take $f = \varphi^{-1} g$. Then $f \in E_i(\Omega_1)$ and $\psi^{-1}(g(\Omega_2)) = f(\Omega_1)$, which proves that ψ is continuous. Since φ is an isomorphism, the same argument works to prove that ψ^{-1} is also continuous, and thus ψ is a homeomorphism.

Therefore the domains Ω_1, Ω_2 are homeomorphic, and hence [8] they have the same dimension, i. e. $n = m$.

3. Localization

We need the following lemma.

Lemma 1. *Suppose H is a semigroup with identity, and f an element of H with the following two properties:*

(i) $hf = fh$, for every h in H;

(ii) $h_1 f = h_2 f$ implies $h_1 = h_2$, for every h_1 and h_2 in H.

Then there exist a semigroup S_f and a monomorphism $i : \; H \to S_f$ such that $i(f)$ is invertible in S_f and commutes with all elements of S_f. Moreover, the semigroup S_f satisfies the following universal property: for every semigroup S_1 with a monomorphism $i_1 : \; H \to S_1$ such that $i_1(f)$ is invertible in S_1 and commutes with all elements of S_1, there exists a unique monomorphism $\hat{i}_1 : \; S_f \to S_1$ such that $i_1 = \hat{i}_1 \circ i$.

Remark 1. Uniqueness of $\hat{i}_1$ implies that the semigroup S_f with the universal property is unique up to an isomorphism.

Proof. We construct S_f as follows. First we consider formal expressions of the form hf^k, where $h \in H$ and k is an integer (it may be positive, negative or zero). Then we define a multiplication on this set: $h_1 f^{k_1} * h_2 f^{k_2} = h_1 h_2 f^{k_1 + k_2}$. Next we consider a relation on the set of formal expressions: $h_1 f^{k_1} \sim h_2 f^{k_2}$ if $k_1 \leq k_2$ and $h_1 = h_2 f^{k_2 - k_1}$ in H, or $k_2 \leq k_1$ and $h_2 = h_1 f^{k_1 - k_2}$ in H. It is easy to verify that this is an equivalence relation and it is compatible with the operation $*$; that is, $x \sim y, \; u \sim v$ implies $x * u \sim y * v$.

Lastly, let S_f be the set of equivalence classes with the binary operation induced by $*$. For S_f to be a semigroup, we need to show that the binary operation $*$ is associative. Let $h_1 f^{k_1} \sim h_1' f^{k_1'}$, $h_2 f^{k_2} \sim h_2' f^{k_2'}$ and $h_3 f^{k_3} \sim h_3' f^{k_3'}$. We need to show that $(h_1 f^{k_1} * h_2 f^{k_2}) * h_3 f^{k_3} \sim h_1' f^{k_1'} * (h_2' f^{k_2'} * h_3' f^{k_3'})$. By the definition of the operation $*$, the last equivalence is the same as $h_1 h_2 h_3 f^{k_1 + k_2 + k_3} \sim h_1' h_2' h_3' f^{k_1' + k_2' + k_3'}$. Assuming that $k_1 + k_2 + k_3 \leq k_1' + k_2' + k_3'$, we have essentially one possibility to consider (the others are either similar or trivial): $k_1 \le k_1'$, $k_2 \le k_2'$, $k_3' \le k_3$. In this case $h_1h_2h_3f^{k_3-k_3'} = h_1'h_2'h_3'f^{k_1'-k_1+k_2'-k_2}$. Now we can use the cancellation property (ii) to get the desired equivalence.

The semigroup H is embedded into S_f via $i: \ h \mapsto [hf^0]$. The element $i(f) = [\mathrm{id}f]$, where id is the identity in H, is invertible in S_f with the inverse $[\mathrm{id}f^{-1}]$. Clearly, $[\mathrm{id}f]$ commutes with all elements of S_f.

Now, suppose that S_1, $i_1: \ H \to S_1$ is a semigroup and a monomorphism, such that $i_1(f)$ is invertible in S_1 and commutes with all elements of S_1. Then we define

$$\hat{i}_1([hf^k]) = i_1(h)(i_1(f))^k.$$

This definition does not depend on a representative of $[hf^k]$. Indeed, suppose $h_1f^{k_1} \sim h_2f^{k_2}$ and assume $k_1 \le k_2$. Then $h_1 = h_2f^{k_2-k_1}$, and thus $i_1(h_1) = i_1(h_2)i_1(f)^{k_2-k_1}$. Hence $i_1(h_1)i_1(f)^{k_1} = i_1(h_2)i_1(f)^{k_2}$.

So defined, $\hat{i}_1$ is a homomorphism:

$$\begin{aligned}\hat{i}_1([h_1f^{k_1}][h_2f^{k_2}]) &= \hat{i}_1([h_1h_2f^{k_1+k_2}])\\ &= i_1(h_1h_2)i_1(f)^{k_1+k_2} = i_1(h_1)i_1(h_2)i_1(f)^{k_1}i_1(f)^{k_2}\\ &= i_1(h_1)i_1(f)^{k_1}i_1(h_2)i_1(f)^{k_2} = \hat{i}_1([h_1f^{k_1}])\hat{i}_1([h_2f^{k_2}]).\end{aligned}$$

The relation $\hat{i}_1 \circ i = i_1$ holds, since $\hat{i}_1([hf^0]) = i_1(h)$ for all $h \in H$.
Uniqueness of $\hat{i}_1$ is clear. Lemma 1 is proved. □

4. Extension of φ

Following [4], we say that for a bounded domain Ω an element $f \in E(\Omega)$ is *good at* $z \in \Omega$, denoted by $f \in G_z(\Omega)$, if

1. z is a unique fixed point of f;
2. $f(\Omega)$ has compact closure in Ω;
3. f is injective in Ω.

Property 3 of a good element was already stated in terms of the semigroup structure of Ω. Since the topology on Ω was described using only the semigroup structure, Property 2 can also be stated in these terms. Property 1 can be expressed in terms of the semigroup structure as

$$(f \circ c_z = c_z) \wedge ((f \circ c_\zeta = c_\zeta) \Rightarrow (c_\zeta = c_z)).$$

Since f is an endomorphism of a domain, all eigenvalues λ of its linear part at z satisfy $|\lambda| \le 1$ [10]. Moreover, $|\lambda| < 1$ because the closure of $f(\Omega)$ is a compact set in Ω. The injectivity of f implies [2] that it is biholomorphic onto $f(\Omega)$ and the Jacobian determinant of f does not vanish at any point of Ω.

It is clear that for every $z \in \Omega$ a good element f at z exists. For example, we can take $f(\zeta) = z + \lambda(\zeta - z)$ with sufficiently small $|\lambda|$.

Consider a good element $f \in G_z(\Omega)$ and its commutant $H_f(\Omega)$ in $E(\Omega)$:

$$H_f(\Omega) = \{h \in E(\Omega): \quad hf = fh\}.$$

Clearly $H_f(\Omega)$ is a subsemigroup of $E(\Omega)$. The element f, being good (hence injective), satisfies the cancellation property (*ii*) of Lemma 1 in $H_f(\Omega)$. Thus, by Lemma 1, we have the extension S_f of $H_f(\Omega)$ in which f is invertible and commutes with all elements of S_f. In the case of analytic endomorphisms we can embed $H_f(\Omega)$ into the subsemigroup of A_z, the semigroup of germs of analytic mappings at z under composition, consisting of elements that commute with the germ of f

and containing the germ of f^{-1}. We use the universal property of Lemma 1 to conclude that S_f is isomorphic to a subsemigroup of A_z. We identify S_f with this semigroup, i. e. we consider elements of S_f as germs of analytic mappings at z.

In proving that ψ is (anti-) biholomorphic we need to show that it is so in a neighborhood of every point of Ω_1. Since an (anti-) biholomorphic type of a domain is preserved by translations in $\mathbb{C}^n$, it is enough to show that ψ is (anti-) biholomorphic in a neighborhood of $0 \in \mathbb{C}^n$, assuming that Ω_1 and Ω_2 contain 0 and $\psi(0) = 0$.

Let $\varphi : E(\Omega_1) \to E(\Omega_2)$ be an isomorphism of the semigroups, f a good element, $f \in G_0(\Omega_1)$, and $H_f(\Omega_1)$ the commutant of f. Then clearly $H_g(\Omega_2) = \varphi(H_f(\Omega_1))$ is the commutant of $g = \varphi f$. By Lemma 1, we have the extensions S_f, S_g of $H_f(\Omega_1)$ and $H_g(\Omega_2)$ respectively, and by the universal property of this lemma the isomorphism φ extends to an isomorphism

$$\Phi : \quad S_f \to S_g.$$

5. System of projections and linearization

Let Ω be a bounded domain in $\mathbb{C}^n$. We say that a good element $f \in G_0(\Omega)$ is *very good at* 0, and write $f \in VG_0(\Omega)$, if the corresponding semigroup $S_f \subset A_0$ constructed in Section 4 contains a system of elements, which we call a system of projections, $\{p_i\}_{i=1}^n$ with the following properties:

(a) $\forall\ (i = 1, \dots, n), \quad (p_i \neq 0)$;

(b) $\forall\ (i = 1, \dots, n), \quad (p_i^2 = p_i)$;

(c) $\forall\ (i,\ j = 1, \dots, n,\ i \neq j), \quad (p_i p_j = 0)$.

There does exist a very good element, since we can take f to be a homothetic transformation at 0 with sufficiently small coefficient, p_i a projection on the i'th coordinate of the standard coordinate system. Clearly, $p_i f = f p_i$ and there exists k such that $p_i f^k \in E(\Omega)$, and hence $p_i \in S_f$. From now on, we fix a very good element $f \in VG_0(\Omega)$, associated semigroups $H_f(\Omega)$, S_f and a system of projections $\{p_i\}$.

We introduce another subsemigroup of $E(\Omega)$:

$$P_f(\Omega) = \{h \in G_0(\Omega) \cap H_f(\Omega), \quad hp_i = p_i h, \quad i = 1, \dots, n\},$$

where the commutativity relations are in $S_f \subset A_0$. Notice that $P_f(\Omega) \neq \emptyset$ since f belongs to it.

Lemma 2. *For every $h \in P_f(\Omega)$ there exists a biholomorphic germ θ_h at $0 \in \mathbb{C}^n$ such that $\theta_h h = \Lambda \theta_h$, where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is an invertible diagonal matrix which is similar to $dh(0)$ in $GL(n, \mathbb{C})$.*

Proof. The relations $p_i \neq 0,\ p_i^2 = p_i,\ p_i p_j = 0,\ i \neq j$, imply that for $P_i = dp_i(0)$, the linear part of p_i at 0, we have $P_i \neq 0,\ P_i^2 = P_i,\ P_i P_j = 0,\ i \neq j$. Since the matrices P_i commute, there exists [7] a matrix $A \in GL(n, \mathbb{C})$ such that $P_i' = AP_iA^{-1} = \Delta_i = \operatorname{diag}(0, \dots, 1, \dots, 0)$, where the only non-zero entry appears in the i'th place.

Since $p_i^2 = p_i,\ i = 1, \dots, n$, we can use the argument given in [10] to linearize p_i, i.e. there exists a biholomorphic germ ξ_i at 0 such that $\xi_i p_i = P_i \xi_i,\ d\xi_i(0) = \mathrm{id},\ i = 1, \dots, n$. The map ξ_i is constructed in [10] as follows:

$$\xi_i = \mathrm{id} + (2P_i - \mathrm{id})(p_i - P_i), \quad i = 1, \dots, n.$$

If we take $\xi_i' = A\xi_i$, we have $\xi_i' p_i = P_i' \xi_i'$. For simplicity of notations, we assume that ξ_i itself conjugates p_i to a diagonal matrix, that is, $P_i = P_i'$ (in this case P_i is not necessarily $dp_i(0)$, but rather $Adp_i(0)A^{-1}$; $d\xi_i(0) = A$). For every $i = 1, \dots, n$ we have $h_i P_i = P_i h_i$, where $h_i = \xi_i h \xi_i^{-1}$. Let $H_i = dh_i(0)$. Then $H_i P_i = P_i H_i$, and hence in the i'th row and the i'th column the matrix H_i has only one non-zero entry, λ_i, which is located at their intersection. Thus λ_i has to be an eigenvalue of H_i, and hence of the linear part of h. In particular, $0 < |\lambda_i| < 1$.

Let $I_i : \mathbb{C} \to \mathbb{C}^n$ be the embedding $z \mapsto (0, \dots, z, \dots, 0)$, where the only non-zero entry is z, which is in the i'th place; and $\pi_i : \mathbb{C}^n \to \mathbb{C}$, a projection $(z_1, \dots, z_n) \mapsto z_i$, corresponding to the i'th axis. For every $i = 1, \dots, n$, the map $\pi_i h_i I_i$ sends a neighborhood of 0 in $\mathbb{C}$ into $\mathbb{C}$, and its derivative at 0, λ_i, is an eigenvalue of h. Hence ([3], p. 31) $\pi_i h_i I_i$ is linearized by the unique solution $\eta_{h,i}$ of the Schröder equation

$$\eta(\pi_i h_i I_i) = \lambda_i \eta, \quad \eta(0) = 0, \ \eta'(0) = 1. \tag{6}$$

Since $P_i I_i = I_i$, $\pi_i P_i I_i = \mathrm{id}_{\mathbb{C}}$, we can rewrite (6) as

$$\eta_{h,i}\pi_i h_i P_i I_i = \lambda_i \eta_{h,i}\pi_i P_i I_i, \quad \text{or} \quad \eta_{h,i}\pi_i h_i P_i = \lambda_i \eta_{h,i}\pi_i P_i.$$

But $h_i P_i = P_i h_i$, and so

$$\eta_{h,i}\pi_i P_i h_i = \lambda_i \eta_{h,i}\pi_i P_i. \tag{7}$$

The equation (7), in its turn, is equivalent to

$$\eta_{h,i}\pi_i \xi_i p_i h = \lambda_i \eta_{h,i}\pi_i \xi_i p_i. \tag{8}$$

We denote

$$\theta_{h,i} = \eta_{h,i}\pi_i \xi_i p_i, \tag{9}$$

a map from a neighborhood of $0 \in \mathbb{C}^n$ into $\mathbb{C}$. Then (8) becomes $\theta_{h,i} h = \lambda_i \theta_{h,i}$. Now we define

$$\theta_h = (\theta_{h,1}, \dots, \theta_{h,n}),$$

which is a germ of an analytic map at 0. This germ linearizes h:

$$\theta_h h = (\theta_{h,1} h, \dots, \theta_{h,n} h) = (\lambda_1 \theta_{h,1}, \dots, \lambda_n \theta_{h,n}) = \Lambda \theta_h,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is an invertible diagonal matrix, which has eigenvalues of $dh(0)$ on its diagonal.

The germ θ_h is biholomorphic. Indeed,

$$\theta_{h,i} = \eta_{h,i}\pi_i \xi_i p_i = \eta_{h,i}\pi_i P_i \xi_i, \quad i = 1, \dots, n.$$

Using the chain rule, we see that $d\theta_h(0) = A$, where A is an invertible diagonal matrix that diagonalizes P_i. We conclude that θ_h is biholomorphic. Lemma 2 is proved. □

6. Simultaneous linearization

Using Lemma 2, we can linearize elements of $P_f(\Omega)$. Namely, for every $h \in P_f(\Omega)$ there exists θ_h (constructed in Section 5) such that $\theta_h h = \Lambda_h \theta_h$, where Λ_h is an invertible diagonal matrix. In particular, we can linearize f:

$$\theta_f f = \Lambda_f \theta_f,$$

where the germ θ_f is biholomorphic at 0, and Λ_f is an invertible diagonal matrix.

Lemma 3. *For every $h \in P_f(\Omega)$ we have $\theta_h = \theta_f$.*

Proof. Let us consider the germ

$$\theta = \Lambda_f^{-1}\theta_h f, \tag{10}$$

which is clearly biholomorphic. We have

$$\theta h = \Lambda_f^{-1}\theta_h f h = \Lambda_f^{-1}\theta_h h f = \Lambda_f^{-1}\Lambda_h\theta_h f = \Lambda_h\Lambda_f^{-1}\theta_h f = \Lambda_h\theta.$$

Using (10), we write the equation $\theta h = \Lambda_h\theta$ in the coordinate form:

$$(1/\lambda_{f,i})\theta_{h,i} f h = (\lambda_{h,i}/\lambda_{f,i})\theta_{h,i} f, \quad i = 1, \ldots, n.$$

By (9) and the definition of ξ_i,

$$(1/\lambda_{f,i})\eta_{h,i}\pi_i P_i f_i h_i = (\lambda_{h,i}/\lambda_{f,i})\eta_{h,i}\pi_i P_i f_i, \quad i = 1, \ldots, n,$$

where $f_i = \xi_i f \xi_i^{-1}$. Using the commutativity relations $f_i P_i = P_i f_i,\ h_i P_i = P_i h_i$, which hold since $\{p_i\} \subset S_f,\ h \in P_f(\Omega)$, we get

$$(1/\lambda_{f,i})\eta_{h,i}\pi_i f_i h_i P_i = (\lambda_{h,i}/\lambda_{f,i})\eta_{h,i}\pi_i f_i P_i, \quad \text{or}$$
$$(1/\lambda_{f,i})\eta_{h,i}\pi_i f_i h_i I_i = (\lambda_{h,i}/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i, \quad i = 1, \ldots, n.$$

This is the same as

$$((1/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i)(\pi_i h_i I_i) = \lambda_{h,i}((1/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i), \quad i = 1, \ldots, n,$$

since h_i locally preserves the i'th coordinate axis ($h_i P_i = P_i h_i$). It is easily seen that

$$((1/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i)(0) = 0,$$
$$((1/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i)'(0) = 1.$$

A normalized solution to a Schröder equation is unique, though; thus we have

$$\eta_{h,i}(\pi_i f_i I_i) = \lambda_{f,i}\eta_{h,i}, \quad \eta_{h,i}(0) = 0,\ \eta'_{h,i}(0) = 1.$$

Using the uniqueness argument again, we obtain $\eta_{h,i} = \eta_{f,i}$, and hence $\theta_h = \theta_f$. The lemma is proved. □

According to Lemma 3, the single biholomorphic germ θ_f conjugates the subsemigroup $P_f(\Omega)$ to some subsemigroup D_f of invertible diagonal matrices in D_n, the set of all $n \times n$ diagonal matrices with entries in $\mathbb{C}$. We show that D_f contains all invertible diagonal matrices with sufficiently small entries. To do this, first we extend θ_f to an analytic map on the whole domain Ω using the formula

$$\theta_f = \Lambda_f^{-l}\theta_f f^l,$$

where l is chosen so large that $\mathrm{Cl}\{f^l(\Omega)\}$ is contained in a neighborhood of 0 where θ_f is originally defined and biholomorphic; the symbol Cl denotes closure. From the procedure of extending θ_f to Ω we see that it is one-to-one and bounded in the whole domain.

Now, let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ be a matrix such that $\mathrm{Cl}\{\Lambda\theta_f(\Omega)\} \subset W$, where W is a neighborhood of $0 \in \mathbb{C}^n$ for which $\mathrm{Cl}\{\theta_f^{-1}W\} \subset \Omega$. Such a matrix Λ exists since θ_f is bounded in Ω. Consider $h = \theta_f^{-1}\Lambda\theta_f$, which belongs to $G_0(\Omega)$. The map h commutes with f and all p_i's. Indeed, using the formula $\theta_f f \theta_f^{-1} = \Lambda_f$, we conclude that $hf = fh$ is equivalent to $\Lambda\Lambda_f = \Lambda_f\Lambda$, which is a true relation since both matrices Λ and Λ_f are diagonal. The relations $hp_i = p_i h,\ i = 1, \ldots, n$, are verified similarly, using the formula $\theta_f p_i \theta_f^{-1} = P_i$, which follows from the definition of θ_f.

7. Solving a matrix equation

We proved that for an element $f \in VG_0(\Omega)$ there exists a biholomorphic germ θ_f conjugating the semigroup $P_f(\Omega)$ to a subsemigroup $D_f \subset D_n$, which contains all invertible diagonal matrices with sufficiently small entries.

Let $f \in VG_0(\Omega_1)$ and $g = \varphi f$. Then $g \in VG_0(\Omega_2)$, and there is an isomorphism

$$\Phi: \quad S_f \to S_g.$$

For the mappings f and g we have

$$\theta_f f = \Lambda_f \theta_f, \quad \theta_g g = M_g \theta_g,$$

where $\Lambda_f,\ M_g$ are invertible diagonal matrices.

Let us consider the germ $L = \theta_g \psi \theta_f^{-1}$. This germ conjugates the semigroups $D_f,\ D_g$:

$$\begin{aligned} L\Lambda L^{-1} &= \theta_g \psi \theta_f^{-1} \Lambda \theta_f \psi^{-1} \theta_g^{-1} \\ &= \theta_g \psi h \psi^{-1} \theta_g^{-1} = \theta_g j \theta_g^{-1} = M, \end{aligned}$$

where $h \in P_f,\ \theta_f h = \Lambda \theta_f;\ j = \varphi h,\ \theta_g j = M\theta_g$.

Define $R(\Lambda) = L\Lambda L^{-1}$. Then $R: \ D_f \to D_g$,

$$R(\Lambda_1 \Lambda_2) = R(\Lambda_1) R(\Lambda_2), \quad \Lambda_1,\ \Lambda_2 \in D_f.$$

In what follows, we will identify D_n with the multiplicative semigroup $\mathbb{C}^n$ ($D_n \cong \mathbb{C}^n$) in the obvious way and consider a topology on D_n induced by the standard topology on $\mathbb{C}^n$.

We are going to extend R to an isomorphism of D_n. First, we denote by $\overline{D}_f,\ \overline{D}_g$ the closures of $D_f,\ D_g$ in D_n, and for $\Lambda \in \overline{D}_f$ we set

$$R(\Lambda) = \lim R(\Lambda_k), \quad \Lambda_k \to \Lambda,\ \Lambda_k \in D_f.$$

This limit exists and does not depend on the sequence $\{\Lambda_k\}$, which follows from the fact that $\psi^{\pm 1}, \theta_f^{\pm 1}, \theta_g^{\pm 1}$ are continuous. The map R is an isomorphism of topological semigroups $\overline{D}_f$ and $\overline{D}_g$ (the inverse of R has a similar representation).

Next, we extend the map R to D_n as

$$R(\Gamma) = R(\Gamma\Lambda) R(\Lambda)^{-1}, \quad \Gamma \in D_n,$$

where $\Lambda \in D_f$ is chosen so that $\Gamma\Lambda \in \overline{D}_f$. This definition does not depend on the choice of Λ. Indeed, since all matrices in question are diagonal (hence commute), the relation $R(\Gamma\Lambda_1)R(\Lambda_1)^{-1} = R(\Gamma\Lambda_2)R(\Lambda_2)^{-1}$ is equivalent to $R(\Gamma\Lambda_1)R(\Lambda_2) = R(\Gamma\Lambda_2)R(\Lambda_1)$, which holds.

The extended map R is clearly an isomorphism of D_n onto itself. Thus we have

$$R(\Lambda'\Lambda'') = R(\Lambda')R(\Lambda''), \quad \Lambda',\ \Lambda'' \in D_n. \tag{11}$$

Injectivity of R and (11) imply that $R(\Delta_i) = \Delta_j$ for all i, where $j = j(i)$ depends on i; $j(i)$ is a permutation on $\{1, \dots, n\}$ (we recall that $\Delta_i = \operatorname{diag}(0, \dots, 1, \dots, 0)$). This is because $\{\Delta_i\}_{i=1}^n$ is the only system in D_n with the following relations: $\Delta_i \neq 0,\ \Delta_i^2 = \Delta_i,\ \Delta_i \Delta_j = 0,\ i \neq j$.

Since all matrices Λ and their images $R(\Lambda)$ are diagonal, we can consider the matrix equation (11) as n scalar equations:

$$r_j(\lambda_1'\lambda_1'', \dots, \lambda_n'\lambda_n'') = r_j(\lambda_1', \dots, \lambda_n')r_j(\lambda_1'', \dots, \lambda_n''), \quad j = 1, \dots, n, \tag{12}$$

where r_j are components of R. If we rewrite the equation $R(\Delta_i\Lambda) = \Delta_j R(\Lambda)$ in the coordinate form, we see that $r_j(\lambda_1, \dots, \lambda_n) = r_j(0, \dots, \lambda_i, \dots, 0) = q_j(\lambda_i)$; that is, each r_j depends on only one of the λ_i's. For each j the corresponding equation in (12) in terms of the q_j's becomes

$$q_j(\lambda_i'\lambda_i'') = q_j(\lambda_i')q_j(\lambda_i'').$$

This equation has ([4], p. 130) either the constant solution $q_j(\lambda_i) = 1$, or

$$q_j(\lambda_i) = \lambda_i^{\alpha_{ij}}\overline{\lambda}_i^{\beta_{ij}}, \quad \alpha_{ij},\ \beta_{ij} \in \mathbb{C}, \quad \alpha_{ij} - \beta_{ij} = \pm 1.$$

Going back to the function L, we have

$$\begin{aligned} L\mathrm{diag}(\lambda_1, \dots, \lambda_n) = \mathrm{diag}(\lambda_{i(1)}^{\alpha_1}\overline{\lambda}_{i(1)}^{\beta_1}, \dots, \lambda_{i(n)}^{\alpha_n}\overline{\lambda}_{i(n)}^{\beta_n})L, \\ \alpha_i - \beta_i = \pm 1, \quad i = 1, \dots, n, \end{aligned} \tag{13}$$

where $i(j)$ is the inverse permutation to $j(i)$.

Let us choose and fix $(\mu_1, \dots, \mu_n)$ such that $(1/\mu_1, \dots, 1/\mu_n)$ belongs to a neighborhood W_0 of $0 \in \mathbb{C}^n$ where L is defined, and let W_1 be a neighborhood of $0 \in \mathbb{C}^n$ such that $(\mu_1 z_1, \dots, \mu_n z_n) \in W_0$, whenever $(z_1, \dots, z_n) \in W_1$. Then from (13) we have

$$\begin{aligned} L(z_1, \dots, z_n) &= L\mathrm{diag}(\mu_1 z_1, \dots, \mu_n z_n)(1/\mu_1, \dots, 1/\mu_n) \\ &= \mathrm{diag}((\mu_{i(1)}z_{i(1)})^{\alpha_1}(\overline{\mu_{i(1)}z_{i(1)}})^{\beta_1}, \dots, (\mu_{i(n)}z_{i(n)})^{\alpha_n}(\overline{\mu_{i(n)}z_{i(n)}})^{\beta_n}) \\ &\times L(1/\mu_1, \dots, 1/\mu_n) = B(z_1^{\alpha_1}\overline{z}_1^{\beta_1}, \dots, z_n^{\alpha_n}\overline{z}_n^{\beta_n}), \end{aligned}$$

where B is a constant matrix. The last formula is the explicit expression for L.

8. Proving that ψ is (anti-) biholomorphic

To prove that ψ is (anti-) biholomorphic is the same as to prove that L is (anti-) biholomorphic, because the relation $L = \theta_g \circ \psi \circ \theta_f^{-1}$ holds. We showed that

$$L(z_1, \dots, z_n) = B(z_1^{\alpha_1}\overline{z}_1^{\beta_1}, \dots, z_n^{\alpha_n}\overline{z}_n^{\beta_n}), \quad \alpha_i - \beta_i = \pm 1, \quad i = 1, \dots, n, \tag{14}$$

in a neighborhood W_1 of 0. From the representation (14) we see that L is $\mathbb{R}$-differentiable and non-degenerate in $W_1 \setminus \bigcup_{k=1}^n \{(z_1, \dots, z_n) : \ z_k = 0\}$. Since this is true for every point in the domain Ω_1, the map ψ is $\mathbb{R}$-differentiable and non-degenerate everywhere, with the possible exception of an analytic set. Let us remove this set from Ω_1, as well as its image under ψ from Ω_2. We call the domains obtained in this way Ω', Ω''. Now the map $\psi :\ \Omega' \to \Omega''$ is $\mathbb{R}$-differentiable and non-degenerate everywhere. It is clear that if we prove that ψ is (anti-) biholomorphic between Ω', Ω'', then it is (anti-) biholomorphic between Ω_1, Ω_2 due to a standard continuation argument [11]. So we can think that ψ is $\mathbb{R}$-differentiable and non-degenerate in Ω_1 itself. The map L thus has to be $\mathbb{R}$-differentiable and non-degenerate at 0. However, this is the case if and only if $\alpha_i + \beta_i = 1,\ i = 1, \dots, n$. Together with the equation $\alpha_i - \beta_i = \pm 1$ it gives us that either $\alpha_i = 1,\ \beta_i = 0$, or $\alpha_i = 0,\ \beta_i = 1$.

It remains to show that either $\alpha_i = 1$ and $\beta_i = 0$, or $\alpha_i = 0$ and $\beta_i = 1$, simultaneously for all i. Suppose, by way of contradiction, that we have $L(z_1, \dots, z_n) = B(\dots, z_i, \dots, \overline{z}_j, \dots)$. Then

$$L^{-1}(w_1, \dots, w_n) = (\dots, l_i(w_1, \dots, w_n), \dots, l_j(\overline{w}_1, \dots, \overline{w}_n), \dots),$$

where $l_i,\ l_j$ are linear analytic functions. Let us look at an endomorphism f_0 of Ω_1 in the form

$$f_0 = \theta_f^{-1}\lambda(\dots, \theta_{f,i}\theta_{f,j}, \dots, \theta_{f,j}, \dots)\theta_f,$$

where $\theta_{f,i}\theta_{f,j}$ is in the i'th place and $\theta_{f,j}$ in the j'th; $|\lambda|$ is sufficiently small. Using (1) and the definition of L, we have

$$\theta_g\varphi f_0\theta_g^{-1} = \theta_g\psi f_0\psi^{-1}\theta_g^{-1} = L\theta_f f_0\theta_f^{-1}L^{-1}.$$

So,

$$\theta_g\varphi f_0\theta_g^{-1}(w_1, \dots, w_n) \\ = B'(\dots, l_i(w_1, \dots, w_n)l_j(\overline{w}_1, \dots, \overline{w}_n), \dots, \bar{l}_j(w_1, \dots, w_n), \dots)$$

for some constant matrix B'. This map, and hence φf_0, is not analytic, though, in a neighborhood of 0, which is a contradiction. Thus L, and hence ψ, is either analytic or antianalytic in a neighborhood of 0.

Theorem 1 is proved completely.

9. Proof of Theorem 2

Since φ is an epimorphism, it takes constant endomorphisms of Ω_1 to constant endomorphisms of Ω_2, which follows from (2). Thus we can define a map $\psi:\ \Omega_1 \to \Omega_2$ as in (3). Following the same steps as in verifying (1), we obtain

$$\varphi f \circ \psi = \psi \circ f, \quad \text{for all } f \in E(\Omega_1). \tag{15}$$

We will show that (15) implies bijectivity of ψ. The map ψ is surjective. Indeed, let $w \in \Omega_2$, and let c_w be the corresponding constant endomorphism. Since φ is an epimorphism, there exists $f \in E(\Omega_1)$ such that $\varphi f = c_w$. If we plug this f into (15), we get

$$\psi f(z) = w$$

for all $z \in \Omega_1$. Thus ψ is surjective.

To prove that ψ is injective, we show that for every $w \in \Omega_2$, the full preimage of w under ψ, $\psi^{-1}(w)$, consists of one point.

Assume for contradiction that $S_w = \psi^{-1}(w)$ consists of more than one point for some $w \in \Omega_2$. The set S_w cannot be all of Ω_1, since ψ is surjective. For $z_0 \in \partial S_w \cap \Omega_1$ we can find $z_1 \in S_w$ and $\zeta \notin S_w$ which are arbitrarily close to z_0. Let z_2 be a fixed point of S_w different from z_1. Consider a homothetic transformation h such that $h(z_1) = z_1,\ h(z_2) = \zeta$. Since the domain Ω_1 is bounded, we can choose points z_1 and ζ sufficiently close to each other so that h belongs to $E(\Omega_1)$. Applying (15) to h, we obtain

$$\varphi h(w) = \varphi h \circ \psi(z_1) = \psi \circ h(z_1) = \psi(z_1) = w;$$
$$\varphi h(w) = \varphi h \circ \psi(z_2) = \psi \circ h(z_2) = \psi(\zeta) \neq w.$$

The contradiction shows injectivity of ψ. Thus we have proved that ψ is bijective.

According to (15) we have

$$\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1),$$

which implies that φ is an isomorphism.

Theorem 2 is proved.

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