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SIGNED SUMS OF POLYNOMIAL VALUES

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ABSTRACT. We give a generalization of Bleicher's result on signed sums of kth powers. Let $f(x)$ be an integral-valued polynomial of degree k satisfying the necessary condition that there exists no integer $d > 1$ dividing the values $f(x)$ for all integers x. Then, for every positive integer n and every integer l, there are infinitely many integers $m \geq l$ and choices of $\varepsilon_i = \pm 1$ such that

$$n = \sum_{i=l}^{m} \varepsilon_i f(i).$$

In [1] Bleicher proved, among other things, the following interesting result: every positive integer n can be written in the form

$$n = \sum_{i=1}^{m} \varepsilon_i i^k \quad \text{with suitable } \varepsilon_i = \pm 1,$$

where $k \geq 2$ is any given integer, and $m \geq 1$ is an integer, depending on n and k. The purpose of this note is to consider an analogous problem for polynomials (cf. [2], [3], [5, Chap.12]). We shall prove:

Theorem. *Let $f(x)$ be an integral-valued polynomial of degree k satisfying the necessary condition that there exists no integer $d > 1$ dividing the values $f(x)$ for all integers x. Then, for any given integer l, every positive integer n can be represented in the form*

$$n = \sum_{i=l}^{m} \varepsilon_i f(i) \quad \textit{with suitable } \varepsilon_i = \pm 1, \tag{1}$$

where $m \geq l$ is an integer, depending on l, n and $f(x)$.

For the proof of the theorem we begin with some preliminaries. It is well-known (cf. [2]) that the condition of the theorem is equivalent to $f(x)$ being of the form

$$f(x) = a_k \binom{x}{k} + \cdots + a_1 \binom{x}{1} + a_0, \tag{2}$$

where $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$ $(1 \leq i \leq k)$, and $a_0, a_1, \cdots, a_k$ are integers satisfying

$$a_k \neq 0 \quad \text{and} \quad (a_0, a_1, \cdots, a_k) = 1. \tag{3}$$

Our starting-point is the following two results.

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Lemma 1. *There exists a partition of* $\{0, 1, \cdots, 2^k - 1\}$ *as a disjoint union of two sets* A *and* B*, such that the polynomial identity*

$$\sum_{a\in A} f(x+a) - \sum_{b\in B} f(x+b) = 2^{\frac{k(k-1)}{2}} a_k \tag{4}$$

holds.

Proof. This result is related to the Tarry-Escott problem. It is known that, for each positive integer h, there is a partition of $\{0, 1, \cdots, 2^h - 1\}$ into sets A, B with $|A| = |B|$, such that

$$\sum_{a\in A} a^j - \sum_{b\in B} b^j = 0 \quad \text{for} \quad j = 1, \cdots, h-1, \tag{5}$$

and

$$\sum_{a\in A} a^h - \sum_{b\in B} b^h = h! 2^{\frac{h(h-1)}{2}}. \tag{6}$$

(For a simple proof of this fact see [4]; cf. also [1, §2].)

Let $f(x) = \alpha_k x^k + \cdots + \alpha_1 x + \alpha_0$; then $\alpha_k = \frac{a_k}{k!}$. By (5), (6) (with $h = k$) and the binomial theorem we infer that the left side of (4) equals $k! 2^{\frac{k(k-1)}{2}} \alpha_k = 2^{\frac{k(k-1)}{2}} a_k$, as stated. □

Lemma 2. *Write* $M = 2^{\frac{k(k-1)}{2}} a_k$*. If for a positive integer* n *there exists an integer* $s \geq l$ *such that*

$$\sum_{i=l}^{s} \varepsilon_i' f(i) \equiv n \pmod{M} \quad \textit{with suitable } \varepsilon_i' = \pm 1, \tag{7}$$

then the integer n *is representable in the form* (1) *for a certain* $m \geq l$*.*

Proof. We denote by $g(x)$ the polynomial of the left side of (4). Then, for any integer u, $g(u)$ is a signed sum of the values of $f(x)$ at consecutive integers $u, u + 1, \cdots, u + 2^k - 1$. The assumptions imply that for a certain integer L we have

$$\sum_{i=l}^{s} \varepsilon_i' f(i) = n + LM.$$

Thus by Lemma 1,

$$n = \sum_{i=l}^{s} \varepsilon_i' f(i) - (\text{sign } L) \sum_{t=0}^{|L|-1} g(s + t \cdot 2^k + 1),$$

and this gives the desired representation of n (cf. [1, §3]). □

Thus to prove the theorem it suffices to establish the solubility of (7) for every integer $n \geq 1$. For this purpose we need two more lemmas.

Lemma 3. *Let* l *be any given integer. Then* $(f(l), f(l+1), \cdots, f(l+k)) = 1$*.*

Proof. Let $d = (f(l), f(l+1), \cdots, f(l+k))$, and let $h(x) = \frac{f(x)}{d}$. Then $\deg h = k$ and the values of $h(x)$ at $k+1$ consecutive integers $l, l+1, \cdots, l+k$ are integers. Thus $h(x)$ is an integral-valued polynomial. In particular, $h(0), h(1), \cdots, h(k)$ are all integers. It follows from (2) that $d | a_i$ $(i = 0, 1, \cdots, k)$. Thus $d = 1$ by (3). □

Lemma 4. *Let $b_1, \cdots, b_k$ be integers, not all zero, and let $d = (b_1, \cdots, b_k)$. If d is an odd number, then there exist odd $x_1, \cdots, x_r$ and even $x_{r+1}, \cdots, x_k$ $(1 \leq r \leq k)$ such that*

$$b_1x_1 + \cdots + b_kx_k = d.$$

Proof. We proceed by induction on k. When $k = 2$ the result is trivial. In fact, it is easily verified that if b_1 is odd, we can choose x_1 odd and x_2 even; if b_1 is even, we can choose x_1 and x_2 both odd.

Suppose that $k \geq 3$ and that result is true with k replaced by $k-1$. Clearly, we may assume that $b_2, \cdots, b_k$ are not all zero, and let $d_1 = (b_2, \cdots, b_k)$. If b_1 is odd, since $(b_1, d_1) = d$ is odd, it follows that there exist odd y_1 and even y_2 such that $b_1y_1 + d_1y_2 = d$. Moreover, there are integers $x'_2, \cdots, x'_k$ such that $\sum_{i=2}^{k} x'_ib_i = d_1$. Thus

$$b_1y_1 + \sum_{i=2}^{k}(y_2x'_i)b_i = d,$$

as required.

If b_1 is even, then d_1 must be odd. By the induction hypothesis, there are odd $x'_2, \cdots, x'_v$ and even $x'_{v+1}, \cdots, x'_k$ $(2 \leq v \leq k)$ such that $\sum_{i=2}^{k} x'_ib_i = d_1$. Furthermore, since b_1 is even, there exist y_1 and y_2, both odd, satisfying $b_1y_1 + d_1y_2 = d$. Thus

$$b_1y_1 + \sum_{i=2}^{k}(y_2x'_i)b_i = d,$$

and this gives the desired result. □

Now we prove that (7) is soluble for every integer $n \geq 1$, and thus complete the proof of the theorem. Let $q > 0$ be the least common denominator of the coefficients of $f(x)$. Write $M^* = q|M|$. Then $k \leq M^*$ and

$$f(uM^* + i) \equiv f(i) \pmod{M} \quad \text{for any integers } u \text{ and } i. \tag{8}$$

Moreover, by Lemmas 2 and 3, there exist odd $x_0, \cdots, x_r$ and even $x_{r+1}, \cdots, x_k$ $(0 \leq r \leq k)$ such that

$$\sum_{i=0}^{k} x_if(i + l) = 1. \tag{9}$$

In the following we fix such integers $x_0, \cdots, x_k$, and put

$$\eta_i = \operatorname{sign} x_i \quad (i = 0, \cdots, k). \tag{10}$$

We distinguish two cases.

(i) n is even. Let $L_{-1} = 0$ and $L_i = \frac{n}{2}(|x_0| + \cdots + |x_i|)$ $(i = 0, \cdots, k)$. By (8) we have, for any integers u and $0 \leq i \leq k$,

$$\begin{aligned} &\sum_{\substack{j=0 \\ j\neq i}}^{M^*-1} f(2uM^* + l + j) - \sum_{\substack{j=0 \\ j\neq i}}^{M^*-1} f((2u+1)M^* + l + j) \\ &\quad + \eta_if(2uM^* + l + i) + \eta_if((2u+1)M^* + l + i) \\ &\equiv 2\eta_if(l + i) \pmod{M}. \end{aligned}$$

Now, summing for u from L_{i-1} to L_i, we see that there exist $\varepsilon'_j = \pm 1$ such that

$$\sum_{j=2L_{i-1}M^*+l}^{2L_iM^*+l-1} \varepsilon'_j f(j) \equiv 2 \cdot \frac{n}{2}|x_i|\eta_i f(l+i) \equiv n x_i f(l+i) \pmod{M}. \tag{11}$$

Summing for $i = 0, 1, \cdots, k$ and using (9), we obtain the desired result:

$$\sum_{j=l}^{2L_kM^*+l-1} \varepsilon'_j f(j) \equiv n \sum_{i=0}^{k} x_i f(l+i) = n \pmod{M}.$$

(ii) n is odd. We define $\delta_i = \frac{n|x_i|-1}{2}$ for $0 \le i \le r$ and $\delta_i = \frac{n|x_i|}{2}$ for $r+1 \le i \le k$. Then all δ_i are integers. Further, let $L_{-1} = 0$ and $L_i = \delta_0 + \cdots + \delta_i$ $(i = 0, \cdots, k)$. In analogy to (11), there exist $\varepsilon'_j = \pm 1$ such that

$$\sum_{j=2L_{i-1}M^*+l}^{2L_iM^*+l-1} \varepsilon'_j f(j) \equiv n x_i f(l+i) - \eta_i f(l+i) \pmod{M}$$

for $0 \le i \le r$; and

$$\sum_{j=2L_{i-1}M^*+l}^{2L_iM^*+l-1} \varepsilon'_j f(j) \equiv n x_i f(l+i) \pmod{M}$$

for $r+1 \le i \le k$. Thus

$$\sum_{j=l}^{2L_kM^*+l-1} \varepsilon'_j f(j) \equiv n - \sum_{i=0}^{r} \eta_i f(l+i) \pmod{M}. \tag{12}$$

Finally, by (8) we have

$$\sum_{i=0}^{r} \eta_i f(2L_kM^* + l + i) \equiv \sum_{i=0}^{r} \eta_i f(l+i) \pmod{M}. \tag{13}$$

The desired result follows from (10), (12) and (13). The proof of the theorem is now complete. □

Remarks. 1. By the result cited in the proof of Lemma 1, it is easy to see that, for any integers $h > k$ and x, there exist $\varepsilon_i = \pm 1$ such that

$$\sum_{i=x}^{x+2^h-1} \varepsilon_i f(i) = 0. \tag{14}$$

It follows that, for every integer $n \neq 0$, there are infinitely many m such that (1) holds (cf. the proof of Corollary 2 in [1]). Moreover, (14) (with $x = l$) shows that for $n = 0$ a representation of the form (1) still exists; in fact, infinitely many exist.

2. When $f(x)$ and l are fixed and n grows to infinity, it seems to be more difficult to estimate the minimal value of m in (1). For the case $f(x) = x^k$, this has been done by Bleicher in [1, §4].

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