

SIGNED SUMS OF POLYNOMIAL VALUES

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ABSTRACT. We give a generalization of Bleicher's result on signed sums of k th powers. Let $f(x)$ be an integral-valued polynomial of degree k satisfying the necessary condition that there exists no integer $d > 1$ dividing the values $f(x)$ for all integers x . Then, for every positive integer n and every integer l , there are infinitely many integers $m \geq l$ and choices of $\varepsilon_i = \pm 1$ such that

$$n = \sum_{i=l}^m \varepsilon_i f(i).$$

In [1] Bleicher proved, among other things, the following interesting result: every positive integer n can be written in the form

$$n = \sum_{i=1}^m \varepsilon_i i^k \quad \text{with suitable } \varepsilon_i = \pm 1,$$

where $k \geq 2$ is any given integer, and $m \geq 1$ is an integer, depending on n and k . The purpose of this note is to consider an analogous problem for polynomials (cf. [2], [3], [5, Chap.12]). We shall prove:

Theorem. *Let $f(x)$ be an integral-valued polynomial of degree k satisfying the necessary condition that there exists no integer $d > 1$ dividing the values $f(x)$ for all integers x . Then, for any given integer l , every positive integer n can be represented in the form*

$$(1) \quad n = \sum_{i=l}^m \varepsilon_i f(i) \quad \text{with suitable } \varepsilon_i = \pm 1,$$

where $m \geq l$ is an integer, depending on l , n and $f(x)$.

For the proof of the theorem we begin with some preliminaries. It is well-known (cf. [2]) that the condition of the theorem is equivalent to $f(x)$ being of the form

$$(2) \quad f(x) = a_k \binom{x}{k} + \cdots + a_1 \binom{x}{1} + a_0,$$

where $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$ ($1 \leq i \leq k$), and a_0, a_1, \dots, a_k are integers satisfying

$$(3) \quad a_k \neq 0 \quad \text{and} \quad (a_0, a_1, \dots, a_k) = 1.$$

Our starting-point is the following two results.

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Lemma 1. *There exists a partition of $\{0, 1, \dots, 2^k - 1\}$ as a disjoint union of two sets A and B , such that the polynomial identity*

$$(4) \quad \sum_{a \in A} f(x + a) - \sum_{b \in B} f(x + b) = 2^{\frac{k(k-1)}{2}} a_k$$

holds.

Proof. This result is related to the Tarry-Escott problem. It is known that, for each positive integer h , there is a partition of $\{0, 1, \dots, 2^h - 1\}$ into sets A, B with $|A| = |B|$, such that

$$(5) \quad \sum_{a \in A} a^j - \sum_{b \in B} b^j = 0 \quad \text{for } j = 1, \dots, h - 1,$$

and

$$(6) \quad \sum_{a \in A} a^h - \sum_{b \in B} b^h = h! 2^{\frac{h(h-1)}{2}}.$$

(For a simple proof of this fact see [4]; cf. also [1, §2].)

Let $f(x) = \alpha_k x^k + \dots + \alpha_1 x + \alpha_0$; then $\alpha_k = \frac{a_k}{k!}$. By (5), (6) (with $h = k$) and the binomial theorem we infer that the left side of (4) equals $k! 2^{\frac{k(k-1)}{2}} \alpha_k = 2^{\frac{k(k-1)}{2}} a_k$, as stated. □

Lemma 2. *Write $M = 2^{\frac{k(k-1)}{2}} a_k$. If for a positive integer n there exists an integer $s \geq l$ such that*

$$(7) \quad \sum_{i=l}^s \varepsilon'_i f(i) \equiv n \pmod{M} \quad \text{with suitable } \varepsilon'_i = \pm 1,$$

then the integer n is representable in the form (1) for a certain $m \geq l$.

Proof. We denote by $g(x)$ the polynomial of the left side of (4). Then, for any integer u , $g(u)$ is a signed sum of the values of $f(x)$ at consecutive integers $u, u + 1, \dots, u + 2^k - 1$. The assumptions imply that for a certain integer L we have

$$\sum_{i=l}^s \varepsilon'_i f(i) = n + LM.$$

Thus by Lemma 1,

$$n = \sum_{i=l}^s \varepsilon'_i f(i) - (\text{sign } L) \sum_{t=0}^{|L|-1} g(s + t \cdot 2^k + 1),$$

and this gives the desired representation of n (cf. [1, §3]). □

Thus to prove the theorem it suffices to establish the solubility of (7) for every integer $n \geq 1$. For this purpose we need two more lemmas.

Lemma 3. *Let l be any given integer. Then $(f(l), f(l + 1), \dots, f(l + k)) = 1$.*

Proof. Let $d = (f(l), f(l + 1), \dots, f(l + k))$, and let $h(x) = \frac{f(x)}{d}$. Then $\deg h = k$ and the values of $h(x)$ at $k + 1$ consecutive integers $l, l + 1, \dots, l + k$ are integers. Thus $h(x)$ is an integral-valued polynomial. In particular, $h(0), h(1), \dots, h(k)$ are all integers. It follows from (2) that $d|a_i$ ($i = 0, 1, \dots, k$). Thus $d = 1$ by (3). □

Lemma 4. *Let b_1, \dots, b_k be integers, not all zero, and let $d = (b_1, \dots, b_k)$. If d is an odd number, then there exist odd x_1, \dots, x_r and even x_{r+1}, \dots, x_k ($1 \leq r \leq k$) such that*

$$b_1x_1 + \dots + b_kx_k = d.$$

Proof. We proceed by induction on k . When $k = 2$ the result is trivial. In fact, it is easily verified that if b_1 is odd, we can choose x_1 odd and x_2 even; if b_1 is even, we can choose x_1 and x_2 both odd.

Suppose that $k \geq 3$ and that result is true with k replaced by $k - 1$. Clearly, we may assume that b_2, \dots, b_k are not all zero, and let $d_1 = (b_2, \dots, b_k)$. If b_1 is odd, since $(b_1, d_1) = d$ is odd, it follows that there exist odd y_1 and even y_2 such that $b_1y_1 + d_1y_2 = d$. Moreover, there are integers x'_2, \dots, x'_k such that $\sum_{i=2}^k x'_i b_i = d_1$. Thus

$$b_1y_1 + \sum_{i=2}^k (y_2x'_i)b_i = d,$$

as required.

If b_1 is even, then d_1 must be odd. By the induction hypothesis, there are odd x'_2, \dots, x'_v and even x'_{v+1}, \dots, x'_k ($2 \leq v \leq k$) such that $\sum_{i=2}^k x'_i b_i = d_1$. Furthermore, since b_1 is even, there exist y_1 and y_2 , both odd, satisfying $b_1y_1 + d_1y_2 = d$. Thus

$$b_1y_1 + \sum_{i=2}^k (y_2x'_i)b_i = d,$$

and this gives the desired result. □

Now we prove that (7) is soluble for every integer $n \geq 1$, and thus complete the proof of the theorem. Let $q > 0$ be the least common denominator of the coefficients of $f(x)$. Write $M^* = q|M|$. Then $k \leq M^*$ and

$$(8) \quad f(uM^* + i) \equiv f(i) \pmod{M} \text{ for any integers } u \text{ and } i.$$

Moreover, by Lemmas 2 and 3, there exist odd x_0, \dots, x_r and even x_{r+1}, \dots, x_k ($0 \leq r \leq k$) such that

$$(9) \quad \sum_{i=0}^k x_i f(i + l) = 1.$$

In the following we fix such integers x_0, \dots, x_k , and put

$$(10) \quad \eta_i = \text{sign } x_i \quad (i = 0, \dots, k).$$

We distinguish two cases.

(i) n is even. Let $L_{-1} = 0$ and $L_i = \frac{n}{2}(|x_0| + \dots + |x_i|)$ ($i = 0, \dots, k$). By (8) we have, for any integers u and $0 \leq i \leq k$,

$$\begin{aligned} & \sum_{\substack{j=0 \\ j \neq i}}^{M^*-1} f(2uM^* + l + j) - \sum_{\substack{j=0 \\ j \neq i}}^{M^*-1} f((2u + 1)M^* + l + j) \\ & \quad + \eta_i f(2uM^* + l + i) + \eta_i f((2u + 1)M^* + l + i) \\ & \equiv 2\eta_i f(l + i) \pmod{M}. \end{aligned}$$

Now, summing for u from L_{i-1} to L_i , we see that there exist $\varepsilon'_j = \pm 1$ such that

$$(11) \quad \sum_{j=2L_{i-1}M^*+l}^{2L_iM^*+l-1} \varepsilon'_j f(j) \equiv 2 \cdot \frac{n}{2} |x_i| \eta_i f(l+i) \equiv nx_i f(l+i) \pmod{M}.$$

Summing for $i = 0, 1, \dots, k$ and using (9), we obtain the desired result:

$$\sum_{j=l}^{2L_kM^*+l-1} \varepsilon'_j f(j) \equiv n \sum_{i=0}^k x_i f(l+i) \equiv n \pmod{M}.$$

(ii) n is odd. We define $\delta_i = \frac{n|x_i|-1}{2}$ for $0 \leq i \leq r$ and $\delta_i = \frac{n|x_i|}{2}$ for $r+1 \leq i \leq k$. Then all δ_i are integers. Further, let $L_{-1} = 0$ and $L_i = \delta_0 + \dots + \delta_i$ ($i = 0, \dots, k$). In analogy to (11), there exist $\varepsilon'_j = \pm 1$ such that

$$\sum_{j=2L_{i-1}M^*+l}^{2L_iM^*+l-1} \varepsilon'_j f(j) \equiv nx_i f(l+i) - \eta_i f(l+i) \pmod{M}$$

for $0 \leq i \leq r$; and

$$\sum_{j=2L_{i-1}M^*+l}^{2L_iM^*+l-1} \varepsilon'_j f(j) \equiv nx_i f(l+i) \pmod{M}$$

for $r+1 \leq i \leq k$. Thus

$$(12) \quad \sum_{j=l}^{2L_kM^*+l-1} \varepsilon'_j f(j) \equiv n - \sum_{i=0}^r \eta_i f(l+i) \pmod{M}.$$

Finally, by (8) we have

$$(13) \quad \sum_{i=0}^r \eta_i f(2L_kM^* + l + i) \equiv \sum_{i=0}^r \eta_i f(l+i) \pmod{M}.$$

The desired result follows from (10), (12) and (13). The proof of the theorem is now complete. □

Remarks. 1. By the result cited in the proof of Lemma 1, it is easy to see that, for any integers $h > k$ and x , there exist $\varepsilon_i = \pm 1$ such that

$$(14) \quad \sum_{i=x}^{x+2^h-1} \varepsilon_i f(i) = 0.$$

It follows that, for every integer $n \neq 0$, there are infinitely many m such that (1) holds (cf. the proof of Corollary 2 in [1]). Moreover, (14) (with $x = l$) shows that for $n = 0$ a representation of the form (1) still exists; in fact, infinitely many exist.

2. When $f(x)$ and l are fixed and n grows to infinity, it seems to be more difficult to estimate the minimal value of m in (1). For the case $f(x) = x^k$, this has been done by Bleicher in [1, §4].

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