

ON THE DUAL OF ORLICZ–LORENTZ SPACE

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ABSTRACT. A description of the Köthe dual of the Orlicz–Lorentz space $\Lambda_{\varphi,w}$ generated by an Orlicz function φ and a regular weight function w is presented. It is also shown that in the case of separable Orlicz–Lorentz spaces the regularity condition on w is necessary and sufficient for the coincidence of the Banach dual space with the described Köthe dual space.

1. INTRODUCTION

Let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a measure space with the complete and σ -finite measure μ , and let $L^0(\mu)$ denote the space of all μ -equivalence classes of Σ -measurable functions on Ω with the topology of convergence in measure on μ -finite sets.

A Banach space $(E, \|\cdot\|_E)$ is said to be a *Banach function space* on (Ω, μ) if it is a subspace of $L^0(\mu)$ such that there exists $h \in L^0(\mu)$ with $h > 0$ a.e. in Ω and the assumptions that $f \in L^0(\mu)$, $g \in E$ and $|f| \leq |g|$ a.e. in Ω imply $f \in E$ and $\|f\|_E \leq \|g\|_E$. If in addition the unit ball $B_E = \{f : \|f\|_E \leq 1\}$ is closed in $L^0(\mu)$, then we say that E has the *Fatou property*. A Banach function space defined on $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with the counting measure μ is called a *Banach sequence space* (on \mathbb{N}).

A Banach function space E on (Ω, μ) is said to be *symmetric* if for every $f \in L^0(\mu)$ and $g \in E$ with $\mu_f = \mu_g$, we have $f \in E$ and $\|f\|_E = \|g\|_E$, where for any $h \in L^0(\mu)$, μ_h is the *distribution function* defined by

$$\mu_h(t) = \mu(\{\omega \in \Omega : |h(\omega)| > t\}), \quad t \geq 0.$$

If E is a Banach function space on (Ω, μ) , then the *Köthe dual* E' of E is a Banach function space, which can be identified with the space of all functionals possessing an integral representation, that is,

$$E' = \{g \in L^0(\mu) : \|g\|_{E'} = \sup_{\|f\|_E \leq 1} \int_{\Omega} |fg| d\mu < \infty\}.$$

It is well known that if E has order continuous norm (i.e., $\|f_n\|_E \rightarrow 0$ whenever $E \ni f_n \downarrow 0$), then the dual space E^* can be naturally identified with E' ([8]).

In this paper we are interested in the description of the Köthe duals for symmetric *Orlicz–Lorentz* spaces defined on either nonatomic or purely atomic measure space. Since after minor modifications the proofs presented in the paper work in essentially

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the same way in both cases, for simplicity, we consider only the case of Orlicz–Lorentz spaces defined on (I, m) , where either $I = (0, 1)$ or $I = (0, \infty)$ and m is the Lebesgue measure.

We recall that if $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an *Orlicz function* (i.e., a convex function which assumes value zero only at zero) and $w : I \rightarrow (0, \infty)$ is a *weight function* (i.e., nonincreasing and locally integrable with respect to the measure m and such that $\int_0^\infty w \, dm = \infty$ if $I = (0, \infty)$), then the *Orlicz–Lorentz function space* $\Lambda_{\varphi, w}$ on (Ω, μ) is the set of all $f \in L^0(\mu)$ such that

$$\int_{\Omega} \varphi(\lambda f^*) w \, dm < \infty$$

for some $\lambda > 0$, where for any $f \in L^0(\mu)$, f^* denotes the *nonincreasing rearrangement* of f defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$$

for any $t > 0$ (by convention $\inf \emptyset = \infty$).

In the case of counting measure on $2^{\mathbb{N}}$ the Orlicz–Lorentz sequence space $\lambda_{\varphi, w}$ on \mathbb{N} is defined by

$$\lambda_{\varphi, w} = \left\{ x = \{x(k)\} : \sum_{k=1}^{\infty} \varphi(\lambda x^*(k)) w(k) < \infty \text{ for some } \lambda > 0 \right\}.$$

Here $w = \{w(k)\}$ is a *weight sequence*, a nonincreasing sequence of positive reals such that $\sum_{k=1}^{\infty} w(k) = \infty$.

It is easy to check that $\Lambda_{\varphi, w}$ (resp., $\lambda_{\varphi, w}$) is a symmetric function space (resp., symmetric sequence space) with the Fatou property, equipped with the norm

$$\|f\| = \inf \left\{ \lambda > 0 : \int_I \varphi(f^*/\lambda) w \, dm \leq 1 \right\},$$

respectively

$$\|x\| = \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \varphi(x^*(k)/\lambda) w(k) \leq 1 \right\}.$$

Note that if $w \equiv 1$ (resp., $w(k) = 1$ for all $k \in \mathbb{N}$), then $\Lambda_{\varphi, w}$ (resp., $\lambda_{\varphi, w}$) is the Orlicz function space L_{φ} (resp., Orlicz sequence space ℓ_{φ}). If $\varphi(t) = t$, then $\Lambda_{\varphi, w}$ (resp., $\lambda_{\varphi, w}$) is the Lorentz space Λ_w (resp., λ_w). We recall that an Orlicz function φ satisfies the Δ_2 -condition ($\varphi \in \Delta_2$) if there exists $C > 0$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t > 0$. We will further say that φ is an *N-function* whenever $\lim_{t \rightarrow 0} \varphi(t)/t = 0$ and $\lim_{t \rightarrow \infty} \varphi(t)/t = \infty$. We refer to [5] and [7] to study the basic properties of Orlicz–Lorentz spaces as well to the references included therein.

In what follows by a *regular weight* we mean a weight function w such that, if we denote $S(t) = \int_0^t w(s) \, ds$ for $t \in I$, then $S(2t) \geq KS(t)$ for any $t > 0$ ($t \in (0, 1/2)$ in the case $I = (0, 1)$), where $K > 1$ is independent of t . In the sequence case a weight $w = \{w(k)\}$ is *regular* if $S(2n) \geq KS(n)$ for any $n \in \mathbb{N}$, where $S(n) = \sum_{k=1}^n w(k)$ and $K > 1$ is a constant independent of n . It is well known and easy to show that w is regular iff there exists $C > 0$ such that $tw(t) \leq S(t) \leq Ctw(t)$ for all $t \in I$ in the function case and analogously $nw(n) \leq S(n) \leq Cnw(n)$ for all $n \in \mathbb{N}$ in the sequence case.

Recall that if $\rho : I \rightarrow (0, \infty)$ is a concave function, then the *Marcinkiewicz space* M_ρ is defined by

$$M_\rho = \left\{ f \in L^0(m) : \|f\|_{M_\rho} = \sup_{t \in I} \frac{\int_0^t f^*(s) ds}{\rho(t)} < \infty \right\},$$

and the Marcinkiewicz space M_S with $S(t) = \int_0^t w(s) ds$ is the Köthe dual of Λ_w , that is,

$$(\Lambda_w)' = M_S$$

with equality of norms (see [3] or [9]). In what follows we will write $f \asymp g$ for nonnegative functions f and g whenever $C_1 f \leq g \leq C_2 f$ for some $C_j > 0$, $j = 1, 2$.

The problem of the description of dual spaces for Lorentz sequence spaces has been considered in [1] and [2]. It was proved there that the regularity of w is a necessary and sufficient condition, in the case $p = 1$, and a sufficient condition in the case $p > 1$, in order that the Köthe dual of classical Lorentz sequence space $d(w, p) = \lambda_{\varphi, w}$ with $\varphi(t) = t^p$ for $t \geq 0$ to consist exactly of those sequences $\{x(k)\}$ for which $\{x^*(k)/w(k)^{1/p}\} \in \ell_{p'}$, where $1/p + 1/p' = 1$.

In [1] Allen also raises the question of whether the similar description of the Köthe dual space is also true for Lorentz function spaces. Reisner [12] answered this question positively and gave a proof which works in essentially the same way in the sequence and in the function spaces. He also proved that regularity of w is necessary for $p > 1$.

Following the ideas from [12] we prove in this paper, under the assumption that φ is an N -function satisfying the Δ_2 -condition, that the regularity of the weight function w is a necessary and sufficient condition for the dual of the Orlicz–Lorentz space $\Lambda_{\varphi, w}$ on (I, m) to consist exactly of those functions f for which f^*/w belongs to the Orlicz function space L_{φ_*} on $(I, w dm)$, where φ_* is the *Young conjugate* of φ , i.e.,

$$\varphi_*(t) = \sup\{st - \varphi(s) : s \geq 0\}$$

for $t \geq 0$. We also obtain some partial results for φ being an arbitrary Orlicz function. A similar result is also true for the symmetric Orlicz–Lorentz sequence space $\lambda_{\varphi, w}$ on \mathbb{N} .

In the proof of the main result we will use the Lozanovskii theorem on the representation of the Köthe dual space for the Calderón–Lozanovskii space (see [11]). Let us recall that if (E_0, E_1) is any couple of Banach function spaces on (Ω, μ) and \mathcal{U} denotes the set of all concave and positively homogeneous functions $\psi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ such that $\psi(s, t) = 0$ if and only if $s = t = 0$, then the Calderón–Lozanovskii space $\psi(E_0, E_1)$ generated by the couple (E_0, E_1) and a function $\psi \in \mathcal{U}$ consists of all $f \in L^0(\mu)$ such that $|f| \leq \lambda \psi(|f_0|, |f_1|)$ a.e. for some $\lambda > 0$ and $f_j \in E_j$, $j = 0, 1$. The space $\psi(E_0, E_1)$ is a Banach function space on (Ω, μ) (cf. [10], [11]) equipped with the following equivalent norms:

$$\|f\|_\psi = \inf\{\lambda > 0 : |f| \leq \lambda \psi(|f_0|, |f_1|) \text{ a.e., } \|f_j\|_{E_j} \leq 1, j = 0, 1\}$$

and

$$\|f\|_\psi^1 = \inf\{\|f_0\|_{E_0} + \|f_1\|_{E_1} : |f| = \psi(|f_0|, |f_1|)\},$$

satisfying the inequalities

$$\|f\|_\psi \leq \|f\|_\psi^1 \leq 2\|f\|_\psi$$

for all $f \in \psi(E_0, E_1)$. In the case of the power function $\psi_\theta(s, t) = s^{1-\theta}t^\theta$ with $0 < \theta < 1$, $\psi_\theta(E_0, E_1)$ is the well-known Calderón space $E_0^{1-\theta}E_1^\theta$ (see [4]).

For any $\psi \in \mathcal{U}$ the *involution* $\widehat{\psi}$ of ψ is defined on \mathbb{R}_+^2 by

$$\widehat{\psi}(s, t) = \inf \left\{ \frac{\alpha s + \beta t}{\psi(\alpha, \beta)} : \alpha, \beta > 0 \right\}.$$

Lozanovskii [11] proved the following theorem (see also [10], [13]).

Theorem 1. a) *If E_0, E_1 are two Banach function spaces on the same measure space, then for all $\psi \in \mathcal{U}$ we have*

$$(\psi(E_0, E_1), \|\cdot\|_\psi)' = (\widehat{\psi}(E_0', E_1'), \|\cdot\|_{\widehat{\psi}}^1)$$

with equality of the norms.

b) *For every $0 \leq f \in L^1(\mu)$ and $\varepsilon > 0$, there exist $0 \leq g \in E$, $0 \leq h \in E'$ such that $f = gh$ and*

$$\|g\|_E \|h\|_{E'} \leq (1 + \varepsilon) \|f\|_{L^1}.$$

If E has the Fatou property, we may take $\varepsilon = 0$ in the above inequality.

2. RESULTS

The main aim of this section is to prove a representation theorem for the Köthe dual of Orlicz–Lorentz spaces. The proof for sequence spaces is analogous and more simple than in the function case. Thus we limit ourselves to a proof for function spaces only. We begin with the lemma which will be useful in the sequel.

Lemma 1. *Let $\psi \in \mathcal{U}$ and let $f, g \in L^0(\mu)$ be such that $\mu_f(t) < \infty$ and $\mu_g(t) < \infty$ for every $t > 0$. If $h \in L^0(\mu)$ is such that $|h| \leq \psi(|f|, |g|)$ a.e., then for all $t > 0$*

$$h^*(t) \leq 2\psi(f^*(t/2), g^*(t/2)).$$

Proof. Since each $\psi \in \mathcal{U}$ is nondecreasing in each variable, we have $\psi(s, t) \leq \max\{s/\alpha, t/\beta\}\psi(\alpha, \beta)$, and in consequence

$$\psi(s, t) \leq \left(\frac{s}{\alpha} + \frac{t}{\beta}\right)\psi(\alpha, \beta)$$

for all $s, t \geq 0$ and $\alpha, \beta > 0$. This implies that

$$\psi(s, t) \leq \widetilde{\psi}(s, t) \leq 2\psi(s, t)$$

holds for all $s, t \geq 0$, where

$$\widetilde{\psi}(s, t) := \inf_{\alpha, \beta > 0} (s/\alpha + t/\beta)\psi(\alpha, \beta)$$

for all $s, t \geq 0$. Now fix $\alpha, \beta > 0$. Then applying the above inequalities, we conclude that for all $\alpha, \beta > 0$

$$|h| \leq \left(\frac{|f|}{\alpha} + \frac{|g|}{\beta}\right)\psi(\alpha, \beta) \text{ a.e.}$$

and in view of the inequality $(f + g)^*(t) \leq f^*(t/2) + g^*(t/2)$,

$$h^*(t) \leq \left(\frac{f^*(t/2)}{\alpha} + \frac{g^*(t/2)}{\beta}\right)\psi(\alpha, \beta)$$

for all $t > 0$. Hence

$$h^*(t) \leq \tilde{\psi}(f^*(t/2), g^*(t/2)) \leq 2\psi(f^*(t/2), g^*(t/2))$$

for all $t > 0$, which completes the proof. □

In what follows, given an Orlicz function φ , we define $I(f) = \int_I \varphi_*(f^*/w)w \, dm$ for $f \in L^0(m)$ and

$$M_{\varphi_*,w} = \{f \in L^0(m) : I(f/\lambda) < \infty \text{ for some } \lambda > 0\}.$$

In the space $M_{\varphi_*,w}$ we define an order monotone and homogenous functional

$$\|f\|_{M_{\varphi_*,w}} = \inf\{\lambda > 0 : I(f/\lambda) \leq 1\}.$$

One can show that if w is regular, then the functional $\|\cdot\|_{M_{\varphi_*,w}}$ is a quasinorm. We also observe that if $\varphi(t) = t$, then

$$M_{\varphi_*,w} = \left\{f \in L^0(m) : \|f\|_{M_{\varphi_*,w}} = \sup_{t \in I} \frac{f^*(t)}{w(t)} < \infty\right\}$$

and $M_{\varphi_*,w} = M_S$ with $\|\cdot\|_{M_{\varphi_*,w}} \asymp \|\cdot\|_{M_S}$, by $tw(t) \asymp S(t)$, whenever w is regular. In a similar way we define the sequence space $m_{\varphi_*,w}$ for a sequence weight w .

Before we prove the main theorem we will need the following lemma which, in the case of $\varphi(t) = t^p$ with $1 \leq p < \infty$, has been proved in [12] as Lemma 2 with means of much more involved arguments.

Lemma 2. *Let an Orlicz function φ and a weight function w be such that $(\Lambda_{\varphi,w})' = M_{\varphi_*,w}$. Then there is $K > 0$ so that*

$$\|g\|_{M_{\varphi_*,w}} \leq K \|g\|_{(\Lambda_{\varphi,w})'} \text{ for all } g \in (\Lambda_{\varphi,w})'.$$

Proof. For a contradiction assume that there exists a sequence $\{g_k\}$ with

$$\|g_k\|_{(\Lambda_{\varphi,w})'} = 1 \text{ and } \|g_k\|_{M_{\varphi_*,w}} > k2^k \text{ for all } k \in \mathbb{N}.$$

Obviously for the function g defined by

$$g = \sum_{k=1}^{\infty} \frac{1}{2^k} |g_k|,$$

we have $g \in (\Lambda_{\varphi,w})' = M_{\varphi_*,w}$. Since $g \geq |g_k|/2^k$, we have $g^*(t) \geq g_k^*(t)/2^k$ for all $k \in \mathbb{N}$ and $t \in I$. Thus from the order monotonicity and homogeneity of the functional $\|\cdot\|_{M_{\varphi_*,w}}$, we get

$$\|g\|_{M_{\varphi_*,w}} \geq k$$

for all $k \in \mathbb{N}$. This contradiction finishes the proof of the lemma. □

Now we are ready to prove the main result of the paper.

Theorem 2. *Let w be a weight function and let either $\varphi(t) = t$ or φ be an N -function. Then the following holds true:*

- (i) *If w is a regular weight, then $(\Lambda_{\varphi,w})' = M_{\varphi_*,w}$ and $\|\cdot\|_{(\Lambda_{\varphi,w})'} \asymp \|\cdot\|_{M_{\varphi_*,w}}$.*
- (ii) *If φ satisfies the Δ_2 -condition and $(\Lambda_{\varphi,w})' = M_{\varphi_*,w}$, then w is regular.*

Proof. At first we notice that if E is any Banach function space on a measure space (Ω, μ) , $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an Orlicz function and $\psi \in \mathcal{U}$ is defined by $\psi(s, t) = 0$ if $t = 0$ and $\psi(s, t) = t\varphi^{-1}(s/t)$ if $t > 0$, then

$$\psi(E, L^\infty) = \{f \in L^0 : \varphi \circ (\lambda f) \in E \text{ for some } \lambda > 0\}$$

and

$$\|f\|_\psi = \inf\{\lambda > 0 : \|\varphi \circ (f/\lambda)\|_E \leq 1\}.$$

In particular, we obtain $\Lambda_{\varphi, w} = \psi(\Lambda_w, L^\infty)$. Hence by Theorem 1, we get

$$(\Lambda_{\varphi, w})' = \widehat{\psi}((\Lambda_w)', (L^\infty)') = \widehat{\psi}(M_S, L^1),$$

with equality of the norms, where $\widehat{\psi}(M_S, L^1)$ is considered with the norm $\|\cdot\|_{\widehat{\psi}}^1$. It is also easy to check that $\widehat{\psi}(s, t) = 0$ for $s = 0$ and $\widehat{\psi}(s, t) = s\varphi_*^{-1}(t/s)$ for $s > 0$. Notice by the assumption that φ is an N -function, φ_* is an Orlicz function and thus $\widehat{\psi} \in \mathcal{U}$.

Now we are ready to prove (i). If $\varphi(t) = t$, then (i) is obvious by the observation stated before Lemma 2. Assume now that φ is an N -function and let $f \in \widehat{\psi}(M_S, L^1)$ with $\|f\|_{\widehat{\psi}}^1 \leq 1$. Since $\|f\|_{\widehat{\psi}} \leq \|f\|_{\widehat{\psi}}^1$, we get $\|f\|_{\widehat{\psi}} \leq 1$. This yields

$$|f| \leq \widehat{\psi}(|f_0|, |f_1|) \text{ a.e.}$$

for some $f_0 \in M_S$ and $f_1 \in L^1$ with $\|f_0\|_{M_S} \leq 1$ and $\|f_1\|_{L^1} \leq 1$. In consequence, it follows by Lemma 1 that

$$f^*(t) \leq 2\widehat{\psi}(f_0^*(t/2), f_1^*(t/2))$$

for all $t \in I$. By regularity of w for any $t \in I$,

$$f_0^*(t/2)/Cw(t) \leq t f_0^*(t/2)/S(t) \leq \|f_0^*(\cdot/2)\|_{M_S} \leq 2\|f_0\|_{M_S} \leq 2$$

and hence for $t \in I$,

$$f_0^*(t/2) \leq 2Cw(t).$$

Combining the above inequalities, we obtain

$$f^*(t) \leq 2\widehat{\psi}(2Cw(t), f_1^*(t/2))$$

for all $t \in I$. Therefore

$$f^*(t) \leq 4Cw(t)\varphi_*^{-1}(f_1^*(t/2)/2Cw(t))$$

or equivalently

$$\varphi_*\left(\frac{f^*(t)}{4Cw(t)}\right)w(t) \leq \frac{1}{2C}f_1^*(t/2)$$

for any $t \in I$. Since $\|f_1\|_{L^1} \leq 1$, $\|f_1^*(\cdot/2)\|_{L^1} \leq 2$, we obtain that $f^* \in M_{\varphi_*, w}$ and $\|f\|_{M_{\varphi_*, w}} \leq 4C$. In consequence we proved that for any $f \in (\Lambda_{\varphi, w})'$, we have

$$\|f\|_{M_{\varphi_*, w}} \leq 4C\|f\|_{(\Lambda_{\varphi, w})'}.$$

Now suppose that $f \in M_{\varphi_*, w}$ and $\|f\|_{M_{\varphi_*, w}} \leq 1$. This implies that

$$\int_I \varphi_*\left(\frac{f^*(t)}{w(t)}\right)w(t) dt \leq 1.$$

Taking $f_0 = w$ and $f_1 = \varphi_*(f^*/w)w$, we have $f_0 \in M_S$, $f_1 \in L^1$, $\|f_0\|_{M_S} = 1$ and $\|f_1\|_{L^1} \leq 1$. Since

$$f^*(t) = \widehat{\psi}(f_0(t), f_1(t))$$

for all $t \in I$, we get $f^* \in \widehat{\psi}(M_S, L^1)$ and $\|f^*\|_{(\Lambda_{\varphi,w})'} = \|f^*\|_{\widehat{\psi}}^1 \leq 2$. In consequence $f^* \in (\Lambda_{\varphi,w})'$, and since the Köthe dual of a symmetric space is also a symmetric space (see [9]), we get $f \in (\Lambda_{\varphi,w})'$. Combining the above inequalities, we obtain

$$2^{-1}\|f\|_{(\Lambda_{\varphi,w})'} \leq \|f\|_{M_{\varphi_*,w}} \leq 4C\|f\|_{(\Lambda_{\varphi,w})'}$$

for every $f \in (\Lambda_{\varphi,w})'$, where C is the regularity constant of w .

(ii). Fix $x \in I$ and let $f = \chi_{(0,x)}/x$. Since $\Lambda_{\varphi,w}$ has the Fatou property, it follows by Theorem 1 b) that there exist functions $g \in E := \Lambda_{\varphi,w}$, $h \in E'$ such that $f = gh$ and $\|g\|_E = \|h\|_{E'} = 1$. Thus, we have $\|f/h\|_E = \|h\|_{E'} = 1$. Without loss of generality we can assume that $h = h^*$ and $\text{supp } h = (0, x)$ (cf. [6, pp. 38-41]). In consequence

$$\begin{aligned} \int_0^x \varphi\left(\frac{1}{xh(t)}\right)w(x-t) dt &= \int_0^x \varphi\left(\frac{1}{xh(x-t)}\right)w(t) dt \\ &= \int_I \varphi\left(\left(\frac{\chi_{(0,x)}}{xh}\right)^*(t)\right)w(t) dt \leq 1. \end{aligned}$$

Moreover, since $\|h\|_{E'} = 1$, we have $\|h\|_{M_{\varphi_*,w}} \leq K$ by Lemma 2, and so

$$\int_I \varphi_*\left(\frac{h(t)}{Kw(t)}\right)w(t) dt \leq 1.$$

From $\varphi \in \Delta_2$, there exists $C \geq 2$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t > 0$. This implies that $\varphi(\lambda t) \leq 2^p\lambda^p\varphi(t)$ for $p = \ln C/\ln 2 \geq 1$ and all $\lambda \geq 1, t \geq 0$. Since w is nonincreasing, we also have

$$\frac{w(x-t)}{w(t)} \geq 1 \text{ for any } t \in [x/2, x].$$

Combining the above inequalities with the definition of the conjugate function φ_* , we obtain

$$\begin{aligned} \frac{1}{xK} \int_{x/2}^x \left(\frac{w(x-t)}{w(t)}\right)^{1/p} dt &\leq \int_{x/2}^x \varphi_*\left(\frac{h(t)}{Kw(t)}\right)w(t) dt \\ &+ \int_{x/2}^x \varphi\left(\frac{1}{xh(t)}\left(\frac{w(x-t)}{w(t)}\right)^{1/p}\right)w(t) dt \\ &\leq 1 + 2^p \int_{x/2}^x \varphi\left(\frac{1}{xh(t)}\right)w(x-t) dt \leq 1 + 2^p. \end{aligned}$$

Clearly

$$\frac{1}{x} \int_0^{x/2} \left(\frac{w(x-t)}{w(t)}\right)^{1/p} dt \leq \frac{1}{2}.$$

Hence for $K_1 = (1 + 2^p)K + 1$,

$$\sup_{x \in I} \frac{1}{x} \int_0^x \left(\frac{w(x-t)}{w(t)}\right)^{1/p} dt \leq K_1.$$

This is equivalent to regularity of $w^{1/p}$.

In the case when $\varphi(t) = t, p = 1$ and obviously w is regular.

Now we assume that $\varphi \in \Delta_2$ is an N -function and we shall show that w is regular. It is easy to check that the regularity of w^a for some $0 < a < \infty$ implies (cf. [12], Lemma 3)

$$w(t) \asymp w(2t) \quad \text{for all } t \in I \text{ with } 2t \in I.$$

Now let $h \in M_S$ with $\|h\|_{M_S} \leq 1$. For any $f \in L^1$ with $\|f\|_{L^1} \leq 1$, we have

$$\widehat{\psi}(|h|, |f|) \in (\Lambda_{\varphi, w})' = \widehat{\psi}(M_S, L^1)$$

with $\|\widehat{\psi}(|h|, |f|)\|_{(\Lambda_{\varphi, w})'} \leq 2$. Now if $(\Lambda_{\varphi, w})' = M_{\varphi^*, w}$, it follows by Lemma 2 that there is $K > 0$ so that

$$\sup_{\|f\|_{L^1} \leq 1} \|\widehat{\psi}(|h|, |f|)\|_{M_{\varphi^*, w}} \leq 2K.$$

Taking $f = (1/x)\chi_{(0,x)}$ with $x \in I$, we get by the homogeneity of $\widehat{\psi}$

$$\int_0^x \varphi_* \left(\frac{\widehat{\psi}(xh^*(t), 1)}{2Kxw(t)} \right) w(t) dt \leq \int_I \varphi_* \left(\frac{\widehat{\psi}(|h|, |f|)^*(t)}{2Kw(t)} \right) w(t) dt \leq 1.$$

Combining these with $w(t) \leq cw(2t)$ (by $w(t) \asymp w(2t)$) gives that for all $x \in I$,

$$\frac{x}{2} \varphi_* \left(\frac{\widehat{\psi}(xh^*(x), 1)}{2cKxw(x)} \right) w(x) \leq \int_{x/2}^x \varphi_* \left(\frac{\widehat{\psi}(xh^*(t), 1)}{2Kxw(t)} \right) w(t) dt \leq 1.$$

Hence for all $x \in I$,

$$\begin{aligned} xh^*(x)\varphi_*^{-1} \left(\frac{1}{xh^*(x)} \right) &= \widehat{\psi}(xh^*(x), 1) \\ &\leq 2cKxw(x)\varphi_*^{-1} \left(\frac{2}{xw(x)} \right). \end{aligned}$$

Now since for any N -function it holds $u \leq \varphi^{-1}(u)\varphi_*^{-1}(u) \leq 2u$ for all $u \geq 0$, we get

$$\varphi_*^{-1} \left(\frac{2}{xw(x)} \right) \leq 8cK\varphi^{-1} \left(\frac{1}{xh^*(x)} \right) \quad \text{for all } x \in I,$$

which in view of the Δ_2 -condition implies that

$$\varphi^{-1} \left(\frac{2}{xw(x)} \right) \leq \varphi^{-1} \left(\frac{C}{xh^*(x)} \right)$$

for all $x \in I$ and a constant $C > 0$, and finally it yields that

$$\sup_{x \in I} \frac{h^*(x)}{w(x)} < \infty.$$

In consequence we proved that

$$(\Lambda_w)' = M_S = \left\{ h \in L^0(m) : \frac{h^*}{w} \in L^\infty \right\}.$$

Thus, by the above proof for $\varphi(t) = t$, it follows that w is regular. □

If the Orlicz function φ is neither linear nor an N -function, we have the following description of the Köthe dual spaces of Orlicz–Lorentz spaces.

Theorem 3. *Let φ be an Orlicz function and let w be a regular weight. Then the following holds true:*

- (i) *If $\lim_{t \rightarrow 0} \varphi(t)/t > 0$ and $\lim_{t \rightarrow 0} \varphi(t)/t < \infty$, then $\varphi(t) \asymp t$ and*

$$(\Lambda_{\varphi, w})' = M_S.$$

(ii) If $\lim_{t \rightarrow 0} \varphi(t)/t > 0$ and $\lim_{t \rightarrow \infty} \varphi(t)/t = \infty$, then there exists an N -function ϕ such that $\phi(t) \asymp t^2$ for t small enough and $\phi(t) \asymp \varphi(t)$ for t large enough, and

$$(\Lambda_{\varphi,w})' = M_S + M_{\phi_*,w}.$$

(iii) If $\lim_{t \rightarrow 0} \varphi(t)/t = 0$ and $\lim_{t \rightarrow \infty} \varphi(t)/t < \infty$, then there exists an N -function ϕ such that $\phi(t) \asymp \varphi(t)$ for t small enough, and $\phi(t) \asymp t$ for t large enough, and

$$(\Lambda_{\varphi,w})' = M_S \cap M_{\phi_*,w}.$$

Proof. We will use two particular cases of the Lozanovskii duality result from which it follows that for any couple (E_0, E_1) of Banach function spaces defined on the same measure space, we have

$$(E_0 \cap E_1)' = E_0' + E_1' \quad \text{and} \quad (E_0 + E_1)' = E_0' \cap E_1',$$

where $E_0 \cap E_1$ and $E_0 + E_1$ are equipped with the natural interpolation norms (cf. [3] or [9]).

(i). It is obvious since in this case $\Lambda_{\varphi,w} = \Lambda_w$.

(ii). We may assume without loss of generality that $\varphi(1) = 1$. Then it is easy to check that

$$\varphi(t) \asymp \max\{t, \phi(t)\},$$

where $\phi(t) = t^2$ for $t \leq 1$ and $\phi(t) = 2\varphi(t) - 1$ for $t > 1$. Clearly

$$\Lambda_{\varphi,w} = \Lambda_w \cap \Lambda_{\phi,w}.$$

Thus the required result follows by Theorem 2 and the Lozanovskii result mentioned above.

(iii). We define $\varphi_1(t) := \varphi(t)$ if $t \leq 1$ and $\varphi_1(t) := \varphi'_+(1)t^2 + (1 - \varphi'_+(1))$ for $t > 1$ and

$$\phi(t) := \int_0^t \min\{s, \varphi_1(s)\} \frac{ds}{s}.$$

Again it is easy to check that ϕ satisfies the required conditions. Since $\varphi(t) \asymp \min\{t, \phi(t)\}$, we immediately obtain

$$\Lambda_{\varphi,w} = \Lambda_w + \Lambda_{\phi,w}.$$

Thus again Theorem 2 and the Lozanovskii result apply. □

By [7] (cf. also [5]), we have that $\Lambda_{\varphi,w}$ is separable if and only if φ satisfies the corresponding Δ_2 -condition. Thus if the weight function w is regular, then by Theorems 2 and 3 we obtain a description of the Banach dual space of the Orlicz–Lorentz space. In fact the following corollary of Theorem 2 holds true.

Theorem 4. *Let either $\varphi(t) = t$ or φ be an N -function satisfying the Δ_2 -condition. Then the regularity of the weight function w is a necessary and sufficient condition for the coincidence of the Banach dual space of the Orlicz–Lorentz space $\Lambda_{\varphi,w}$ with the space $M_{\varphi_*,w}$.*

Analogous results characterizing the Köthe dual or dual spaces may also be stated for Orlicz–Lorentz sequence spaces.

Finally we notice that the description of the dual of the Lorentz space $\Lambda_{p,w}$ given by Reisner [12] is a particular case of the above theorem for $\varphi(t) = t^p$ for $t \geq 0$, $1 \leq p < \infty$.

REFERENCES

- [1] G.D. Allen, *Duals of Lorentz spaces*, Pacific J. Math. **77** (1978), 287-291. MR **80b**:46015
- [2] G.D. Allen and L.C. Shen, *On the structure of principal ideals of operators*, Trans. Amer. Math. Soc. **238** (1978), 253-270. MR **57**:7230
- [3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics **129**, Academic Press, Boston 1988. MR **89e**:46001
- [4] A.P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113-190. MR **29**:5097
- [5] H. Hudzik, A. Kamińska and M. Mastyło, *Geometric properties of some Calderón-Lozanovskii spaces and Orlicz-Lorentz spaces*, Houston J. Math. **22** (1996), 639-663. MR **97i**:46057
- [6] N.J. Kalton, *Nonlinear commutators in interpolation theory*, Memoirs Amer. Math. Soc., Providence, RI **373** (1988). MR **89h**:47104
- [7] A. Kamińska, *Some remarks on Orlicz-Lorentz spaces*, Math. Nachr. **147** (1990), 29-38. MR **92h**:46034
- [8] L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, 2nd rev. ed., "Nauka", Moscow 1977 (in Russian); English transl.: Pergamon Press 1982. MR **58**:23465; MR **83h**:46002
- [9] S.G. Krein, Ju.I. Petunin and E.M. Semenov, *Interpolation of Linear Operators*, Amer. Math. Soc., Trans. of Math. Monog. **54**, Providence, RI, 1982. MR **84j**:46103
- [10] G.Ya. Lozanovskii, *On some Banach lattices IV*, Sibirsk. Math. Zh. **14** (1973), 140-155; English transl.: Sibirsk. Math. J. **14** (1973), 97-108. MR **49**:1089
- [11] G.Ya. Lozanovskii, *Transformations of ideal Banach spaces by means of concave functions*, Qualitative and approximate methods for the investigation of operator equations, Gos. Univ., Yaroslavl. **3** (1978), 122-148 (in Russian). MR **83a**:46042a
- [12] S. Reisner, *On the duals of Lorentz function and sequence spaces*, Indiana Univ. Math. J. **31** (1982), 65-72. MR **83m**:46044
- [13] S. Reisner, *On two theorems of Lozanovskii concerning intermediate Banach lattices*, Lecture Notes in Math. **1317** (1988), 67-83. MR **89j**:46019

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