PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 6, Pages 1859–1866
S 0002-9939(02)06272-X
Article electronically published on January 16, 2002

L^p VERSION OF HARDY'S THEOREM ON SEMISIMPLE LIE GROUPS

E. K. NARAYANAN AND S. K. RAY

(Communicated by Christopher D. Sogge)

ABSTRACT. We prove an analogue of the L^p version of Hardy's theorem on semisimple Lie groups. The theorem says that on a semisimple Lie group, a function and its Fourier transform cannot decay very rapidly on an average.

1. INTRODUCTION

Uncertainty principles related to decay of a function and its Fourier transform have a long history, starting from a theorem due to Hardy (see [10], [8]). Hardy's theorem consists of two parts: in the first it characterises the heat kernel in terms of its decay and that of its Fourier transform and in the second it shows that a nonzero function and its Fourier transform can have no faster decay. Several generalisations of the second part of Hardy's theorem has appeared since, most notable among them being the results of Cowling-Price and of Beurling (see [3], [14]). The theorem of Cowling and Price says roughly that it is not possible for a nonzero function and its Fourier transform both to decay very rapidly. To be precise, let $e_a(x) = e^{ax^2}$; then if $\|e_a f\|_p < \infty$, $\|e_b \hat{f}\|_q < \infty$, $\min(p,q) < \infty$ and $ab \geq \frac{1}{4}$, then $f = 0$. The case $p = q = \infty$ was treated by Hardy.

Recently, considerable attention has been paid to proving analogues of Hardy's theorem and its L^p version in the setup of noncommutative groups (see [2], [4], [5], [6], [7], [8], [16], [17], [18], [19], [21] and references therein). In [20], Sitaram and Sundari generalised the second part of Hardy's theorem to connected semisimple Lie groups with one conjugacy class of Cartan subgroups and to the K-invariant case for general connected semisimple Lie groups. This result was extended to all semisimple Lie groups with finite centre by Cowling et al. in [5]. In this paper our aim is to discuss the analogue of the theorem of Cowling and Price on semisimple Lie groups. It turns out that the obvious analogue of the Cowling-Price theorem is false on semisimple Lie groups for $p > 2$ (see the example in section 3), that is, for $p > 2$ there may exist a nonzero $K-$bi-invariant function f such that for $ab = \frac{1}{4}$, $\int_G |f(x)|^p e^{pa|x|_G^2} d\mu(x) < \infty$ and $\int_{\mathcal{A}^*} |\hat{f}(\lambda)|^q e^{q|\lambda|^2} |c(\lambda)|^{-2} d\lambda < \infty$ (for the meaning of the symbols see section 2). So to get an analogue of the Cowling-Price theorem we need to find the correct decay condition, and our work shows that the correct

Received by the editors July 25, 2000 and, in revised form, January 2, 2001.
2000 *Mathematics Subject Classification.* Primary 22E30; Secondary 22E46, 43A30.
Key words and phrases. Hardy's theorem, uncertainty principle, semisimple Lie groups.
This research was supported by NBHM, India.

decay condition depends on the index p. In the course of the proof we will make use of the main lemma proved in [5]. We also mention that the final result proved in [5] can be deduced from our result as in the Euclidean spaces.

This paper is organised as follows: In section 2 we describe the notation and background material from the theory of semisimple Lie groups along with some complex analytic results needed for the Cowling-Price theorem. In section 3 we prove our version of the Cowling-Price theorem on semisimple Lie groups.

We are grateful to the referee, whose suggestions and comments improved the presentation of the paper.

2. Notation and preliminaries

In this section we set up the notation that we subsequently employ and recall some basic facts from the theory of semisimple Lie groups. Our discussion of the latter will be brief and we refer the reader to [9], [12], [15] for details. Let G be a connected, non-compact, real semi-simple Lie group with finite center and K be a fixed maximal compact subgroup of G. Let $\mathcal{G}$, $\mathcal{K}$ denote the Lie algebras of G and K respectively. Let B be the Cartan Killing form on $\mathcal{G}$, and let $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be the Cartan decomposition of $\mathcal{G}$. It is known that $B\big|_{\mathcal{P}\times\mathcal{P}}$ is positive definite, thus it gives an inner product and hence a norm on $\mathcal{P}$. Let $\mathcal{A}$ be a fixed maximal abelian subspace of $\mathcal{P}$. Let Δ denote the set of nonzero roots corresponding to $(\mathcal{G}, \mathcal{A})$ and Δ_+ the set of positive roots with respect to some ordering. Let $\mathcal{A}_+$ be the positive Weyl chamber and $A_+ = \exp \mathcal{A}_+$. If $\bar{A_+}$ denotes the closure of A_+ in G, then one gets the polar decomposition $G = K\bar{A_+}K$, that is, each $x \in G$ can be uniquely written as $x = k_1(x)a(x)k_2(x)$ with $k_1(x), k_2(x) \in K$, $a(x) \in \bar{A_+}$. If $\mathcal{G}_\alpha$ denotes the root space corresponding to $\alpha \in \Delta$ with $m_\alpha = \dim \mathcal{G}_\alpha$, then one can choose a Haar measure dx on G such that relative to the polar decomposition it is given by $dx = J(a)dk_1 da dk_2$ where $J(a) = J(\exp H) = \prod_{\alpha\in\Delta_+}\left(e^{\alpha(H)} - e^{-\alpha(H)}\right)^{m_\alpha}$ and da is a Haar measure on A. If $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} m_\alpha \alpha$, then one has, $J(a) = J(\exp H) \le Ce^{2\rho(H)}$ for $H \in \bar{A_+}$. Using the polar decomposition we define $|x|_G = |k_1(x)a(x)k_2(x)| = B(\log a(x), \log a(x))^{\frac{1}{2}}$. For $\lambda \in \mathcal{A}^*$, $H_\lambda \in \mathcal{A}$ is the unique vector such that $\lambda(H) = B(H, H_\lambda) = \langle H, H_\lambda\rangle$ for all $H \in \mathcal{A}$. For λ, $\mu \in \mathcal{A}^*$ we thus have $\langle \lambda, \mu\rangle = B(H_\lambda, H_\mu)$. The bilinear extension of $\langle .,.\rangle$ to $\mathcal{A}^*_{\mathbb{C}}$ is also denoted by $\langle .,.\rangle$. For $\lambda \in \mathcal{A}^*$, $|\lambda|$ will denote $\langle\lambda,\lambda\rangle^{\frac{1}{2}}$.

Let $G = KAN$ be the Iwasawa decomposition of G. Then we have the projection mappings $\kappa : G \to K$, $\mathbf{a} : G \to A$, $\eta : G \to N$ such that $x = \kappa(x)\mathbf{a}(x)\eta(x) = \kappa(x)\exp H(x)\eta(x)$ where $H(x) = \log \mathbf{a}(x) \in \mathcal{A}$. Let M denote the centralizer of A in K. Let $\delta \in \hat{M}$ and $\mathbf{H}_\delta$ be the Hilbert space corresponding to δ. For $\delta \in \hat{M}$ and $\lambda \in \mathcal{A}^*_{\mathbb{C}}$ we have a representation $\pi_{\delta,\lambda}$ acting on the Hilbert space $L^2(K/M, dk, \mathbf{H}_\delta)$ and the action is given by

$$(\pi_{\delta,\lambda}(g)f)(k) = e^{-(i\lambda+\rho)H(g^{-1}k)}\ f(\kappa(g^{-1}k)).$$

The $\pi_{\delta,\lambda}$'s are called the *principal series representations induced from the minimal parabolic subgroup.* $\pi_{\delta,\lambda}$ is unitary if $\lambda \in \mathcal{A}^*$. It is known that given $\delta \in \hat{M}$ there exists a dense open subset $O_\delta \subset \mathcal{A}^*$ such that for $\lambda \in O_\delta$, $\pi_{\delta,\lambda}$ is irreducible and the only identification among the above-described representations are given by the obvious action of the Weyl group on $\hat{M}$ and $\mathcal{A}^*$.

If δ is the trivial representation of M, then $\pi_{\delta,\lambda}$ is denoted by π_λ, the so-called *spherical principal series representations*, which are realised on the Hilbert space $L^2(K/M)$. Note that $\pi_\lambda|K$ are given by left translation on $L^2(K/M)$, in particular the K-fixed vectors are given by constant functions. Let ϕ_λ be the elementary spherical function corresponding to $\lambda \in \mathcal{A}^*_{\mathbb{C}}$, that is, for $\lambda \in \mathcal{A}^*_{\mathbb{C}}$

$$\begin{aligned}\phi_\lambda(x) &= \langle \pi_\lambda(x)1\ ,\ 1\rangle \\ &= \int_K e^{-(i\lambda+\rho)(H(x^{-1}k))}\ dk = \int e^{(i\lambda-\rho)H(xk)}\ dk,\end{aligned}$$

where 1 denotes the constant function $1(x) = 1$ for all $x \in K/M$. The following properties of ϕ_λ are crucial for us and can be found in ([9], [12]).

Proposition 2.1. i) *$\phi_\lambda(x)$ is K-bi-invariant in $x \in G$ and W invariant in $\lambda \in \mathcal{A}^*_{\mathbb{C}}$.*
ii) *$\phi_\lambda(x)$ is a C^∞ function of x and a holomorphic function of λ.*
iii) *We have*

$$e^{-\rho(H)} \le \phi_0(\exp H) \le C(1+\|H\|)^m e^{-\rho(H)}$$

for $H \in \bar{\mathcal{A}}_+$ and some constants $C,\ m > 0$.
iv) *We have*

$$0 < \phi_{i\lambda}(\exp H) \le e^{\lambda(H)}\phi_0(\exp H)$$

for $H \in \bar{\mathcal{A}}_+\ ,\ \lambda \in \bar{\mathcal{A}}_+{}^*$.

The following lemma, proved in [3], is crucial for us.

Lemma 2.2. *Suppose that g is analytic in the region $Q = \{z = re^{i\theta} : 0 < \theta < \frac{\pi}{2}\}$ and continuous up to the boundary. Suppose also that for $p \in [1,\infty)$ and constants A, a*

$$\begin{aligned}&|g(x+iy)| \le A\ e^{ax^2}\ (x+iy \in \bar{Q}),\\ &\left(\int_0^\infty |g(x)|^p\ dx\right)^{\frac{1}{p}} \le A.\end{aligned}$$

Then

$$\int_\sigma^{\sigma+1} |g(\rho e^{i\psi})|\ d\rho \le A\ max\ (e^a, (\sigma+1)^{\frac{1}{p}})$$

for $\psi \in [0, \frac{\pi}{2}]$ and $\sigma \in \mathbb{R}^+$.

Lemma 2.3. *Suppose $f : \mathbb{C}^n \to \mathbb{C}^n$ be an entire function such that for positive a and m*
i) *$|f(z)| \le C\ e^{a\|Re\, z\|^2}\ (1+\|Imz\|)^m$,*
ii) *$\int_{\mathbb{R}^n} |f(x)|^p\ |Q(x)|\ dx < \infty$*
for some p , $1 \le p < \infty$, where Q is a measurable function on $\mathbb{R}^n$ with the property that, for almost all $\tilde{x} \in \mathbb{R}^{n-1}$, $Q(\tilde{x},t)$ is bounded away from zero as a function of $t \in \mathbb{R}$ outside a compact set. Then $f = 0$.

Proof. Fix some $\tilde{x} \in \mathbb{R}^{n-1}$ with the property that $Q_{\tilde{x}}(t) = Q(\tilde{x},t)$ is bounded away from 0 for large t and define $g_{\tilde{x}}(z) = f(\tilde{x},z)$ for $z \in \mathbb{C}$. Then $g_{\tilde{x}}$ is an entire function on $\mathbb{C}$ and satisfies i) and ii). Assuming the lemma to be true for $n = 1$, we get $f(\tilde{x}, z) = 0$ for almost every $\tilde{x} \in \mathbb{R}^{n-1}$. Thus $f = 0$ almost everywhere on $\mathbb{R}^n$ and hence on $\mathbb{C}^n$. So it is enough to prove the case $n = 1$. In this case ii) implies that $\int_{\mathbb{R}} |f(x)|^p dx$ is finite. Let $g(z) = \frac{f(z)}{(i+z)^m}$ for $\mathrm{Im} z > 0$. Then g satisfies the conditions of Lemma 2.2, which gives the estimate

$$\int_{\sigma}^{\sigma+1} |g(\rho e^{i\psi})| d\rho \leq C \max\{e^a, (\sigma+1)^{\frac{1}{p}}\}$$

for $\psi \in [0, \frac{\pi}{2}]$. Considering $g_1(z) = \frac{f(-z)}{(i+z)^m}$, $g_2(z) = \frac{\overline{f(\bar{z})}}{(i+z)^m}$, $g_3(z) = \frac{\overline{f(-\bar{z})}}{(i+z)^m}$, we have for large σ

$$\int_{\sigma}^{\sigma+1} |f(\rho e^{i\psi}) d\rho \leq C(\sigma+1)^m (\sigma+1)^{\frac{1}{p}},$$

for $\psi \in [0, 2\pi]$. Now by Cauchy's integral formula,

$$|f^{(n)}(0)| \leq n!(2\pi)^{-1} \int_0^{2\pi} |f(\rho e^{i\psi})| \rho^{-n} d\psi.$$

Consequently, for large σ,

$$\begin{aligned} |f^{(n)}(0)| &\leq n!(2\pi)^{-1} \int_0^{2\pi} \left(\int_{\sigma}^{\sigma+1} |f(\rho e^{i\psi})| \rho^{-n} d\rho \right) d\psi \\ &\leq Cn!(2\pi)^{-1} \sigma^{-n} (1+\sigma)^m (1+\sigma)^{\frac{1}{p}}. \end{aligned}$$

So $f^{(n)}(0) = 0$ for n large. Thus f is a polynomial. Then it follows from ii) that f is zero. $\square$

3. Uncertainty principles

Before we can embark on our extension of the L^p analogue, we need the following lemma which is essentially in [5].

Lemma 3.1. *Let $f \in L^1(G) \cap L^2(G)$. Suppose f decays sufficiently rapidly so that $\hat{f}(\pi_{\delta,\lambda})$ makes sense for all $\delta \in \hat{M}$, $\lambda \in \mathcal{A}^*_{\mathbb{C}}$ (see the remark below), and*

$$\hat{f}(\pi_{\delta,\lambda}) = 0 \quad \textit{for all } \delta \in \hat{M},\ \lambda \in \mathcal{A}^*_{\mathbb{C}},$$

then $f = 0$ almost everywhere.

Remark. For a rapidly decreasing integrable function f, even though $\hat{f}(\pi_{\delta,\lambda})$ may fail to make sense as an operator on $L^2(K/M, dk, \mathbf{H}_\delta)$ when $\lambda \in \mathcal{A}^*_{\mathbb{C}}$ (note that, in general, for $\lambda \in \mathcal{A}^*_{\mathbb{C}} \setminus \mathcal{A}^*$, $\pi_{\delta,\lambda}$ is not unitary), we use the notation $\hat{f}(\pi_{\delta,\lambda})$ in the sense of an infinite matrix whose elements are

$$\hat{f}(\pi_{\delta,\lambda})_{m,n} = \int_G f(x)\ \langle \pi_{\delta,\lambda}(x) e_m, e_n \rangle_{\mathbf{H}_{\delta,\lambda}}\ d\mu(x)\ e_m, e_n \in \mathbf{H}_{\delta,\lambda}.$$

For more details we refer to [5].

For each $\delta \in \hat{M}$ let ν_δ denote the density of the Plancherel measure on the set of representations $\{\pi_{\delta,\lambda} : \lambda \in \mathcal{A}^*\}$. Let $\|T\|$ denote the operator norm of T.

With this preparation we proceed to prove the following theorem.

Theorem 3.2. *Let $f : G \to \mathbb{C}$ be a measurable function. Suppose that $1 \leq p \leq \infty$, $1 \leq q < \infty$, and a and b are positive numbers such that $ab \geq \frac{1}{4}$. If*

i) $\int_G \left(|f(x)|\ e^{a|x|_G^2}\ \phi_0(x)^{\frac{2}{p}-1}\right)^p\ d\mu(x)\ <\ \infty$,

ii) $\int_{\mathcal{A}^*}\ e^{qb|\lambda|^2}\ \|\hat{f}(\pi_{\delta,\lambda})\|^q\ \nu_\delta(\lambda)\ d\lambda < \ \infty$ *for all* δ,

then $f = 0$.

Proof. It suffices to prove the case $ab = \frac{1}{4}$. For $\lambda \in \mathcal{A}^*_{\mathbb{C}}$, let $F_\delta^{m,n}(\lambda) = \int_G\ f(x)\ \phi_{\delta,\lambda}^{m,n}(x)\ d\mu(x)$, that is, $F_\delta^{m,n}(\lambda)$ is the (m,n)-th matrix entry of $\hat{f}(\pi_{\delta,\lambda})$. If we can show that $F_\delta^{m,n}(\lambda) = 0$, then it will follow from Lemma 3.1 that $f = 0$. Now,

$$\begin{aligned}
|F_\delta^{m,n}(\lambda)| &\leq \int_{K\bar{\mathcal{A}}_+K} |f(k_1ak_2)|\ e^{\lambda_I^+(\log a)}\ \phi_0(a)\ J(a)\ dk_1\ da\ dk_2 \\
&= \int_{K\bar{\mathcal{A}}_+K} |f(k_1 \exp Hk_2)|\ e^{a\|H\|^2}\ \phi_0(\exp h)^{\frac{2}{p}-1}\ e^{\lambda_I^+(H)}\ e^{-a\|H\|^2} \\
&\qquad \phi_0(\exp H)^{2-\frac{2}{p}}\ J(\exp H)\ dk_1 dH dk_2 \\
&\leq \mathit{Const.}\left(\int_{\bar{\mathcal{A}}^+} (e^{-a\|H\|^2}\ e^{\lambda_I^+(H)}\ e^{-(2-\frac{2}{p})\rho(H)}\ (1+\|H\|)^r)^{p'}\ e^{2\rho(H)} dH\right)^{\frac{1}{p'}} \\
&\qquad\quad \text{(by i) and Proposition 2.1)} \\
&\leq \left(\int_{\bar{\mathcal{A}}^+} e^{-p'a\|H\|^2 + p'\lambda_I^+(H)}\ (1+\|H\|)^{rp'}\ dH\right)^{\frac{1}{p'}} \\
&= \mathit{Const}\ e^{\frac{1}{4a}\|H_{\lambda_I}\|^2}\ \left(\int_{\bar{\mathcal{A}}_+} e^{-p'a\|H - \frac{1}{2a}H_{\lambda_I^+}\|^2}\ (1+\|H\|)^{rp'}\ dH\right)^{\frac{1}{p'}} \\
&\leq \mathit{Const}\ e^{\frac{1}{4a}\|H_{\lambda_I}\|^2}\left(\int_{\mathcal{A}} e^{-p'a\|H\|^2}\ (1+\|H + \frac{1}{2a}H_{\lambda_I^+}\|)^{rp'}\ dH\right)^{\frac{1}{p'}}.
\end{aligned}$$

Hence $|F_\delta^{m,n}(\lambda)| \leq C\ e^{\frac{1}{4a}\|\operatorname{Im}\lambda\|^2}\ (1+\|\mathrm{Im}\lambda\|)^s$. Define $G_\delta(\lambda) = F_\delta^{m,n}(\lambda)\ e^{\frac{1}{4a}\langle\lambda,\lambda\rangle}$ for $\lambda \in \mathcal{A}^*_{\mathbb{C}}$. Then it follows that $G_\delta(\lambda)$ is an entire function in λ and

$$|G_\delta(\lambda)| \leq C\ e^{\frac{1}{4a}\|\operatorname{Re}\lambda\|^2}\ (1+\|\ \mathrm{Im}\lambda\|)^s. \tag{3.1}$$

As $b = \frac{1}{4a}$ and $|F_\delta^{m,n}(\lambda)| \leq \|\hat{f}(\pi_{\delta,\lambda})\|$, from ii) it follows that

$$\int_{\mathcal{A}^*} |G_\delta(\lambda)|^q\ \nu_\delta(\lambda)\ d\lambda < \infty. \tag{3.2}$$

Theorem 3.2 will be proved if we can show that each $\nu_\delta(\lambda)$ satisfies the conditions on Q given in Lemma 2.3. When G is a complex semisimple Lie group it is known that the Plancherel measure is a polynomial in λ and it is easily verified that the conditions of Lemma 2.3 are satisfied. If real-rank $G = 1$, then there are two cases, either rank G >rank K or rank G =rank K. In the first case it is known that for every fixed δ, $\nu_{\delta,\lambda}$ is a polynimial in λ, and in the second case it follows from the explicit description of the Plancherel measure given in [23] (see p. 420) that it satisfies the condition of Lemma 2.3. Thus the theorem is true if real-rank $G = 1$. For the general case, let us assume that δ is the trivial representation of M. Other cases can be similarly dealt with. From [11] we have $\nu_\delta(\lambda) = |c(\lambda)|^{-2} = \prod_{\alpha\in\Delta_{++}} |c_\alpha(\lambda_\alpha)|^{-2}$ (here Δ_{++} is the set of short roots) where each $c_\alpha(\lambda_\alpha)$ is the Plancherel measure of certain rank one subgroup of G. From the explicit expression for the Plancherel measure of the rank one group (see [23]) it is clear that $|c_\alpha(\lambda_\alpha)|^{-2}$ is bounded away from zero whenever $\langle\lambda_\alpha, \alpha_\alpha\rangle_\alpha$ is large enough. Since $\langle\lambda_\alpha, \alpha_\alpha\rangle_\alpha = \langle\lambda, \alpha\rangle$ (see [12]), $|c_\alpha(\lambda_\alpha)|^{-2}$ is bounded away from zero whenever λ is away from the hyperplane $X_\alpha = \{\beta \in \mathcal{A}^* : \langle\beta, \alpha\rangle = 0\}$. Without loss of generality we assume that no α is in the direction of any axis in $\mathcal{A}^*$. Let us write $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $\tilde\lambda = (\lambda_1, \cdots, \lambda_{n-1})$. Then it is clear that for any fixed $\tilde\lambda$ and $N > 0$ the intersection of $\{\lambda : |\langle\lambda, \alpha\rangle| \le N\}$ and the axis $X_n = \{\beta \in \mathcal{A}^* : \beta = (\tilde\lambda, \lambda_n)\}$ lies in a bounded set contained in X_n, say S_α. Now from the above remarks it follows that, for N large, if λ_n is outside a compact set containing $\bigcup_{\alpha\in\Delta_{++}} S_\alpha$, then $|c(\lambda)|^{-2}$ is bounded away from 0. This completes the proof of Theorem 3.2. $\square$

Remark. The above theorem continues to hold even if we replace $\phi_0^{\frac{2}{p}-1}$ by ϕ_0^α where $\alpha \le \frac{2}{p} - 1$. We also observe that for $p < 2$ our result is stronger than the obvious analogue stated in the Introduction.

Now we give an example to show that the index $\frac{2}{p} - 1$ is optimal.

Example. Let $G = SL(2, \mathbb{C})$. Then

$$\mathrm{A} = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} \quad \text{and } \mathcal{A} = \left\{ \mathbf{a}_t = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in \mathbb{R} \right\}.$$

$\lambda \in \mathcal{A}^*$ is identified with $\lambda \in \mathbb{R}$ via $\mathbf{a}_t \mapsto \lambda t$. With this identification the Jacobian of the Haar measure is $\sinh^2 2t$, $|c(\lambda)|^{-2} = \lambda^2$, $\phi_\lambda(t) = \text{Const.}\ \frac{\sin \lambda t}{\lambda \sinh 2t}$ (see [12], [22]) and thus $\phi_0 = \text{Const.}\ \frac{t}{\sinh 2t}$. Also $|\lambda|_{\mathcal{A}^*} = \frac{|\lambda|}{4}$ and $\|a_t\|_G = 4|t|$. Fix an $s > \frac{2}{p} - 1$. We will give an example of a nonzero K-bi-invariant function f such that:

a) $\int_0^\infty |f(a_t)|^p e^{pt^2} \left(\frac{t}{\sinh 2t}\right)^{sp} \sinh^2 2t dt < \infty$,

b) $\int_{\mathcal{A}^*} |\hat f(\phi_\lambda)|^q e^{q\frac{\lambda^2}{4}} \lambda^2 d\lambda < \infty$.

Let $\psi \in C_c^\infty(-\beta, \beta)$ be even. Define $f = \mathbf{A}^{-1}(\psi) * h$ where h is the K-bi-invariant function defined by $\hat h(\lambda) = e^{-\frac{\lambda^2}{4}}$ and $\mathbf{A}$ is the Abel transform (see [13]). f is well defined by the Paley-Weiner theorem, belongs to L^p-Schwartz class for $0 < p \le 2$ (see [1], [9] for L^p-Schwartz class functions) and $\hat f(\lambda) = \tilde\psi(\lambda)\ e^{-\frac{\lambda^2}{4}}$, where $\tilde\psi$ is the Euclidian Fourier transform of ψ. Obviously f satisfies b). Using the spherical Fourier inversion and the explicit description of ϕ_λ we can easily prove that $|f(a_t)| \le c\frac{e^{-t^2}e^{2\beta t}}{\sinh 2t}$ for large t (see [17]). So it is enough to show that for large N

$$I = \int_N^\infty \frac{e^{2p\beta t}}{(\sinh 2t)^p} \left(\frac{t}{\sinh 2t}\right)^{sp} \sinh^2 2t dt < \infty.$$

If $s > 0$, then

$$I \le \int_N^\infty e^{2t\{p\beta - (p+sp-2)\}} t^{sp} dt;$$

as $s > \frac{2}{p} - 1$ we can choose a $\beta > 0$ such that $p\beta - p - sp + 2 < 0$. For such a choice of β, I is finite. If $s \leq 0$, that is, $p \geq 2$, then

$$I \leq \int_N^\infty e^{2t\{p\beta - (p+sp-2)\}} dt.$$

Again due to positivity of $sp + p - 2$ one can choose a positive β making the above integral finite. Notice that in this case if we take $s = 0$, then it follows that the version of the Cowling-Price theorem without the factor of ϕ_0 is false for $p \geq 2$.

Acknowledgement

We are grateful to our teacher Professor A. Sitaram for several useful discussions. The first author wishes to thank N.B.H.M, India, for financial support.

References

[1] Anker J. P. *The Spherical Fourier transform of Rapidly decreasing functions. A simple proof of a characterisation due to Harish-Chandra, Helgason, Trombi and Varadarajan*, J. Funct. Anal. 96 (1991), 331-349. MR **92d:**22008

[2] Astengo F. , Cowling M. G. , Di Blasio B. , Sundari M. *Hardy's uncertainty principle on some Lie groups*, J. London Math. Soc.(2) 62 (2000).no:2 461-472. CMP 2001:01

[3] Cowling M. G. , Price J. F. *Generalizations of Heisenberg's Inequality*, Harmonic Analysis (G. Mauceri, F. Ricci, G. Weiss, eds.), LNM, no. 992, Springer, Berlin, 1983, 443-449. MR **86g:**42002b

[4] Bagchi S. C. , Ray S. *Uncertainty principles like Hardy's theorem on some Lie groups*, J. Austral. Math. Soc. (series A) 65(1998), 289-302. MR **99k:**43001

[5] Cowling M. G. , Sitaram A. , Sundari M. *Hardy's Uncertainty Principle on Semisimple Lie Groups*, Pacific. J. Math. 192 (2000), 293-296. MR **2001c:**22007

[6] Ebata M. , Eguchi M. , Koizumi S. , Kumahara K. *A generalisation of the Hardy theorem to semisimple Lie groups*, Proc. Japan. Acad. , Ser. A, Math. Sci. 75 (1999), 113-114. MR **2001c:**22008

[7] Eguchi M. , Koizumi S. , Kumahara K. *L^p version of the Hardy theorem for motion groups*, J. Austral. Math. Soc. (series A), 68 (2000), 55-67. MR **2001f:**43005

[8] Folland G. B. , Sitaram A. *The Uncertainty Principle: A Mathematical Survey*, J. Fourier Anal. Appl. 3 (1997), 207-238. MR **98f:**42006

[9] Gangolli R. , Varadarajan V. S. *Harmonic Analysis of Spherical functions on Real Reductive Groups* (Springer-Verlag, 1988). MR **89m:**22015

[10] Hardy G. *A theorem concerning Fourier transforms*, J. London Math. Soc. 8 (1933), 227-231.

[11] Harish-Chandra *On the theory of the Eisenstein integral*, Harish-Chandra collected papers, Vol. 4, Narosa Publishing House, New Delhi, 1985.

[12] Helgason S. *Groups and Geometric Analysis. Integral Geometry, Invariant Differential operators and Spherical functions*, Academic Press, Orlando, 1984. MR **86c:**22017

[13] Helgason S. *Geometric Analysis on Symmetric spaces*, AMS, Mathematical Surveys and Monographs, Vol 39, 1994. MR **96h:**43009

[14] Hörmander L. *A uniqueness theorem of Beurling for Fourier transform pairs*, Arkiv för Matematik, 29 (2)(1991), 237-240. MR **93b:**42016

[15] Knapp A. W. *Representation Theory of Semisimple Lie Groups, an overview based on examples*, Princeton Univ. Press, Princeton, NJ, 1986. MR **87j:**22022

[16] Kaniuth E. , Kumar A. *Hardy's theorem for simply connected nilpotent Lie groups*, Preprint, 2000.

[17] Narayanan E. K., Ray S. *Hardy's theorem on Symmetric spaces of noncompact type*, Preprint, 2000.

[18] Ray S. *Uncertainty principles on two step nilpotent Lie groups*, to appear in Proc. Ind. Acad. Sci.

[19] Sarkar R. P. Revisiting Hardy's theorem on $SL(2, \mathbb{R})$, Preprint, 2000.

[20] Sitaram A., Sundari M. *An analogue of Hardy's theorem for very rapidly decreasing functions on semisimple Lie groups*, Paciffic J. Math. 177 (1997), 187-200. MR **99a:**22018

[21] Sengupta J. *An analogue of Hardy's theorem on semisimple Lie groups* Proc. AMS. 128 (2000), no:8, 2493-2499. MR **2000k:**43004

[22] Terras A. *Harmonic Analysis on Symmetric Spaces and Applications*, Vols. 1 and 2, Springer-Verlag, 1988. MR **87f:**22010; MR **89k:**22017

[23] Warner G. , *Harmonic analysis on Semisimple Lie groups*, Vol. 2, Springer-Verlag, 1972. MR **58:**16980

Stat-Math Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore 560 059, India

E-mail address: naru@isibang.ac.in

Current address: Department of Mathematics & Computer Sciences, Bar-Ilan University, 52900 Ramat-Gan, Israel

E-mail address: naru@macs.biu.ac.il

Stat-Math Unit, Indian Statistical Institute, 203 B. T. Road, Calcutta 700035, India

E-mail address: res9601@www.isical.ac.in

Current address: Department of Mathematics, Indian Institute of Technology, Kanpur, U.P.-208016, India

E-mail address: skray@iitk.ac.in