

## EXPLICIT CONTINUED FRACTIONS WITH EXPECTED PARTIAL QUOTIENT GROWTH

TAKESHI OKANO

(Communicated by David E. Rohrlich)

ABSTRACT. For  $0 < x < 1$  let  $[0, a_1(x), a_2(x), \dots]$  be the continued fraction expansion of  $x$ . Write

$$L_N(x) = \max_{1 \leq n \leq N} a_n(x).$$

We construct some numbers  $x$ 's with

$$\lim_{N \rightarrow \infty} \inf N^{-1} L_N(x) \log \log N = 1/\log 2.$$

### 1. INTRODUCTION

For  $0 < x < 1$  let  $[0, a_1(x), a_2(x), \dots]$  be the continued fraction expansion of  $x$ . Write

$$L_N(x) = \max_{1 \leq n \leq N} a_n(x).$$

W. Philipp [1] proved the following theorem: *For almost all  $x$*

$$\lim_{N \rightarrow \infty} \inf N^{-1} L_N(x) \log \log N = 1/\log 2.$$

Previously no explicit numbers were known that satisfied

$$\lim_{N \rightarrow \infty} \inf N^{-1} L_N(x) \log \log N = 1/\log 2.$$

In this paper we give a method that may be used to construct examples of such numbers.

### 2. MAIN RESULT

**Theorem.** *Let  $k$ ,  $K$  and  $M_0$  be arbitrary fixed positive numbers with  $k > 1$  and  $K \geq 2$ . Let  $x = [0, a_1, a_2, \dots]$  be an irrational number such that*

$$\max\{a_n \mid (\log k)^2 M K^{k^M} < n \leq (\log k)^2 (M+1) K^{k^M}\} = [K^{k^{M+1}}]$$

and

$$\max\{a_n \mid (\log k)^2 (M+1) K^{k^M} < n \leq (\log k)^2 (M+1) K^{k^{M+1}}\} \leq [K^{k^{M+1}}],$$

---

Received by the editors January 2, 2001.

2000 *Mathematics Subject Classification.* Primary 11A55; Secondary 11K50.

*Key words and phrases.* Continued fractions, measure theory.

where  $M$  is a positive integer with  $M > M_0$  and  $[K^{k^{M+1}}]$  is the integral part of  $K^{k^{M+1}}$ . Then

$$\liminf_{N \rightarrow \infty} N^{-1} L_N(x) \log \log N = 1/\log k.$$

*Proof.* Put  $N_0 = [(\log k)^2 M_0 K^{k^{M_0}}]$  and choose  $M' > M_0$  such that  $L_{N_0}(x) \leq [K^{k^{M'+1}}]$ . Let  $M$  be any integer greater than  $M'$ . We now consider two cases.

*Case 1.*  $(\log k)^2 M K^{k^M} < N \leq (\log k)^2 (M + 1) K^{k^M}$ .

Suppose that  $M$  is a sufficiently large integer. In this case, we find that  $L_N(x) \geq [K^{k^M}]$  and

$$\log \log N = M \log k + O(1) \quad (M \rightarrow \infty).$$

Hence

$$N^{-1} L_N(x) \log \log N \geq \frac{[K^{k^M}](M \log k + O(1))}{(\log k)^2 (M + 1) K^{k^M}} \quad (M \rightarrow \infty).$$

*Case 2.*  $(\log k)^2 (M + 1) K^{k^M} < N \leq (\log k)^2 (M + 1) K^{k^{M+1}}$ .

In this case, we find that  $L_N(x) = [K^{k^{M+1}}]$  and

$$\log \log N = M \log k + O(1) \quad (M \rightarrow \infty).$$

Hence

$$\begin{aligned} & \inf\{N^{-1} L_N(x) \log \log N \mid (\log k)^2 (M + 1) K^{k^M} < N \leq (\log k)^2 (M + 1) K^{k^{M+1}}\} \\ &= \frac{[K^{k^{M+1}}](M \log k + O(1))}{(\log k)^2 (M + 1) K^{k^{M+1}}} \quad (M \rightarrow \infty). \end{aligned}$$

Then by these two cases, we can deduce that

$$\liminf_{N \rightarrow \infty} N^{-1} L_N(x) \log \log N = 1/\log k.$$

This completes the proof.

### 3. EXAMPLE

Let  $x = [0, a_1, a_2, \dots]$  be an irrational number. Let  $K$  and  $M$  be as in the Theorem, and let  $k = 2$  in the Theorem. Define

$$\mu(M) := [(\log 2)^2 M K^{2^M}] + 1.$$

By the Theorem, we define  $a_n$  as follows:

$$a_n := \begin{cases} [K^{2^{M+1}}] & \text{if } n = \mu(M) \text{ for } M \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Now putting  $K = 2$ , we find that  $\mu(1) = 2, \mu(2) = 16, \mu(3) = 369, \mu(4) = 125948, \dots$ . Hence, we can construct  $x$  as follows:

$$\begin{aligned} x &= [0, a_1, a_2, a_3, \dots, a_{15}, a_{16}, a_{17}, \dots, a_{368}, a_{369}, a_{370}, \dots, a_{125947}, a_{125948}, a_{125949}, \dots] \\ &= [0, 1, 2^{2^2}, 1, \dots, 1, 2^{2^3}, 1, \dots, 1, 2^{2^4}, 1, \dots, 1, 2^{2^5}, 1, \dots]. \end{aligned}$$

We can easily see that  $x$  is an example of the numbers defined in the Theorem.

### ACKNOWLEDGMENTS

The author thanks the referee for many helpful comments and suggestions.

## REFERENCES

- [1] W. Philipp, *A conjecture of Erdős on continued fractions*, Acta. Arith. **28** (1976), 379–386.  
MR **52**:8069

DEPARTMENT OF MATHEMATICS, SAITAMA INSTITUTE OF TECHNOLOGY, OKABE-MACHI, SAITAMA  
369-0293, JAPAN

*E-mail address:* `okano@sit.ac.jp`