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LINEARITY OF DIMENSION FUNCTIONS FOR SEMILINEAR G-SPHERES

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Dedicated to the memory of Professor Katsuo Kawakubo

ABSTRACT. In this paper, we show that the dimension function of every semilinear G-sphere is equal to that of a linear G-sphere for finite nilpotent groups G of order $p^n q^m$, where p, q are primes. We also show that there exists a semilinear G-sphere whose dimension function is not virtually linear for an arbitrary nonsolvable compact Lie group G.

1. INTRODUCTION

Let G be a compact Lie group and V an orthogonal G-representation. The unit sphere $S(V)$ of V is a closed G-manifold whose H-fixed point set is a sphere (or possibly the empty set) for every closed subgroup H of G. We call this G-manifold a linear G-sphere. A *semilinear G-sphere* is defined as a natural generalization of a linear G-sphere, namely it is a (smooth) closed G-manifold such that the H-fixed point set is a homotopy sphere or the empty set for every closed subgroup H of G.

Many authors studied linear G-spheres, in particular K. Kawakubo [9] studied the G-homotopy classification of linear G-spheres for compact abelian groups, and T. tom Dieck [3] studied these spheres for finite p-groups.

T. tom Dieck and T. Petrie [6] introduced the notion of G-homotopy representations and developed the (stable) homotopy theory of G-homotopy representations. E. Laitinen [10] developed the unstable theory of G-homotopy representations. The G-homotopy representation X is defined as a G-CW complex satisfying the following properties (1)–(3):

(1) For every closed subgroup H, the H-fixed point set X^H is homotopy equivalent to a sphere $S^{\underline{n}(H)-1}$, where we define the (-1)-dimensional sphere S^{-1} as the empty set.
(2) $\underline{n}(H) - 1 = \dim X^H$ for every closed subgroup H. (We set $\dim \emptyset = -1$.)
(3) X has finitely many orbit types.

Note that a semilinear G-sphere is an example of a G-homotopy representation since any (smooth) G-manifold has a G-CW complex structure. We say that a

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G-homotopy representation X is linear if X is G-homotopy equivalent to a linear G-sphere.

There exist many nonlinear G-homotopy representations for groups G. (See for example [16], [17].)

For any G-homotopy representation X, we define the integer-valued function $\operatorname{Dim} X$ on the set of closed subgroups by setting $\operatorname{Dim} X(H) = \dim X^H + 1$ for $H \leq G$. This function is called the *dimension function* of X. Clearly the dimension function is a G-homotopy invariant, and the additivity of dimension functions holds, i.e., $\operatorname{Dim} X * Y = \operatorname{Dim} X + \operatorname{Dim} Y$ for G-homotopy representations, where $*$ means join. (Note that $X * Y$ is also a G-homotopy representation up to G-homotopy.) Dimension functions of G-homotopy representations are studied by [4] and [2] for finite groups and by [1] for compact Lie groups. We say that the dimension function $\underline{n}$ is *linear* if $\underline{n} = \operatorname{Dim} S(V)$ for some G-representation V, and that $\underline{n}$ is *virtually linear* (or stably linear) if $\underline{n} = \operatorname{Dim} S(V) - \operatorname{Dim} S(W)$ for some V and W.

The purpose of this paper is to study the dimension functions of semilinear G-spheres from the viewpoint of linearity.

For certain groups G, it is known that a semilinear G-sphere with nonlinear dimension function is constructed by the equivariant surgery. (See Remark 1.3 below.) On the other hand, the following result is known for p-groups.

Theorem 1.1 ([7], [4])**.** *For any p-group P, every P-homotopy representation has a linear dimension function.*

Our first result is the following.

Theorem A. *Let G be a finite nilpotent group of order $p^n q^m$, where p, q are primes. Then every semilinear G-sphere has a linear dimension function.*

Remark 1.2. Theorem A does not hold for G-homotopy representations unless G is a p-group.

Remark 1.3. Further generalization of Theorem A cannot be expected, because the following results are known.

(1) There exists a semilinear C_{pqr}-sphere with nonlinear dimension function for cyclic groups C_{pqr} of order pqr, where p, q, r are distinct odd primes ([14], [11]).
(2) There exists a semilinear $Z_{p,q}$-sphere with nonlinear dimension function for metacyclic groups $Z_{p,q}$ of order pq, where p, q are distinct odd primes and $q|(p-1)$ ([13]).
(3) There exists a semilinear S^1-sphere with nonlinear dimension function for the circle group S^1 ([15]).

We also show the following result on nonlinearity of dimension functions in Section 4.

Theorem B. *Let G be an arbitrary nonsolvable compact Lie group. Then there exists a semilinear G-sphere whose dimension function is not virtually linear.*

2. The case of C_{pq}

We first note the following fact.

Lemma 2.1. *Let G be a compact Lie group. If a semilinear G-sphere Σ has a G-fixed point, then Σ is linear, i.e., G-homotopy equivalent to a linear G-sphere, and in particular its dimension function is linear.*

Proof. Let x be a G-fixed point of Σ. Take a (small) G-invariant neighborhood U of x which is G-diffeomorphic to $\operatorname{Int} D(V)$, where V is the tangential representation at x. Then the projection $\Sigma \to \Sigma/(\Sigma - U) \cong S(V \oplus \mathbb{R})$ is a G-homotopy equivalence by the equivariant Whitehead theorem. □

In this section, we show the following special case of Theorem A.

Proposition 2.2. *Let G be a cyclic group C_{pq} of order pq, where p, q are primes. Then every semilinear G-sphere has a linear dimension function.*

Proof. In case $p = q$, by Theorem 1.1, every semilinear G-sphere has a linear dimension function. We may assume $p > q$. By Lemma 2.1, we may also assume that the action is G-fixed point free. Let Σ be a semilinear C_{pq}-sphere without a C_{pq}-fixed point, and let $\underline{n}$ be the dimension function of Σ.

We first show the inequality $\underline{n}(C_p) + \underline{n}(C_q) \leq \underline{n}(1)$. Put $M := \Sigma^{C_q}$, which is a homotopy sphere with free $C_p(\cong C_{pq}/C_q)$-action. Put $N := \Sigma - N(\Sigma^{C_p})$, where $N(\Sigma^{C_p})$ is an equivariant open tubular neighborhood of Σ^{C_p}. Then N is considered as a (compact) C_p-manifold with free C_p-action and its homology group $H_*(N;\mathbb{Z})$ is isomorphic to $H_*(S^{\underline{n}(1)-\underline{n}(C_q)-1};\mathbb{Z})$ by the Alexander duality. Clearly the natural inclusion $i : M \to N$ is a C_p-map. If necessary, by taking the join of $S(W)$ and N, $N * S(W)$ is simply connected and $\dim N * S(W) \geq 3$, where W is a free C_p-representation of dimension $m > 2$. Then $N * S(W)$ is a C_p-CW complex which is (nonequivariantly) homotopy equivalent to $S^{\underline{n}(C_q)+m-1}$. By Proposition 5.9 of [6], one can see that $N * S(W)$ is C_p-homotopy equivalent to a free C_p-homotopy representation X of dimension $\underline{n}(1) - \underline{n}(C_q) + m - 1$, and $M * S(W)$ is also a free C_p-homotopy representation of dimension $\underline{n}(C_p) + m - 1$. Let $f : N * S(W) \to X$ be a C_p-homotopy equivalence. Then the composite $f \circ (i * id) : M * S(W) \to N * S(W) \to X$ is a C_p-map between free C_p-homotopy representations. By the Borsuk-Ulam theorem for free C_p-actions (cf. [10, Proposition 1.14]), we obtain $\underline{n}(C_p) + m - 1 \leq \underline{n}(1) - \underline{n}(C_q) + m - 1$, and hence $\underline{n}(C_p) + \underline{n}(C_q) \leq \underline{n}(1)$.

Next, since $C_{pq/r} = C_{pq}/C_r$ ($r = p$ or q) acts freely on the homotopy sphere Σ^{C_r}, $\underline{n}(C_r) = \dim \Sigma^{C_r} + 1$ is even unless $pq/r = 2$. Then there exists a free $C_{pq/r}$-representation V_r such that $\dim V_r = \underline{n}(C_r)$. Via the projection $C_{pq} \to C_{pq/r}$, we regard V_r as a C_{pq}-representation with kernel C_r. Put $k := \underline{n}(1) - \underline{n}(C_p) - \underline{n}(C_q) \geq 0$. Then k is even. In fact, since $C_p(\leq C_{pq})$ (p:odd prime by the assumption $p > q$) acts semi-freely on Σ, it follows that $\underline{n}(1) - \underline{n}(C_p)$ is even, and $\underline{n}(C_q)$ is even as mentioned above. Therefore k is even and so we can take a free C_{pq}-representation U of dimension k. Put $V = V_p \oplus V_q \oplus U$. One can easily see that $\operatorname{Dim} \Sigma = \operatorname{Dim} S(V)$. □

3. The general case

Throughout this section, G is a finite group. We first recall well-known results on linear dimension functions. Let $F(G)$ be the free module consisting of integer-valued functions on the set of subgroups of G. Let us denote by $D^\ell_+(G)$ the set of linear dimension functions for G, and by $D^\ell(G)$ the submodule of $F(G)$ generated by $D^\ell_+(G)$. By the additivity of dimension functions, any function in $D^\ell(G)$ has a form of $\underline{n} - \underline{n}'$, where $\underline{n}$, $\underline{n}'$ are linear dimension functions, and hence $D^\ell(G)$ is the free module consisting of virtually linear dimension functions for G.

We say that G-representations V and W are G-Galois conjugate if there exists an integer k prime to $|G|$ such that $\chi_V(g) = \chi_W(g^k)$ for all $g \in G$, where χ_V denotes the character of V. Note that if V and W are G-Galois conjugate, then

$\operatorname{Dim} S(V) = \operatorname{Dim} S(W)$ and $\operatorname{Ker} V = \operatorname{Ker} W$. Let $V_1, V_2, \ldots, V_r$ be the complete representative system of G-Galois conjugate classes of irreducible G-representations, and $\mathcal{B}(G)$ the set of dimension functions of $S(V_i)$.

Proposition 3.1 ([12], [5]). *The set $\mathcal{B}(G)$ is a $\mathbb{Z}$-basis of $D^\ell(G)$.*

We call $\mathcal{B}(G)$ the *standard basis* of $D^\ell(G)$. By Proposition 3.1, any virtually linear function $\underline{n}$ is presented as

$$\underline{n} = \sum_{\underline{m} \in \mathcal{B}(G)} a(\underline{m}, \underline{n})\underline{m}.$$

We call the integer $a(\underline{m}, \underline{n})$ the *multiplicity* of $\underline{m}$ in $\underline{n}$. We also put

$$\mathcal{B}(G, \underline{n}) = \{\, \underline{m} \in \mathcal{B}(G) \mid a(\underline{m}, \underline{n}) \neq 0 \,\}.$$

We define $\operatorname{Ker} \underline{n}$ as $\operatorname{Ker} V$ if $\underline{n} = \operatorname{Dim} S(V)$. For any $\underline{n} \in D^\ell(G)$, we define a function $\underline{n}^K$ by setting $\underline{n}^K(H/K) = \underline{n}(H)$ for $H/K \leq G/K$, where K is any normal subgroup of G. Then $\underline{n}^K \in D^\ell(G/K)$ because $\underline{n}^K = \operatorname{Dim} S(V^K) - \operatorname{Dim} S(W^K)$ if $\underline{n} = \operatorname{Dim} S(V) - \operatorname{Dim} S(W)$. We shall show the following lemmas.

Lemma 3.2. *Let K be an arbitrary normal subgroup of G. If $\underline{m} \in \mathcal{B}(G)$ and $\operatorname{Ker} \underline{m} \not\geq K$, then $\underline{m}^K = 0$ (zero-function).*

Proof. Let $\underline{m} = \operatorname{Dim} S(V)$, where V is an irreducible G-representation. Then V^K is regarded as a G-subrepresentation of V, and $V^K \neq V$ by $\operatorname{Ker} \underline{m} \not\geq K$. Since V is irreducible, we have $V^K = 0$, and hence $\underline{m}^K = 0$. □

Put $B = \{\, \underline{m}^K \mid \underline{m} \in \mathcal{B}(G)$ such that $\operatorname{Ker} \underline{m} \geq K \,\}$.

Lemma 3.3. *The set B is a part of the standard basis $\mathcal{B}(G/K)$.*

Proof. If V is an irreducible G-representation whose kernel contains K, then V^K ($= V$) is an irreducible G/K-representation. It is easy to see that irreducible G-representations V and W are G-Galois conjugate if and only if V^K and W^K are G/K-Galois conjugate. This shows the desired result. □

We now prove Theorem A. Since G is nilpotent of order $p^n q^m$, we may put $G = G_p \times G_q$, where G_p [resp. G_q] is a p- [resp. q-] group. Let $\underline{n}$ be the dimension function of a semilinear G-sphere. Since all dimension functions of G-homotopy representations are virtually linear if (and only if) G is nilpotent ([6], Proposition 10.23), $\underline{n}$ is presented as a linear combination of $\underline{m} \in \mathcal{B}(G)$; $\underline{n} = \sum_{\underline{m} \in \mathcal{B}(G)} a(\underline{m}, \underline{n})\underline{m}$. In order to prove Theorem A, it suffices to show that all $a(\underline{m}, \underline{n})$ are nonnegative.

The proof is done by induction on the order of G. For any $\underline{m} \in \mathcal{B}(G, \underline{n})$ with nontrivial kernel K, it is seen that $a(\underline{m}, \underline{n}) = a(\underline{m}^K, \underline{n}^K)$ by Lemmas 3.2 and 3.3, and hence $a(\underline{m}, \underline{n})$ is nonnegative by the inductive assumption.

We have to show that $a(\underline{m}, \underline{n})$ is nonnegative when $\operatorname{Ker} \underline{m} = 1$. By Theorem 1.1, Theorem A holds for p-groups, and so we assume that p and q are distinct primes and that G_p and G_q are nontrivial.

The proof is divided into two cases: (1) G has a normal subgroup A isomorphic to $C_p \times C_p$ or $C_q \times C_q$, and (2) otherwise.

Case (1): If there is no $\underline{m} \in \mathcal{B}(G, \underline{n})$ with trivial kernel, there is nothing to do. We assume that there is an $\underline{m} \in \mathcal{B}(G, \underline{n})$ with trivial kernel, namely G has a faithful irreducible G-representation. Let $\underline{m} = \operatorname{Dim} S(U)$, where U is a faithful irreducible G-representation. By assumption, G has a normal subgroup A isomorphic to $C_p \times$

C_p or $C_q \times C_q$. We may assume that A is isomorphic to $C_p \times C_p$. Let A_0, A_1, $\dots$, A_p be all subgroups of order p in A. By representation theory, the following is known.

Proposition 3.4. *Under the above situation,*

(1) *One of the A_i, say A_0, is normal in G and the others have the same normalizer $N = N_G(A_i)$ of index p in G. Furthermore A_0 is in the center $Z(G)$ of G, and N is normal in G and contains A.*
(2) *The restriction $\mathrm{Res}_N U$ is the direct sum of irreducible N-representations $U_1, \dots, U_p$ such that $\mathrm{Ker}\,\mathrm{Res}_A V_i = A_i$ $(1 \le i \le p)$.*
(3) *The irreducible summands U_i and U_j are not N-Galois conjugate for $i \ne j$.*
(4) *Let W be another faithful irreducible G-representation and let $\mathrm{Res}_N W = W_1 \oplus \cdots \oplus W_p$ as in (2). If U and W are not G-Galois conjugate, then U_i and W_j are not N-Galois conjugate for every i and j.*

Proof. Statements (1) and (2) are shown by Proposition 5.12 of [5] and its proof. Statements (3) and (4) are shown by Proposition 5.6 of [18]. □

For any $\underline{n} \in D^\ell(G)$ and any subgroup H of G, we define the restriction $r_H \underline{n} \in D^\ell(H)$ by setting $r_H \underline{n}(L) = \underline{n}(L)$ for $L \le H$. Put $\underline{m}_i = \mathrm{Dim}\, S(U_i) \in \mathcal{B}(N)$ $(1 \le i \le p)$. By Proposition 3.4, $r_N \underline{m} = \sum_i \underline{m}_i$.

Lemma 3.5. *Under the above situation, $a(\underline{m}, \underline{n}) = a(\underline{m}_i, r_N \underline{n})$ $(1 \le i \le p)$.*

Proof. Since $\underline{n} = \sum_{\underline{\ell} \in \mathcal{B}(G,\underline{n})} a(\underline{\ell}, \underline{n}) \underline{\ell}$, it follows from the additivity of the restriction that

$$a(\underline{m}_i, r_N \underline{n}) = \sum_{\underline{\ell} \in \mathcal{B}(G,\underline{n})} a(\underline{\ell}, \underline{n}) a(\underline{m}_i, r_N \underline{\ell}). \tag{3.1}$$

We first consider the case where $\underline{\ell} \in \mathcal{B}(G, \underline{n})$ has a nontrivial kernel K. We note that the center $Z(G) = Z(G_p) \times Z(G_q)$ of G is cyclic since G has a faithful irreducible G-representation (see for example [8]). Furthermore any (nontrivial) p-group P has a nontrivial center, and any nontrivial normal subgroup of P intersects the center $Z(P)$ nontrivially. Since $N = N_{G_p}(A_i) \times G_q$, $\mathrm{Ker}\, r_N \underline{\ell} = N \cap K$ is a nontrivial normal subgroup of G, and hence $\mathrm{Ker}\, \underline{s} \ge N \cap K$ for any $\underline{s} \in \mathcal{B}(N, r_N \underline{\ell})$. On the other hand

$$r_{N/N\cap K}(\underline{m}^{N\cap K}) = (r_N \underline{m})^{N\cap K} = \sum_i \underline{m}_i^{N\cap K}.$$

Since $\underline{m}$ has the trivial kernel, it follows from Lemma 3.2 that $m^{N\cap K} = 0$, and hence $\underline{m}_i^{N\cap K} = 0$ for $1 \le i \le p$. Thus we have $\mathrm{Ker}\, \underline{m}_i \not\ge N \cap K$ and $\mathrm{Ker}\, \underline{s} \ne \mathrm{Ker}\, \underline{m}_i$. This shows that $a(\underline{m}_i, r_N \underline{\ell}) = 0$ when $\mathrm{Ker}\, \underline{\ell} \ne 1$.

Next, if $\underline{\ell} \in \mathcal{B}(G, \underline{n})$ has the trivial kernel, by Proposition 3.4 (3) and (4),

$$a(\underline{m}_i, r_N \underline{\ell}) = \begin{cases} 1 & \text{if } \underline{\ell} = \underline{m}, \\ 0 & \text{if } \underline{\ell} \ne \underline{m}. \end{cases}$$

Thus we obtain $a(\underline{m}, \underline{n}) = a(\underline{m}_i, r_N \underline{n})$ $(1 \le i \le p)$ from equation (3.1). □

By Lemma 3.5 and the induction hypothesis, we see that $a(\underline{m}, \underline{n})$ is nonnegative for any $\underline{m}$ with trivial kernel. Thus the inductive argument in Case (1) is completed.

Case (2): Assume $p < q$. In this case, as is well known (see [8]), G_p is cyclic, dihedral, quaternion or semi-dihedral, where the last three cases occur only in case $p = 2$, and G_q is cyclic. All faithful irreducible G_r-representations ($r = p, q$) are G_r-Galois conjugate by investigating the character tables, and hence all faithful irreducible G-representations are G-Galois conjugate. Therefore $\mathcal{B}(G)$ contains only one dimension function with trivial kernel. We denote by $\underline{u}$ this dimension function. Let C be a cyclic subgroup $C = C_p \times C_q$ of order pq in the center $Z(G)$. Then $\mathcal{B}(C)$ consists of four elements $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$, $\underline{\ell}_4$ whose kernels are 1, C_p, C_q, C respectively. We have to show that $a(\underline{u}, \underline{n})$ is nonnegative.

Lemma 3.6. *Under the above situation,* $a(\underline{\ell}_1, r_C\underline{n}) = a(\underline{u}, \underline{n})a(\underline{\ell}_1, r_C\underline{u})$.

Proof. The dimension function $\underline{n}$ is presented as $\underline{n} = \sum_{\underline{m}\in\mathcal{B}(G,\underline{n})} a(\underline{m}, \underline{n})\underline{m}$, and furthermore $r_C\underline{n} = \sum_{\underline{m}\in\mathcal{B}(G,\underline{n})} a(\underline{m}, \underline{n})r_C\underline{m}$ and $r_C\underline{m} = \sum_i a(\underline{\ell}_i, r_C\underline{m})\underline{\ell}_i$. If $\underline{m} \in \mathcal{B}(G, \underline{n})$ has a nontrivial kernel K, then $\operatorname{Ker} r_C\underline{m} = C \cap K$ is nontrivial since K is a nontrivial normal subgroup of G and $C = C_p \times C_q$ is contained in the center $Z(G)$. (Note that $Z(G_r)$ ($r = p, q$) is a cyclic r-group.) Hence we have $a(\underline{\ell}_1, r_C\underline{m}) = 0$ by Lemma 3.2, and so

$$\begin{aligned} a(\underline{\ell}_1, r_C\underline{n}) &= \sum_{\underline{m}\in\mathcal{B}(G,\underline{n})} a(\underline{m}, \underline{n})a(\underline{\ell}_1, r_C\underline{m}) \\ &= a(\underline{u}, \underline{n})a(\underline{\ell}_1, r_C\underline{u}). \end{aligned}$$

□

Lemma 3.7. *The number* $a(\underline{\ell}_1, r_C\underline{u})$ *is positive.*

Proof. Suppose $r_C\underline{u} = \sum_{i=1}^4 b_i\underline{\ell}_i$, where $b_i = a(\underline{\ell}_i, r_C\underline{u})$. Since $\operatorname{Ker}\underline{u} = 1$, $(r_C\underline{u})^C = r_{C/C}\underline{u}^C = 0$ and $(r_C\underline{u})^{C_r} = r_{C/C_r}\underline{u}^{C_r} = 0$ for $r = p$ and q. This shows $b_2 = b_3 = b_4 = 0$, and hence $r_C\underline{u} = b_1\underline{\ell}_1$. Since $r_C\underline{u}(1)$ and $\underline{\ell}_1(1)$ are positive, it follows that $b_1 = a(\underline{\ell}_1, r_C\underline{u})$ is positive. □

By Lemmas 3.6 and 3.7 and the inductive assumption, we obtain that $a(\underline{u}, \underline{n})$ is nonnegative. Thus the inductive argument in Case (2) is completed and Theorem A has been proved. □

4. Semilinear G-spheres with nonlinear dimension functions

In this section we prove Theorem B. Let G be a compact Lie group. For the sake of the proof, we shall show the following.

Lemma 4.1. *Let* $\underline{n}$ *be a virtually linear dimension function. If* $\underline{n}(C) = 0$ *for every* (*finite*) *cyclic subgroup* C *of* G*, then* $\underline{n}$ *is a zero-function.*

Proof. Let $\underline{n} = \operatorname{Dim} S(V_1) - \operatorname{Dim} S(V_2)$. In the case where G is finite, by representation theory, $\underline{n}(H) = \dim V_1^H - \dim V_2^H = \sum_{g\in H}(\chi_{V_1}(g) - \chi_{V_2}(g))$, where χ_{V_i} is the character of V_i. We also note

$$\sum_{g\in H}(\chi_{V_1}(g) - \chi_{V_2}(g)) = \sum_{\substack{C\leq H \\ C:\text{cyclic}}} \sum_{g\in C^*}(\chi_{V_1}(g) - \chi_{V_2}(g)), \tag{4.1}$$

where C^* denotes the set of generators of C. We set, for any cyclic subgroup C,

$$\begin{aligned} h(C) &= \sum_{g\in C}(\chi_{V_1}(g) - \chi_{V_2}(g)), \\ k(C) &= \sum_{g\in C^*}(\chi_{V_1}(g) - \chi_{V_2}(g)). \end{aligned}$$

From equation (4.1), we obtain

$$h(C) = \sum_{D \leq C} k(D). \tag{4.2}$$

Applying the Möbius inversion to (4.2), we obtain

$$k(C) = \sum_{D \leq C} \mu(|C/D|)h(D),$$

where μ is the Möbius function. Since, by assumption, $h(D) = \underline{n}(D) = 0$ for any cyclic subgroup D, we obtain $k(C) = 0$ for any cyclic subgroup C. It follows from (4.1) that $\underline{n}(H) = 0$.

We next consider the case where G is a compact Lie group. Since G_0 is normal, $V_i^{G_0}$ is considered as a G-subrepresentation of V_i. We denote by $V_{i\,G_0}$ the orthogonal complement of $V_i^{G_0}$ in V_i. Since $V_{1\,G_0}$ and $V_{2\,G_0}$ are isomorphic as G-representations by Theorem 1.1 of [20], we have $\underline{n} = \operatorname{Dim} S(V_1^{G_0}) - \operatorname{Dim} S(V_2^{G_0})$, and also $\underline{n}^{G_0} = \operatorname{Dim} S(V_1^{G_0}) - \operatorname{Dim} S(V_2^{G_0})$ by considering $V_i^{G_0}$ as a G/G_0-representation. Let K/G_0 be any cyclic subgroup of G/G_0, $g \in K$ an element representing a generator of K/G_0, and T the closure of the subgroup generated by g. Then $K/G_0 = TG_0/G_0$, and $\underline{n}^{G_0}(K/G_0) = \underline{n}(T)$. Since $\bigcup_{C \leq T, C:\text{finite cyclic}} C$ is dense in T and a G-representation has only finitely many isotropy types, there exists a finite cyclic subgroup C such that $\underline{n}(T) = \underline{n}(C)$. By assumption, $\underline{n}(C) = 0$ and hence $\underline{n}^{G_0}(K/G_0) = 0$. Since this lemma has been proved for finite groups, by applying to $\underline{n}^{G_0}$, we obtain $\underline{n}(H) = \underline{n}^{G_0}(HG_0/G_0) = 0$ for every closed subgroup H. □

By [19], there exists a smooth G-action on a disk D such that D^H is a disk if H is solvable, and the empty set if H is nonsolvable. In particular D has no G-fixed point if G is nonsolvable. We call D a *quasilinear* G-disk. Clearly $D \times D^n$ is also a quasilinear G-disk, where D^n is an n-dimensional disk with trivial G-action, and the boundary $\partial(D \times D^n)$ is a semilinear G-sphere. We put $\Sigma_n = \partial(D \times D^n)$.

The following leads to the proof of Theorem B.

Theorem 4.2. *Let G be a nonsolvable compact Lie group. Then the dimension function of Σ_n is not virtually linear except at most one n.*

Proof. Suppose that $\operatorname{Dim} \Sigma_r$ and $\operatorname{Dim} \Sigma_s$ are virtually linear for two distinct numbers r, s $(r < s)$. Put $\underline{n} = \operatorname{Dim} \Sigma_r$, $\underline{m} = \operatorname{Dim} \Sigma_s$ and $\underline{\ell} = \operatorname{Dim} S(\mathbb{R}^{s-r})$. Then $\underline{u} := \underline{m} - \underline{n} - \underline{\ell}$ is also a virtually linear dimension function. For any finite cyclic subgroup C, $\underline{n}(C) = \operatorname{Dim} \partial D(C) + r$, $\underline{m}(C) = \operatorname{Dim} \partial D(C) + s$ and $\underline{\ell}(C) = s - r$. Therefore $\underline{u}(C) = 0$. By Lemma 4.1, we have $\underline{u} = 0$. On the other hand, since Σ_ℓ^G $(\ell = r,\, s)$ is empty, we have $\underline{u}(G) = s - r \neq 0$. This is a contradiction. □

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