ON THE PREDICTABILITY OF DISCRETE DYNAMICAL SYSTEMS

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Abstract. Let \( X \) be a metric space. A function \( f : X \to X \) is said to be non-sensitive at a point \( a \in X \) if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for any choice of points \( a_0 \in B(a; \delta), a_1 \in B(f(a_0); \delta), a_2 \in B(f(a_1); \delta), \ldots, \) we have that \( d(a_m, f^m(a)) \) < \( \epsilon \) for every \( m \geq 0 \). Let \( H(X) \) be the set of all homeomorphisms from \( X \) onto \( X \) endowed with the topology of uniform convergence. The main goal of the present paper is to prove that for certain spaces \( X \), “most” functions in \( H(X) \) are non-sensitive at “most” points of \( X \).

1. Introduction

Consider a discrete dynamical system \((X, f)\), where \( X \) is a metric space with metric \( d \). If \( a \in X \), the sequence \( a, f(a), f^2(a), \ldots \) can be thought of as the actual behaviour of the system \((X, f)\) at \( a \). However, in concrete situations, we are often unable to compute the initial condition \( a \) exactly. We just compute a value \( a_0 \) close to \( a \). It may also be the case that we cannot compute \( f(a_0) \) exactly, but just a value \( a_1 \) close to \( f(a_0) \). Then we compute a value \( a_2 \) close to \( f(a_1) \) and so on. In this way, we obtain a sequence \( a_0, a_1, a_2, \ldots \) that can be thought of as the predicted behaviour of the system \((X, f)\) at \( a \). It is natural to ask whether or not this predicted behaviour is close to the actual behaviour of the system. This leads to the following definition:

Definition 1. We say that \( f \) is non-sensitive at \( a \) if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for any choice of points

\[
a_0 \in B(a; \delta), \ a_1 \in B(f(a_0); \delta), \ a_2 \in B(f(a_1); \delta), \ldots,
\]

we have that

\[
d(a_m, f^m(a)) < \epsilon \quad \text{for every} \ m \geq 0.
\]

If \( f \) is non-sensitive at \( a \), then the discrete dynamical system \((X, f)\) is “predictable at \( a \)”, in the sense that we can predict the future evolution of \( a \) in the system forever as accurately as we want provided we can compute the initial condition and the values of \( f \) precisely enough.

The above definition may remind the reader of the well-known notion of “shadowing” which, in the case \( X \) is compact, may be defined as follows [1]: \( f \) is said
to have the shadowing property (also called the pseudo-orbit tracing property) if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every sequence $a_0, a_1, a_2, \ldots$ in $X$ satisfying $d(f(a_i), a_{i+1}) < \delta$ for all $i \geq 0$, there is a point $x \in X$ such that

$$d(a_m, f^m(x)) < \epsilon \text{ for all } m \geq 0.$$ 

There is an important difference between “shadowing” and “non-sensitivity” despite the obvious fact that the former is a global notion whereas the latter is a pointwise one. To understand this difference, let $a \in X$ and choose points $a_0 \in B(a; \delta)$, $a_1 \in B(f(a_0); \delta)$, $a_2 \in B(f(a_1); \delta)$, etc. The shadowing property gives us the existence of a point $x \in X$ (possibly different from $a$) such that $d(a_m, f^m(x)) < \epsilon$ for all $m \geq 0$. On the other hand, non-sensitivity at $a$ guarantees that this holds with $x = a$. This is a major difference if we are interested in the problem of predictability: the shadowing property guarantees only that the predicted behaviour of the system $(X, f)$ at $a$ is close to some actual behaviour of the system (but this actual behaviour may be different from the actual behaviour at the initial condition $a$). For results and further references on “shadowing”, see the book [1] by Aoki and Hiraide.

Let $H(X)$ be the set of all homeomorphisms from $X$ onto $X$. If $X$ is compact, we consider $H(X)$ endowed with the supremum metric: $d(f, g) = \sup_{x \in X} d(f(x), g(x))$. $H(X)$ is then a Baire space.

If $M$ is a Baire space, we say that “most elements of $M$” satisfy a certain property $P$ if the set of all $x \in M$ that do not satisfy property $P$ is of the first category in $M$. The terms “typical” and “generic” are often used instead of “most”.

A result related to the notion of predictability was obtained by the author in [2]. Let $B^n$ denote the closed unit ball of $\mathbb{R}^n$. Then, Theorem 20 of [2] can be stated as follows:

**Theorem 2.** Fix $n \geq 1$. Let $X$ be a metrizable compact topological $n$-manifold with (or without) boundary [5] and fix a metric $d$ compatible with the topology of $X$. Then, most functions in $H(X)$ are non-sensitive at most points of $X$.

We shall prove Theorem 2 in Section 4.

In the case $X$ is a metrizable compact smooth manifold without boundary, Odani [6] proved that most functions in $CLD(X)$ (the closure in $H(X)$ of the set of all diffeomorphisms of $X$) have the shadowing property. His arguments were based on tools from Differentiable Dynamics. This work was preceeded by Yano [9], who proved that most functions in $H(S^1)$ have the shadowing property (where $S^1$ is the unit circle). By Munkres [4] and Whitehead [8], $CLD(X) = H(X)$ whenever $\dim X \leq 3$; but Munkres [4] also showed that this is false if $\dim X > 3$. The problem of whether most functions in $H(X)$ have the shadowing property was recently solved in the affirmative for $\dim X$ arbitrary by Pilyugin and Plamenevskaya [7]. However, this problem seems to remain open if $X$ is an arbitrary metrizable compact topological manifold with (or without) boundary.
An interesting problem is to change the notion of “most” on $B^n$ from category sense to measure-theoretic (probability) sense, that is, to consider “most” as meaning “full measure”. In this direction we have the following result:

**Theorem 3.** Fix $n \geq 2$. Most functions in $H(B^n)$ are non-sensitive at every point of a full (Lebesgue) measure subset of $B^n$.

As an immediate consequence of this theorem we have the following:

**Corollary 4.** Let $n \geq 2$. For most functions $f \in H(B^n)$, if $f$ is sensitive to initial conditions on a subset $Y$ of $B^n$, then $Y$ has Lebesgue measure zero.

Recall that $f$ is sensitive to initial conditions on $Y \subseteq B^n$ if there is an $\epsilon > 0$ such that for any $y \in Y$ and any $\delta > 0$, there is an $x \in B^n$ with $\|x - y\| < \delta$ and there is an $m \geq 1$ so that $\|f^m(x) - f^m(y)\| \geq \epsilon$.

We shall prove Theorem 3 in Section 2. Its proof will establish the following result at the same time:

**Theorem 5.** Let $n \geq 2$. For most functions $f \in H(B^n)$, the set $\Omega_f$ of all non-wandering points of $f$ has Lebesgue measure zero.

Theorem 5 improves Theorem 17 of [2], which asserts that for most functions $f \in H(B^n)$, the set $P_f$ of all periodic points of $f$ has Lebesgue measure zero. In this direction, it is interesting to observe the following fact:

**Proposition 6.** Fix $n \geq 2$. For most functions $f \in H(B^n)$, most points of $\Omega_f$ are recurrent and non-periodic.

We shall prove Proposition 6 in Section 3.

## 2. Proof of Theorems 3 and 5

Consider $\mathbb{R}^n$ endowed with the metric given by its euclidean norm $\| \cdot \|$. Given $A \subseteq \mathbb{R}^n$, $\overline{A}$, $\text{Int}A$, $\text{Bd}A$ and $\text{diam}A$ denote the closure, the interior, the boundary and the diameter of $A$ in $\mathbb{R}^n$, respectively, and we define

$$N_\delta(A) = \bigcup_{a \in A} \{x \in \mathbb{R}^n; \|x - a\| < \delta\} \quad \text{for } \delta > 0.$$

Throughout the present section, $X$ denotes the closed unit ball of $\mathbb{R}^n$ and $\mu$ denotes Lebesgue measure on $\mathbb{R}^n$. By an open box we mean a set of the form

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n; \alpha_i < x_i < \alpha_i + \delta \text{ for } 1 \leq i \leq n\},$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\delta > 0$. A closed box is just the closure of an open box. By a tree we mean a finite rooted tree $T$. If $T$ is a tree, $V(T)$ denotes the set of all vertices of $T$. Moreover, if $v_1, v_2 \in V(T)$, we write “$v_1 > v_2$” or “$v_2 < v_1$” to mean that $v_1$ and $v_2$ are adjacent and that the unique path connecting $v_2$ to the root of $T$ passes through $v_1$. A $B$-tree is a pair $(T, \varphi)$, where $T$ is a tree and $\varphi$ is a bijective correspondence between $V(T)$ and a collection of pairwise disjoint closed boxes contained in $\text{Int}X$. If $(T, \varphi)$ is a $B$-tree, we usually omit the correspondence $\varphi$ and speak just of the $B$-tree $T$; moreover, we make no distinction between a vertex of $T$ and its corresponding closed box.

For each $k \geq 1$, let $A_k$ be the set of all functions $f \in H(X)$ for which there are finitely many $B$-trees $T_1, \ldots, T_s$ so that the following properties are satisfied:

(i) $C \cap F = \emptyset$ whenever $C \in V(T_i)$, $F \in V(T_j)$ and $i \neq j$.
Since the following properties:

This implies that $f$ is related to $C$, and we choose $a_1 \in B(a; \delta_k)$, $a_2 \in B(f(a_1); \delta_k)$, etc., then $|a_m - f^m(a)| < 2/k$ for every $m \geq 0$. This implies that $f$ is non-sensitive at every point of $M$.

It remains to show that each $A_k$ is dense in $H(X)$. Fix $k \geq 1$, $f \in H(X)$ and $\epsilon > 0$. Let $0 < \delta < 1/k$ be such that

$$\text{diam} f(Y) < \frac{\epsilon}{2} \quad \text{whenever } Y \subset X \text{ and diam} Y < \delta.$$ 

Let $C$ be a finite collection of pairwise disjoint closed boxes contained in $\text{Int} X$ such that

$$\mu(X - \bigcup_{C \in C} C) < \frac{1}{k} \quad \text{and diam} C < \delta \text{ for every } C \in C.$$ 

Let $r$ be the cardinality of $C$ and choose $\eta > 0$ such that

$$\text{diam} f(Y) < \frac{\epsilon}{2r+1} \quad \text{whenever } Y \subset X \text{ and diam} Y < \eta.$$ 

Our next goal is to construct finitely many $B$-trees $T_1, \ldots, T_s$ satisfying (i) and the following properties:

(a) diam $C < \delta$ whenever $C \in V(T_i)$ for some $1 \leq i \leq s$.
(b) $C \subset V(T_i) \cup \ldots \cup V(T_s)$.
(c) If $C, F \in V(T_i)$ and $C > F$, then we have two possibilities:
   - either $C \subset C$ and $f(F) \subset C$,
   - or $C \notin C$ and $(\text{Int} C) \cap f(\text{Bd} F) \neq \emptyset$. 

(d) For each \(i\), there is a chain \(C_{i,1} > C_{i,2} > \cdots > C_{i,t_i}\) of successive boxes in \(V(T_i)\), beginning with the root \(C_{i,1}\) of \(T_i\), with the following property: 
\[ f(C_{i,1}) \subset C_{i,t_i} \] and/or there is a \(C_{i,t_i+1} \in V(T_i)\) with \(C_{i,t_i} > C_{i,t_i+1}\) such that \(C_{i,1}, \ldots, C_{i,t_i+1} \notin C\) and both \(C_{i,1}\) and \(C_{i,t_i+1}\) are contained in an open box \(W_i\) of diameter \(< \eta\).

In order to explain how to construct the trees \(T_1, \ldots, T_s\), we shall need a variable \(B\), which will denote the set of all closed boxes that have already been used in the construction up to the current step (at the beginning we have \(B = \emptyset\)). It is also important to observe that during the construction the trees will be seen as variables. Even when the construction of a tree \(T_i\) is apparently done, it may be necessary to change it later on in the process.

We begin by choosing a closed box \(C_1 \in C\) and by putting it as a vertex of \(T_1\) (note that now \(B = \{C_1\}\)). Suppose that in a certain moment \(T_1\) consists of the vertices \(C_1 < C_2 < \cdots < C_j\). We look at the set \(f(C_j)\). There are three possibilities:

**Case 1.** \(f(C_j) \subset C\) for some \(C \in B\).

We stop the construction of \(T_1\) (for the time being). So, \(C_j\) is the root of \(T_1\).

**Case 2.** \(f(C_j) \subset C\) for some \(C \in C - B\).

Let \(C_{j+1}\) be such a \(C\). We put \(C_{j+1}\) as a vertex of \(T_1\) adjacent to \(C_j\) and satisfying \(C_j < C_{j+1}\).

**Case 3.** \(f(C_j) \notin \bigcup\{C; C \in C \text{ or } C \in B\}\).

Then we also have that \(f(BdC_j) \notin \bigcup\{C; C \in C \text{ or } C \in B\}\). Hence, we can choose a closed box \(C_{j+1} \subset \text{Int}X\) disjoint from each box in \(C \cup B\) such that \(\text{diam}(C_{j+1}) < \delta\) and \((\text{Int}C_{j+1}) \cap f(BdC_j) \neq \emptyset\). We put \(C_{j+1}\) as a vertex of \(T_1\) adjacent to \(C_j\) and satisfying \(C_j < C_{j+1}\).

If Case 1 never happens, then the construction can go on forever. In this case we will stop the construction of \(T_1\) as soon as we obtain a chain \(C_{1,1} > C_{1,2} > \cdots > C_{1,t_1} > C_{1,t_1+1}\) beginning with the root \(C_{1,1}\) of \(T_1\) so that \(C_{1,1}, \ldots, C_{1,t_1+1} \notin C\) and both \(C_{1,1}\) and \(C_{1,t_1+1}\) are contained in an open box \(W_1\) of diameter \(< \eta\). Since \(C\) is finite and \(X\) is bounded, we will obtain such a chain in a finite number of steps.

Suppose that we have already constructed \(T_1, \ldots, T_{i-1}\). If \(B \supseteq C\), we are done. If this is not the case, we choose a \(C'_{i} \in C - B\) and put it as a vertex of \(T_i\). If in a certain moment \(T_i\) consists of the vertices \(C'_{i} < C'_2 < \cdots < C'_{j}\), we then look at \(f(C'_{j})\). Cases 2 and 3 are treated as before. However, Case 1 should be divided in two possibilities:

**Case 1a.** \(f(C'_{j}) \subset C\) for some \(C \in V(T_i)\).

We stop the construction of \(T_i\) (for the time being). So, \(C'_{j}\) is the root of \(T_i\).

**Case 1b.** \(f(C'_{j}) \subset C\) for some \(C \in B - V(T_i)\).

Let \(\tilde{C}\) be such a \(C\). Then \(\tilde{C}\) is a vertex of a previous tree, say \(\tilde{V} \in V(T_{i_0})\), where \(1 \leq i_0 < i\). In this case we will have no tree \(T_i\) for the time being. We will just enlarge \(T_{i_0}\) by putting the chain \(C'_1 < C'_2 < \cdots < C'_{j}\) as a new branch of it, satisfying the relation \(C'_{j} < \tilde{C}\).
If Cases 1a and 1b never happen, we will stop the construction of $T_i$ when we obtain a chain $C_{i,1} > C_{i,2} > \cdots > C_{i,t_i} > C_{i,t_i+1}$ beginning with the root $C_{i,1}$ of $T_i$ so that $C_{i,1}, \ldots, C_{i,t_i+1} \notin \mathcal{C}$ and both $C_{i,1}$ and $C_{i,t_i+1}$ are contained in an open box $W_i$ of diameter $< \eta$.

By the way the trees $T_1, \ldots, T_s$ were constructed, it is immediate to check that they have all the desired properties. Note also that $s \leq r$.

Let $I = \{i \in \{1, \ldots, s\}; f(C_{i,1}) \subset C_{i,t_i}\}$ and $J = \{1, \ldots, s\} - I$. For each $C \in V(T_1) \cup \cdots \cup V(T_s)$, choose an open box $V_C$ with
\[
\text{diam} V_C < \delta \quad \text{and} \quad C \subset V_C \subset \overline{V_C} \subset \text{Int} X,
\]
so that the family $\{\overline{V_C}\}_{C \in V(T_1) \cup \cdots \cup V(T_s)}$ is pairwise disjoint. We may also assume that
\[
\overline{V_{C,i}} \subset W_i \quad \text{and} \quad \overline{V_{C,i,t_i+1}} \subset W_i \quad \text{for every} \ x \in J.
\]

Now, we are going to define a function $g_0 \in H(X)$ as follows: Suppose that $C, F \in V(T_i)$ for some $i$, and $C > F$. By (c), there is a closed box $B \subset \text{Int} F$ such that $f(B) \subset \text{Int} C$. If $F \notin \{C_{i,t_i+1}; i \in J\}$, choose a $\varphi \in H(V_F)$ such that
\[
\varphi(F) \subset B \quad \text{and} \quad \varphi(x) = x \quad \text{for all} \ x \in \text{Bd} V_F,
\]
and define
\[
g_0(x) = f(\varphi(x)) \quad \text{for all} \ x \in V_F.
\]
If $F \in \{C_{i,t_i+1}; i \in J\}$, then $C \notin \mathcal{C}$ and so $(\text{Int} C) \cap f(\text{Bd} F) \neq \emptyset$. Choose an open box $Z_F \subset V_F$ with $F \subset Z_F \subset \overline{Z_F} \subset V_F$ such that there is a closed box $D_F \subset V_F - Z_F$ with $f(D_F) \subset \text{Int} C$; choose a $\varphi \in H(Z_F)$ such that $\varphi(F) \subset B$ and $\varphi(x) = x$ for all $x \in \text{Bd} Z_F$, and define
\[
g_0(x) = f(\varphi(x)) \quad \text{for all} \ x \in \overline{Z_F}.
\]
We make this definition for all $C, F \in V(T_1)$ with $C > F$ (1 \leq i \leq s). For $i \in I$, there also exists a closed box $B \subset \text{Int} C_{i,1}$ such that $f(B) \subset \text{Int} C_{i,t_i}$. So, choose a $\varphi \in H(\overline{V_{C_{i,1}}})$ such that $\varphi(C_{i,1}) \subset B$ and $\varphi(x) = x$ for all $x \in \text{Bd} V_{C_{i,1}}$, and put
\[
g_0(x) = f(\varphi(x)) \quad \text{for all} \ x \in \overline{V_{C_{i,1}}}.
\]
Let $K$ be the union of all these closed boxes where we have already defined $g_0$. We finally put $g_0(x) = f(x)$ for all $x \in X - K$. Then $g_0 \in H(X)$, $d(g_0, f) < \epsilon/2$, and (iii) holds for every $i \in \{1, \ldots, s\}$ and (iv) holds for every $i \in I$ with $g_0$ in place of $f$.

Now, we need to change $g_0$ a little bit in order to obtain (iv) also for $i \in J$. If $i \in J$ and $1 \leq j \leq t_i + 1$, we denote the open box $V_{C_{i,j}}$ simply by $V_{i,j}$. Moreover, for $i \in J$, we write $Z_i$ and $D_i$ in place of $Z_{C_{i,t_i+1}}$ and $D_{C_{i,t_i+1}}$, respectively. Recall that
\[
D_i \subset V_{i,t_i+1} - \overline{Z_i} \quad \text{and} \quad f(D_i) \subset \text{Int} C_{i,t_i} \quad (i \in J).
\]
Write $J = \{i_1, \ldots, i_w\}$ and put
\[
K_1 = K \cup \overline{V_{i_1,1}} \cup \cdots \cup \overline{V_{i_w,1}} \cup D_{i_2} \cup \cdots \cup D_{i_w}.
\]
Choose
\[
a_1 \in \text{Int} C_{i_1,1} \subset \overline{V_{i_1,1}} \subset W_{i_1} - K_1 \quad \text{and} \quad b_1 \in \text{Int} D_{i_1} \subset W_{i_1} - K_1.
\]
Since $K_1$ is the union of a finite collection of pairwise disjoint closed boxes contained in $\text{Int} X$, $W_{i_1} - K_1$ is connected. So, there is a continuous path $\alpha : [0, 1] \rightarrow W_{i_1} - K_1$ from $a_1$ to $b_1$. Moreover, we may assume that $\alpha([0, 1]) \subset \text{Int} X$. Cover
\( \alpha([0,1]) \) by finitely many open balls \( B_1, \ldots, B_{\ell} \) whose closures are contained in \((W_{i-1} - K_1) \cap \text{Int} X \) so that
\[
    \overline{B_1} \subset \text{Int} C_{i,1}, \quad \overline{B_2} \subset \text{Int} D_{i_2} \quad \text{and} \quad B_i \cap B_{i+1} \neq \emptyset \quad \text{for every } 1 \leq i < \ell.
\]
By working on \( \overline{V_{i,1}} \cup \overline{B_1} \cup \ldots \cup \overline{B_\ell} \), we see that it is possible to construct a \( \varphi \in H(X) \) such that
\[
    \varphi(C_{i,1}) \subset \text{Int} D_{i_2} \quad \text{and} \quad \varphi(x) = x \quad \text{if } x \notin \overline{V_{i,1}} \cup \overline{B_1} \cup \ldots \cup \overline{B_\ell}.
\]
Put \( g_1 = g_0 \circ \varphi \). Then \( g_1 \in H(X) \), \( g_1 = g_0 \) on \( X \setminus (\overline{V_{i,1}} \cup \overline{B_1} \cup \ldots \cup \overline{B_\ell}) \) and
\[
    g_1(C_{i,1}) \subset g_0(D_{i_2}) = f(D_{i_2}) \subset \text{Int} C_{i,1}^{\prime, i_1}.
\]
Moreover, since \( \text{diam}(\overline{V_{i,1}} \cup \overline{B_1} \cup \ldots \cup \overline{B_\ell}) < \eta \), we have \( d(g_1, g_0) < \epsilon/2^{r+1} \) and
\[
    \text{diam} g_1(Y) < \frac{\epsilon}{2^r} \quad \text{whenever } Y \subset X - K \quad \text{and diam} Y < \eta.
\]
Put
\[
    K_2 = K \cup \overline{V_{i,1}} \cup \overline{V_{i_1,1}} \cup \ldots \cup \overline{V_{i_w,1}} \cup D_{i_3} \cup \ldots \cup D_{i_w}.
\]
Choose
\[
a_2 \in \text{Int} C_{i_2,1} \subset \overline{V_{i_2,1}} \subset W_{i_2} - K_2 \quad \text{and} \quad b_2 \in \text{Int} D_{i_2} \subset W_{i_2} - K_2,
\]
and argue as before. We then obtain a \( g_2 \in H(X) \) with \( g_2(C_{i_2,1}) \subset \text{Int} C_{i_2,1}^{\prime, i_2} \).
Moreover, since \( g_2 \) differs from \( g_1 \) only on a set of diameter \( < \eta \), we have \( d(g_2, g_1) < \epsilon/2^r \) and
\[
    \text{diam} g_2(Y) < \frac{\epsilon}{2^{r-1}} \quad \text{if } Y \subset X - K \quad \text{and diam} Y < \eta.
\]
By continuing this process, we will obtain a function \( g_w \in H(X) \) such that
\[
    d(g_w, g_0) < \frac{\epsilon}{2^{r+1} - 1} + \frac{\epsilon}{2^{r+1} - 2} + \cdots + \frac{\epsilon}{2^{r+1}} < \frac{\epsilon}{2},
\]
and so \( d(g_w, f) < \epsilon \). Moreover, properties (iii) and (iv) hold for every \( i \in \{1, \ldots, s\} \) if we replace \( f \) by \( g_w \).

Finally, it remains to deal with property (v). Recall that \( C \subset V(T_1) \cup \ldots \cup V(T_s) \) and that \( \mu(\bigcup_{C \in C} C) > \mu(X) - 1/k \). If we had every box of \( C \) appearing in the first union in property (v) we would be done. However, this may not be the case, since for \( i \in I \) there may exist boxes of \( C \) in the chain \( C_{i,1} > C_{i,2} > \cdots > C_{i,t_i} \). For each \( i \in I \), let \( U_i \) be an open box such that
\[
    C_{i,1} \subset U_i \subset \overline{U_i} \subset V_{i,1} \quad \text{and} \quad g_w(\overline{U_i}) \subset \text{Int} C_{i,1}^{\prime, i_1}.
\]
Let \( \varphi_i \in H(U_i) \) be such that \( \varphi_i(x) = x \) for all \( x \in \partial U_i \) and \( \text{diam} \varphi_i(C_{i,1}) \) is very small. Put \( g = g_w \circ \varphi_i \) on \( U_i \) for each \( i \in I \) and \( g = g_w \) on \( X \setminus \bigcup_{i \in I} U_i \). Then \( g \in H(X) \) and \( d(g, f) < \epsilon \), because \( g_w(\overline{U_i}) \subset f(\overline{V_{i,1}}) \) for each \( i \in I \), by the way the \( g_j \)'s were constructed. Moreover, (iii) and (iv) still hold with \( g \) in place of \( f \). By choosing \( \varphi_i \) so that \( \text{diam} \varphi_i(C_{i,1}) \) is small enough, we will have that the measure of
\[
(C_{i,t_i} - g(C_{i,1})) \cup (C_{i, t_i - 1} - g^2(C_{i,1})) \cup \ldots \cup (C_{i,1} - g^{t_i}(C_{i,1}))
\]
is so close to the measure of \( C_{i,t_i} \cup C_{i, t_i - 1} \cup \ldots \cup C_{i,1} \) that (v) will also hold with \( g \) in place of \( f \). This completes the proof.
3. Proof of Proposition 6

Put $X = B^n$. We know that for most functions $f \in H(X)$, $\Omega_f = \overline{f}$ and the set of all periodic points of $f$ with period $m$ is dense in the set of all periodic points of $f$ with period $q$ whenever $q$ divides $m$ [2]. Fix an $f \in H(X)$ which has these two properties. Let $R_f$ denote the set of all recurrent points of $f$. For each $m \geq 1$ and each $r \geq 1$, let $V_{m,r} = \{x \in X; \|x - f^t(x)\| < 1/m \text{ for some } t \geq r\}$. Then, each $V_{m,r}$ is open and

$$R_f = \bigcap_{m,r,k} (V_{m,r} - F_{f^k})$$

(where $F_{f^k}$ denotes the set of all fixed points of $f^k$). Thus, it remains to show that each set $(V_{m,r} - F_{f^k}) \cap \Omega_f$ is dense in $\Omega_f$. For this purpose, let $U$ be an open set that meets $\Omega_f$. Choose a $y \in P_f \cap U$. Let $p$ be the period of $y$ and choose an integer $t$ of the form $sp$ (for some $s \geq 1$) which is greater than $k$. Now, choose a periodic point $z$ of $f$ with period $t$ which lies in $U$. Then, $z \in (V_{m,r} - F_{f^k}) \cap \Omega_f \cap U$, which completes the proof.

4. Proof of Theorem 2

Let $i(X)$ denote the interior of the manifold $X$. If $A \subset X$, then $\overline{A}$ and $\text{Int} A$ denote the closure and the interior of $A$ in $X$, respectively, and we define $N_\delta(A) = \bigcup_{a \in A} B(a;\delta)$ for $\delta > 0$. Moreover, if $f : X \to X$ is a mapping, we define $(f \circ N_\delta)^0(A) = A$, $(f \circ N_\delta)^1(A) = f(N_\delta(A))$, $(f \circ N_\delta)^2(A) = f(N_\delta(f(N_\delta(A))))$, and so on.

Fix a sequence $z_1, z_2, \ldots$ in $i(X)$ which is dense in $X$. For each $r \geq 1$ and $k \geq 1$, let $\mathcal{O}_{r,k}$ be the set of all functions $f \in H(X)$ for which there is a closed set $V \subset i(X)$ and there are integers $q \geq 0$ and $m \geq 1$ so that $f^q(z_k) \in \text{Int} V$, $f^m(V) \subset \text{Int} V$ and $\text{diam} f^i(V) < 1/r$ for $0 \leq i \leq m - 1$.

Clearly, each $\mathcal{O}_{r,k}$ is open. Let $f \in \bigcap_{k \geq 1} \mathcal{O}_{r,k}$. By definition, for each $r \geq 1$ and each $k \geq 1$, there are a closed set $V_{r,k} \subset i(X)$ and integers $q_{r,k} \geq 0$ and $m_{r,k} \geq 1$ such that $f^{q_{r,k}}(z_k) \in \text{Int} V_{r,k}$, $f^{m_{r,k}}(V_{r,k}) \subset \text{Int} V_{r,k}$ and $\text{diam} f^i(V_{r,k}) < 1/r$ for $0 \leq i \leq m_{r,k} - 1$. Let $W_{r,k}$ be an open ball centered at $z_k$ such that $f^{q_{r,k}}(\overline{W_{r,k}}) \subset \text{Int} V_{r,k}$ and $\text{diam} f^i(\overline{W_{r,k}}) < 1/r$ for $0 \leq i \leq q_{r,k} - 1$.

Choose $0 \leq \delta_{r,k} < 1/r$ such that

$$(f \circ N_{\delta_{r,k}})^{q_{r,k}}(\overline{W_{r,k}}) \subset \text{Int} V_{r,k}, \quad (f \circ N_{\delta_{r,k}})^{m_{r,k}}(V_{r,k}) \subset \text{Int} V_{r,k},$$

$$\text{diam}(f \circ N_{\delta_{r,k}})^i(\overline{W_{r,k}}) < \frac{1}{r} - \delta_{r,k} \quad \text{for } 0 \leq i \leq q_{r,k} - 1$$

and

$$\text{diam}(f \circ N_{\delta_{r,k}})^i(V_{r,k}) < \frac{1}{r} - \delta_{r,k} \quad \text{for } 0 \leq i \leq m_{r,k} - 1.$$
sequence \((f^j(z_k))_{j \geq 0}\) and choose a neighborhood \(W\) of \(a\) in \(X\) for which there is a homeomorphism \(\psi : W \to B^n\) with

\[
\psi(i(X) \cap \text{Int} W) = \{x \in \mathbb{R}^n; ||x|| < 1\}.
\]

Let \(q \geq 0\) be the smallest integer such that \(f^q(z_k) \in \text{Int} W\). Let \(s \geq 1\) be such that \(f^{q+s}(z_k) \in \text{Int} W\). We have two possibilities:

**Case 1.** \(n \geq 2\).

We choose a point \(b \in (i(X) \cap \text{Int} W) \setminus \{f^j(z_k); j \in \mathbb{Z}\}\) so close to \(f^q(z_k)\) that we have \(f^s(b) \in \text{Int} W\), and we let \(m \geq 1\) be the smallest integer such that \(f^m(b) \in \text{Int} W\).

**Case 2.** \(n = 1\).

We may think of \(W\) as being \([-1, 1]\). So, we may define the sets

\[
L = \{x \in W; -1 < x < f^q(z_k)\} \quad \text{and} \quad R = \{x \in W; f^q(z_k) < x < 1\}.
\]

If \(f^{q+s}(z_k) = f^{q+s}(z_k)\), choose \(b \in R \setminus \{f^j(z_k); j \in \mathbb{Z}\}\) so close to \(f^q(z_k)\) that we have \(f^s(b) \) and \(f^{2s}(b) \) in \(\text{Int} W\); then \(f^s(b)\) or \(f^{2s}(b)\) belong to \(R\). If \(f^{q+s}(z_k) \neq f^{q+s}(z_k)\), then either \(f^{q+s}(z_k) \in L\) or \(f^{q+s}(z_k) \in R\); say \(f^{q+s}(z_k) \in R\). In this case, choose \(b \in R \setminus \{f^j(z_k); j \in \mathbb{Z}\}\) so that \(f^s(b) \in R\). Let \(m \geq 1\) be the smallest integer such that \(f^m(b) \in R \subset \text{Int} W\).

Let \(\phi \in H(X)\) be such that

\[
\phi(f^m(b)) = b \quad \text{and} \quad \phi(x) = x \text{ for all } x \in (X - \text{Int} W) \cup \{f^q(z_k)\}.
\]

If \(n = 1\), we also assume that \(\phi(x) = x\) for all \(x \in L\). Define \(g = \phi \circ f \in H(X)\).

Then

\[
g(x) = f(x) \quad \text{whenever } f(x) \in (X - \text{Int} W) \cup \{f^q(z_k)\}
\]

and \(b\) is a periodic point of \(g\) with period \(m\). Moreover,

\[
g^q(z_k) = f^q(z_k) \in \text{Int} W \quad \text{and} \quad g^j(z_k) = f^j(z_k) \notin \text{Int} W \quad \text{for } 0 \leq j \leq q - 1.
\]

Let \(Z\) and \(V\) be closed neighborhoods of \(b\) such that

\[
g^q(z_k) \in \text{Int} V \subset V \subset \text{Int} Z \subset Z \subset \text{Int} W
\]

and \(\psi(Z)\) is a closed ball contained in \(\{x \in \mathbb{R}^n; ||x|| < 1\}\). If \(n = 1\), we also assume that

\[
g(b), \ldots, g^{m-1}(b) \notin Z.
\]

Since \(g(b), \ldots, g^{m-1}(b) \notin Z\), there is a closed neighborhood \(V'\) of \(b\) such that \(V' \subset \text{Int} Z\), \(Z, g(V'), \ldots, g^{m-1}(V')\) are pairwise disjoint, \(g^m(V') \subset \text{Int} V\) and \(\text{diam} g^i(V') < 1/r\) for \(1 \leq i \leq m - 1\). Now, let \(\varphi \in H(X)\) be such that

\[
\varphi(V) \subset V' \quad \text{and} \quad \varphi(x) = x \text{ for all } x \in (X - \text{Int} Z) \cup \{b\}.
\]

Let \(h = g \circ \varphi \in H(X)\). Then

\[
h^q(z_k) = g^q(z_k) \in \text{Int} V,
\]

\[
h^m(V) = h^{m-1}(g(\varphi(V))) \subset g^m(V') \subset \text{Int} V
\]

and

\[
\text{diam} h^i(V) \leq \text{diam} g^i(V') < \frac{1}{r} \quad \text{for } 1 \leq i \leq m - 1.
\]

Moreover, by choosing \(W\) small enough we can also guarantee that \(\text{diam} V < 1/r\) (hence, \(h \in O_{r,k}\)) and \(d(h, f) < \epsilon\). This completes the proof.
REFERENCES


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