A SEMINORM WITH SQUARE PROPERTY ON A COMPLEX ASSOCIATIVE ALGEBRA IS SUBMULTIPLICATIVE

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Dedicated to Professor Franek Szafraniec on the occasion of his sixtieth birthday

Abstract. The result stated in the title is proved as a consequence of an appropriate generalization replacing the square property of a seminorm with a similar weaker property which implies an equivalence to the supnorm of all continuous functions on a compact Hausdorff space also.

Theorem. Let \( p \) be a seminorm with the square property on a complex (associative) algebra \( A \). Then the following hold for all \( a, b \) in \( A \):

1. \( p(ab - ba) = 0 \).
2. \( p(ab) \leq p(a)p(b) \).

This is a uniform seminorm analogue of [3] or Thm. 6 in [6] that a \( \mathbb{C} \)-seminorm is submultiplicative (and the involution is isometric). We answer a problem posed in [3] and solved in the particular case of Banach algebras [4].

A seminorm on \( A \) is a nonnegative function on \( A \) satisfying:

1. \( p(a + b) \leq p(a) + p(b) \) for all \( a, b \) in \( A \) and
2. \( p(\lambda a) = |\lambda|p(a) \) for all \( a \), for all scalars \( \lambda \).

The seminorm \( p \) is submultiplicative if

1. \( p(ab) \leq p(a)p(b) \) for all \( a, b \) in \( A \).

It satisfies the square property [3] [4] if

1. \( p(a^2) = p(a)^2 \) for all \( a \) in \( A \).

The above theorem is a consequence of the following.

Proposition. Let \( p \) be a seminorm on a complex (associative) algebra \( A \) satisfying

1. \( mp(a)^2 \leq p(a^2) \leq Mp(a)^2 \) for all \( a \) in \( A \),
where \( 0 < m \leq M \) are given constants. Then properties (1) and (2)*,

2. \( mp(ab) \leq M^2 p(a)p(b) \) for all \( a, b \) in \( A \),

hold true.
So the first step is done.

enables us to conclude by the first step that

\[ 2(ab + ba) = (a + b)^2 - (a - b)^2 \]

implies that

\[ 2p(ab + ba) \leq p((a + b)^2) + p((a - b)^2) \leq Mp(a + b)^2 + Mp(a - b)^2 \]

\[ \leq 2M(p(a) + p(b))^2, \]

and so

\[ p(ab + ba) \leq 4M \text{ for all } a, b \text{ in } A \text{ with } p(a) \leq 1, p(b) \leq 1. \]

Now, for any \( \varepsilon > 0 \) we find that

\[ p\left(\frac{a}{p(a) + \varepsilon}\right) < 1, \quad p\left(\frac{b}{p(b) + \varepsilon}\right) < 1, \]

hence that

\[ p(ab + ba) \leq 4M(p(a) + \varepsilon)(p(b) + \varepsilon). \]

So the first step is done.

Second step. \( p(bab) \leq 10M^2p(a)p(b)^2 \) for all \( a, b \) in \( A \).

The identity

\[ 2bab = (ab + ba)b - ab^2 + b(ab + ba) - b^2a \]

enables us to conclude by the first step that

\[ 2p(bab) \leq p((ab + ba)b + b(ab + ba)) + p(ab^2 + b^2a) \leq 4Mp(ab + ba)p(b) + 4Mp(a)p(b^2) \leq 16M^2p(a)p(b)^2 + 4M^2p(a)p(b)^2, \]

and the second step follows.

Third step. \( mp(ab - ba)^2 \leq 96M^3p(a)^2p(b)^2 \) for all \( a, b \) in \( A \).

The identity

\[ (ab - ba)^2 = 2[a(bab) + (bab)a] - (ab + ba)^2 \]

implies by the first two steps that

\[ mp(ab - ba)^2 \leq p((ab - ba)^2) \leq 2p(a(bab) + (bab)a) + p((ab + ba)^2) \leq 8Mp(a)p(bab) + Mp(ab + ba)^2 \leq 80M^3p(a)^2p(b)^2 + 16M^3p(a)^2p(b)^2. \]

Fourth step. \( p(ab) \leq (2 + 5\sqrt{M/m})Mp(a)p(b) \) for all \( a, b \) in \( A \).

This is an easy consequence of the former steps:

\[ 2p(ab) = p((ab + ba) + (ab - ba)) \leq p(ab + ba) + p(ab - ba) \leq 4Mp(a)p(b) + 10M\sqrt{M/m}p(a)p(b), \]

since \( 96 \leq 10^2 \) and the step is done.
Fifth (final) step.

According to the fourth step the kernel of \( p, \ker p \), is a two-sided ideal in the algebra \( A \); therefore the norm \( |\cdot| \) on the quotient algebra \( A/\ker p \), defined by

\[
|a + \ker p| := p(a) \text{ for all } a \in A,
\]
satisfies property (iv)*, hence the consequences stated in the first four steps. Define an algebra norm

\[
\|a + \ker p\| := \sup \{|\lambda a + ab + \ker p| : |\lambda| + |b + \ker p| \leq 1; \ \lambda \in \mathbb{C}, \ b \in A\}
\]
for all \( a \) in \( A \)
on \( A/\ker p \) as in 1.1.9 Prop. in [7]. We find that

\[
|a + \ker p| \leq \|a + \ker p\| \leq \left(2 + 5 \sqrt{\frac{M}{m}}\right) M|a + \ker p|
\]
and that \( \|\cdot\| \) satisfies property (iii), i.e. it is submultiplicative.

We note that property (iv)* holds also as follows:

\[
m^{2^{n-1}} p(a) 2^n \leq p(a^{2^n}) \leq M^{2^{n-1}} p(a) 2^n \text{ for all } a \in A \text{ and } n = 1, 2, \ldots.
\]
The norm \( |\cdot| \) on \( A/\ker p \) thus also fulfills

\[
m^{2^{n-1}} |a + \ker p| 2^n \leq |a^{2^n} + \ker p| \leq M^{2^{n-1}} |a + \ker p| 2^n.
\]
We conclude that the spectral radius \( r \) in the normed algebra \( A/\ker p \) satisfies

\[
(\text{v}) \quad \|a + \ker p\| \leq \left(2 + 5 \sqrt{\frac{M}{m}}\right) M r(a + \ker p) \text{ for all } a \in A.
\]
Indeed, we find that with \( C = \left(2 + 5 \sqrt{\frac{M}{m}}\right) M \)

\[
\|a + \ker p\| \leq C|a + \ker p| \leq \frac{C}{m^{1-\frac{1}{2^n}}} |a^{2^n} + \ker p| 2^{-n}
\]

\[
\leq \frac{C}{m^{1-\frac{1}{2^n}}} \|a^{2^n} + \ker p\| 2^{-n}, \quad n = 1, 2, \ldots,
\]
and the statement follows as \( r(a + \ker p) = \lim_{n \to \infty} \|a^{2^n} + \ker p\| 2^{-n} \). The Hirschfeld-Zelazko Theorem (see 3.1.7 Prop. in [7], (B.6.17) Cor. in [5] or [2, Lemma 2, p. 46]) now gives by (v) that \( A/\ker p \) is commutative, i.e. property (1) holds.

Finally property (2)* follows in consequence of the following argument: since

\[
m^{2^{n-1}} p(ab) 2^n \leq p((ab)^{2^n}) = |(ab)^{2^n} + \ker p| \leq |(ab)^{2^n} + \ker p|
\]

\[
= \|a^{2^n} b^{2^n} + \ker p\| \leq \|a^{2^n} + \ker p\| |b^{2^n} + \ker p| \leq C^2 p(a^{2^n}) p(b^{2^n})
\]

\[
\leq C^2 M^{2^{n+1-2p(a)2^n} p(b) 2^n}
\]
holds for \( n = 1, 2, \ldots \) we see that property (2)* holds true indeed.

We conclude with an affirmative answer to a question in [4, Remarks (5)].

**Corollary.** Let \( A = C(K) \) be the Banach algebra with supnorm \( \|\cdot\|_\infty \) of all continuous functions on a compact Hausdorff space \( K \). Let \( |\cdot| \) be a norm on \( C(K) \) with property (iv)*. Then \( |\cdot| \) is equivalent to \( \|\cdot\|_\infty \).
Proof. Noting that $\|\cdot\|_\infty$ is the spectral radius in $C(K)$ we have by the proof of the Proposition that with the algebra norm $\|\cdot\|$ above we have at once that
\[
\frac{m}{C}|a| \leq \frac{m}{C}\|a\|_1 \leq \|a\|_\infty \leq \|a\| \leq C|a| \text{ for all } a \in A.
\]
Here we use the fact that $\|\cdot\|_\infty \leq \|\cdot\|$ automatically holds true (see 2.4.15 Theorem in [7]).

Note added (December, 2000). In the square property (iv), the inequality "< or =" would be more natural for the conclusion (iii). However, this is not true: the numerical radius for Hilbert space operators, e.g. for 2-by-2 matrices, fulfills property (iv) or even the power inequality $p(a^n) \leq p(a)^n$ and is not submultiplicative (see Theorem 3.1 in [1]).

It is also a natural question of whether the submultiplicativity of a seminorm (with the square property) implies subadditivity. A two-dimensional counterexample follows: $\mathbb{C}$ as a two-point function algebra has multiplicative and nonsubadditive seminorm of the form $p(wz) = \sqrt{|wz|}$ since, e.g. $p((1,3)) + p((3,1)) = \sqrt{3} + \sqrt{3} < 4 = p((4,4))$.

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