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JOINT SPECTRUM OF SUBNORMAL n-TUPLES OF COMPOSITION OPERATORS

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Abstract. We compute the joint (Taylor) spectrum of an n-tuple of commuting composition operators acting on the Hardy space H^2.

1. Introduction

Let $A(\Delta)$ denote the space of all analytic functions in the unit disk Δ with the topology of uniform convergence on compact subsets of Δ, and let $\mathcal{H}$ be a linear subspace of $A(\Delta)$. If ψ is an analytic self map of Δ such that $f \circ \psi$ belongs to $\mathcal{H}$ for all $f \in \mathcal{H}$, then ψ induces a linear operator $C_\psi : \mathcal{H} \to \mathcal{H}$ defined as $C_\psi(f) := f \circ \psi$. C_ψ is called the *composition operator* with *symbol* ψ.

In [12] we observed that, according to a result of C. C. Cowen (cf. [4]), if $\psi : \Delta \to \Delta$ is a linear fractional map, and the composition operator C_ψ is hyponormal, then C_ψ is unitarily equivalent to a composition operator with symbol ϕ of the form

$$\phi \equiv \phi_a := \frac{z}{az + (a+1)}, \text{ with } a > 0. \tag{1.1}$$

In fact, any such ψ induces a strongly continuous one-parameter semigroup of subnormal composition operators. More precisely, we may define non-negative powers of C_{ϕ_a} by the formula

$$C_{\phi_a}^{\lambda} := C_{\phi_{(1+a)^\lambda - 1}}. \tag{1.2}$$

Then we proved:

Theorem 1.1 ([12]). *Let $\mathbf{\Phi} = (\phi_1, \phi_2, \ldots, \phi_n)$ be an n-tuple of commuting linear fractional self-maps of Δ, neither one of them having ∞ as a fixed point. Let $C_{\mathbf{\Phi}} := (C_{\phi_1}, \ldots, C_{\phi_n})$. The following statements are equivalent:*

(i) *C_{ϕ_k} is hyponormal for some k.*
(ii) *$C_{\mathbf{\Phi}}$ is (jointly) subnormal.*
(iii) *$C_{\mathbf{\Phi}}$ is (jointly) hyponormal.*
(iv) *There exist positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that $C_{\mathbf{\Phi}} \cong ({C_\psi}^{\lambda_1}, {C_\psi}^{\lambda_2}, \ldots, {C_\psi}^{\lambda_n})$, where $\psi := \frac{z}{z+2}$.*

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Observe that for each $a > 0$, ϕ_a has Denjoy-Wolff fixed point 0, and a single fixed point, 1, on the boundary. It follows from [5, Theorem 7.30, page 289] that the spectrum of C_{ϕ_a} is

$$\sigma(C_{\phi_a}) = \{z \in \mathbb{C} : |z| \leq \frac{1}{(1+a)^{1/2}}\} \cup \{1\}. \tag{1.3}$$

In this note we compute the Taylor spectrum of a commuting n-tuple $(C_{\phi_1}, C_{\phi_2}, \ldots, C_{\phi_n})$, where the ϕ_i's are as in (1.1).

2. Joint spectrum

As pointed out in [7], when dealing with n-tuples of elements in a Banach algebra, the (joint) spectral theory is more complicated. In this case, not only is there a difference between the study of commutative and non-commutative n-tuples, but also there is a distinction between algebraic and spatial joint spectra. The notion of *Taylor spectrum* belongs to the spatial theory.

Definition 2.1 (Taylor Spectrum)**.** Let $\Lambda = \Lambda[e] = \Lambda_n[e]$ be the *exterior algebra* on n generators $e_1, e_2, \ldots, e_n$, with identity $e_0 \equiv 1$. That is, Λ is the algebra of forms in $e_1, e_2, \ldots, e_n$, such that $e_i \wedge e_j = -e_j \wedge e_i$ $(1 \leq i, j \leq n)$. Λ is a graded algebra, for if Λ_n^k is the algebra of k-forms on $e_1, e_2, \ldots, e_n$, then $\Lambda = \bigoplus_{k=-\infty}^{\infty} \Lambda_n^k$, with $\Lambda_n^k \wedge \Lambda_n^l \subset \Lambda_n^{k+l}$.

For each $i \in \{1, 2, \ldots, n\}$ define $E_i : \Lambda \to \Lambda$ by

$$E_i \xi := e_i \wedge \xi.$$

$E_1, E_2, \ldots, E_n$ are called the *creation operators*, and they clearly satisfy the property $E_i E_j = -E_j E_i$ $(1 \leq i, j \leq n)$.

Λ can be regarded as a Hilbert space if one declares the set

$$\{\xi = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{ik} : i_1 < i_2 < \cdots < i_k\}$$

as an orthonormal basis. Then $E_i^* \xi = \xi'$, where $\xi = e_i \xi' + \xi''$ is the unique decomposition of a form ξ as the sum of an element in the range of E_i and an element in the kernel of E_i^* (actually each E_i is a partial isometry, and $E_i^* E_j + E_j E_i^* = \delta_{ij}$).

For $\mathcal{B}$ a Banach space and $A = (A_1, A_2, \ldots, A_n)$ an n-tuple of commuting operators in $\mathcal{L}(\mathcal{B})$, define $D_A : \Lambda(\mathcal{B}) \to \Lambda(\mathcal{B})$ $(\Lambda(\mathcal{B}) := \mathcal{B} \otimes_{\mathbb{C}} \Lambda)$ by

$$D_A := \sum_{i=1}^{n} A_i \otimes E_i.$$

Then

$$D_A^2(x \wedge \xi) = \sum_{i=1}^{n} A_j A_i x \otimes E_j E_i \xi = \sum_{i<j} A_i A_j x \otimes (E_i E_j + E_j E_i)\xi = 0.$$

Set $\Lambda^k(\mathcal{B}) := \mathcal{B} \otimes_{\mathbb{C}} \Lambda_n^k$ and $D_A^k := D_A|_{\Lambda^k(\mathcal{B})}$. Then we can construct a cochain complex $K(A, \mathcal{B})$, called the *Koszul complex* associated to A on $\mathcal{B}$, as follows:

$$K(A, \mathcal{B}) : \quad \mathbf{0} \longrightarrow \Lambda^0(\mathcal{B}) \xrightarrow{D_A^0} \Lambda^1(\mathcal{B}) \xrightarrow{D_A^1} \ldots \xrightarrow{D_A^{n-1}} \Lambda^n(\mathcal{B}) \longrightarrow \mathbf{0}.$$

The Taylor spectrum of A on $\mathcal{B}$ is then the set

$$\sigma_T(A, \mathcal{B}) := \{\lambda \in \mathbb{C}^n : K(A - \lambda, \mathcal{B}) \text{ is not exact}\}.$$

$\sigma_T(A,\mathcal{B})$ generalizes the one variable notion of spectrum (see Example 2.2(i) below); it is a nonempty compact subset of $\mathbb{C}^n$ that carries an analytic functional calculus: If Ω is an open set in $\mathbb{C}^n$ containing $\sigma(A,\mathcal{B})$, then $\sigma_T(f(A),\mathcal{B}) = f(\sigma_T(A,\mathcal{B}))$ $\forall\, f \in A(\Omega)$; see [20], [21]. The reader can find an excellent account of the Taylor spectrum and its relations with other multiparameter spectral theories in [7].

Example 2.2. Let $\mathcal{H}$ be a Hilbert space.

(i) For $A = (A_1)$,

$$K(A,\mathcal{H}): \qquad \mathbf{0} \longrightarrow \mathcal{H} \xrightarrow{D_A^0} \mathcal{H} \longrightarrow \mathbf{0}$$

with $D_A^0 = A_1$. Clearly, $K(A,\mathcal{H})$ is exact if and only if A_1 is one-to-one and onto. Hence in this case, $\sigma_T(A,\mathcal{H}) = \sigma(A)$, the usual spectrum of A.

(ii) Suppose that $A_1, A_2 \in \mathcal{L}(\mathcal{H})$ commute. Then the *Koszul complex* associated to $A = (A_1, A_2)$ on $\mathcal{H}$ is the cochain complex

$$K(A,\mathcal{H}): \qquad \mathbf{0} \longrightarrow \mathcal{H} \xrightarrow{D_A^0} \mathcal{H} \oplus \mathcal{H} \xrightarrow{D_A^1} \mathcal{H} \longrightarrow \mathbf{0},$$

where

$$D_A^0(f) := (A_1 f, A_2 f)$$

and

$$D_A^1(f,g) := -A_2 f + A_1 g.$$

□

Example 2.3. Let (C_ϕ, C_ψ) be a subnormal pair of (commuting operators). Then by Theorem 1.1

$$(C_\phi, C_\psi) \cong (C_{\phi_a}, C_{\phi_a}^\lambda),$$

where $\phi_a \equiv \frac{z}{az+1+a}$, $a > 0$, and $\lambda > 0$.

Assume now that $\lambda \in \mathbb{Q}$, say $\lambda = m/n$, with m, n positive integers. Choose b such that $(1+b)^n = 1 + a$. Then

$$(C_{\phi_a}, C_{\phi_a}^\lambda) = (C_{\phi_b}^n, C_{\phi_b}^m)$$

and therefore, by the functional calculus,

$$\sigma_T((C_\phi, C_\psi), H^2) = F(\sigma(C_{\phi_b}))$$

where $F : \Delta \to \mathbb{C}^2$ is the polynomial mapping defined by $F(z) := (z^n, z^m)$.

By (1.3) it follows that

$$\sigma_T((C_{\phi_a}, C_{\phi_a}^\lambda), H^2) = \{(z^n, z^m) : |z| \le \frac{1}{(1+b)^{1/2}}\} \cup \{(1,1)\}.$$

Now by setting $w := z^n$ (for $z \neq 0$) we can write

$$\sigma_T((C_{\phi_a}, C_{\phi_a}^\lambda), H^2) = \{(w, w^\lambda) : 0 < |w| \le \frac{1}{(1+a)^{1/2}}\} \cup \{(0,0), (1,1)\}.$$

Thus if $f_\lambda : \sigma(C_{\phi_a}) \to \mathbb{C}$ is the (multivalued) function defined by

$$f_\lambda(z) = \begin{cases} 0 & \text{if } z = 0, \\ 1 & \text{if } z = 1, \\ z^\lambda & \text{if z} \neq 0, z \neq 1, \end{cases} \tag{2.1}$$

then we have

$$\sigma_T((C_\phi, C_\psi), H^2) = \sigma_T((C_{\phi_a}, C_{\phi_a}^\lambda), H^2) = f_\lambda(\sigma(C_{\phi_a})).$$

We thus observe that the spectral mapping theorem holds in this particular situation. □

It is interesting to notice that although the functional relation

$$\mathbb{R}^+ \ni \lambda \longrightarrow C_{\phi_a}^\lambda$$

was not achieved via a standard functional calculus, such as the Riesz analytic functional calculus, it was still possible in the example above to apply the spectral mapping theorem. The core of the problem is, of course, the fact that the function $z \to z^\lambda$, $\lambda > 0$, is not analytic on $\sigma(C_{\phi_a})$. To extend the above result to the general case we will need to combine results from general theory of strongly continuous semigroups of operators, theory of semigroups of composition operators and multiparameter spectral theory (of which Theorem 2.13 will be our main tool).

We now begin by recalling some facts about strongly continuous semigroups of contractions.

Let $\mathcal{T} = \{T(s)\}_{s\geq 0}$ be a strongly continuous semigroup of contractions in a Hilbert space $\mathcal{H}$. The *infinitesimal generator* A of $\mathcal{T}$ is defined by

$$Ah := \lim_{s\to +0} \frac{1}{s}[T(s) - I]h$$

whenever this limit exists. A is a closed operator with domain $\mathfrak{D}(A)$ dense in $\mathcal{H}$; also, A determines the semigroup $\mathcal{T}$ uniquely (cf. [18]).

A proof of the next theorem can be found in [1].

Theorem 2.4. *Let A be the infinitesimal generator of the strongly continuous semigroup $\{T(s)\}_{s>0} \subset \mathcal{L}(\mathcal{H})$. Then:*

(i) *$Re(Ah, h) \leq 0$ for $h \in \mathfrak{D}(A)$.*

(ii) *$\lambda I - A$ is invertible whenever $Re(\lambda) > 0$ and, for such λ,*

$$\|(\lambda I - A)^{-1}\| \leq 1/Re(\lambda)$$

and

$$(\lambda I - A)^{-1}h = \int_0^\infty e^{-\lambda s} T(s)h\, ds, \quad h \in \mathcal{H}.$$

Conversely, if A is an operator satisfying $\|(\lambda I - A)^{-1}\| \leq 1/\lambda$ for $\lambda > 0$, then there exists a unique semigroup of contractions whose generator is A.

It follows from Theorem 2.4(ii) that $A - I$ is boundedly invertible and it is easy to verify (cf. [1]) that the operator T defined by

$$T := (A + I)(A - I)^{-1} \tag{2.2}$$

is a contraction. T is called the *co-generator* (cf. [18]) of the semigroup $\mathcal{T}$.

From (2.2) it follows that

$$T - I = (A + I)(A - I)^{-1} - I = 2(A - I)^{-1}.$$

Thus $(T-I)^{-1}$ exists and $A = (T+I)(T-I)^{-1}$. Note that the existence of $(T-I)^{-1}$ is equivalent to the fact that 1 is not an eigenvalue of T. The co-generator T can be used instead of A to study the semigroup $\mathcal{T}$. It has the advantage that T is bounded, while A is generally unbounded, and this actually occurs in the case of the semigroups $\{C_{\phi_a}^\lambda\}$; cf. [2], [17].

The following theorem ([19], Theorem 8.1) shows an explicit functional relation between T and $\mathcal{T}$.

Theorem 2.5. *Let T be a contraction in $\mathcal{H}$. In order that there exists a strongly continuous semigroup $\mathcal{T} = \{T(s)\}_{s\geq 0}$ whose co-generator is T it is necessary and sufficient that 1 not be an eigenvalue of T. In this case T and $\mathcal{T}$ determine one another by the relations*

$$T(s) = e_s(T) \tag{2.3}$$

and

$$T = \lim_{s\to +0} \phi_s(T(s)) \ \ (SOT)$$

where

$$e_s(\lambda) := \exp\left[s\frac{\lambda+1}{\lambda-1}\right] \quad \text{and} \quad \phi_s(T(s)) := \frac{\lambda-1+s}{\lambda-1-s}. \tag{2.4}$$

Corollary 2.6. *Let T be the co-generator of a strongly continuous semigroup $\mathcal{T} = \{T(s)\}_{s\geq 0}$ of contractions in $L(\mathcal{H})$. Then*

$$(I-T)f = 2\int_0^\infty e^{-t} e_t(T) f\, dt, \ \ f \in \mathcal{H}. \tag{2.5}$$

Proof. This is a direct consequence of Theorems 2.4 and 2.5. □

The next example (see [11]) shows the kind of hurdles one might expect when dealing with spectral functional calculus and semigroups.

Example 2.7 (Foiaş-Mlak)**.** Let $\mathcal{H} = L_2(0,1)$ and let $\{T_t\}_{t>0}$ be a strongly continuous semigroup of contractions defined by

$$(T_t f)(x) := \begin{cases} 0 & \text{if } x < t \ \text{ and } \ 0 \leq t < 1, \\ f(x-t) & \text{if } x \geq t \ \text{ and } \ 0 \leq t < 1, \\ 0 & \text{if } t \geq 1. \end{cases} \tag{2.6}$$

Let T be the co-generator of $\{T_t\}$. Then $T_t = e_t(T)$.

Since $e_1(T) \equiv 0$, $\sigma(e_t(T)) = \{0\}$. It can be shown, on the other hand, that $\sigma(T) = \{1\}$. Therefore the expression

$$\sigma(u(T)) = u(\sigma(T)), \quad u \in H^\infty,$$

does not make sense in general. Moreover, since the set L of limit values of $e_1(z)$ at $z = 1$ equals $\bar{\Delta}$, $u(\sigma(T))$ may not even be replaced by the set of limit values of u at $\sigma(T)$. □

In general, weaker spectral mapping relations hold:

Proposition 2.8. (Spectral mapping theorems for semigroups of operators [14].) *Let A be the infinitesimal generator of the strongly continuous semigroup $\{T(s)\}_{s>0} \subset \mathcal{L}(\mathcal{H})$. Then:*

(a) $e^{t\sigma(A)} \subset \sigma(T_t)$ *for* $t \geq 0$.

(b) $e^{t\sigma_p(A)} = \sigma_p(T_t) \setminus \{0\}$, $t \geq 0$, *where σ_p denotes point spectrum.*

(c) *If $\{T(s)\}_{s>0}$ consists of normal operators, then*

$$\sigma(T_t) = \overline{e^{t\sigma(A)}}, \ \text{for} \ t \geq 0.$$

(d) *If* $\{T(s)\}_{s\in\mathbb{R}}$ *is a bounded strongly continuous group, then*

$$\sigma(T_t) = \overline{e^{t\sigma(A)}}, \text{ for } t \in \mathbb{R}.$$

Let $H_0^2 := H^2 - \langle \mathbf{1} \rangle$ (the orthogonal complement of the constant functions in H^2). Since for all $s \in \mathbb{C}$, $C_{\phi_a}(s\mathbf{1}) = s\mathbf{1} \circ \phi_a = s$ and $C^*_{\phi_a}(s\mathbf{1}) = sC^*_{\phi_a}(K_0) = sK_{\phi_a(0)} = s$ it follows that $\langle \mathbf{1} \rangle$ reduces C_{ϕ_a} and therefore that H_0^2 is a reducing subspace for C_{ϕ_a}.

Lemma 2.9. *The* (*relative*) *semigroup* $\{C^\lambda_{\phi_a}|_{H_0^2}\}$ *consists of completely non-unitary contractions* (*i.e. there are no non-zero reducing subspaces* $\mathcal{H}_\lambda$ *of* H_0^2 *such that* $C^\lambda_{\phi_a}|_{\mathcal{H}_\lambda}$ *is a unitary operator*).

Proof. If there exist $\lambda > 0$ and a non-zero subspace $\mathcal{H}_\lambda \subset H_0^2$ such that $C^\lambda_{\phi_a}|_{\mathcal{H}_\lambda}$ is a unitary operator, then for all $0 \neq h \in \mathcal{H}_\lambda$ and $n = 0, 1, 2 \ldots$

$$||(C^\lambda_{\phi_a})^n h|| = ||h|| = ||({C^\lambda_{\phi_a}}^*)^n h||.$$

Now

$$||C^{n\lambda}_{\phi_a} h|| = ||h(\phi_a^{n\lambda})|| = ||h(\phi_{(1+a)^{n\lambda}-1})||.$$

Since $\lim_{n\to\infty} \phi_{(1+a)^{n\lambda}-1} = 0$ (the Denjoy-Wolff fixed point) uniformly on any compact subset of Δ, and $h(0) = 0$, one must necessarily have $||h|| = 0$, a contradiction. □

Proposition 2.10. *Let* $\{T_s\}_{s>0}$ *be a* (s.c.) *semigroup of contractions with infinitesimal generator* A *and co-generator* T. *Then* $\sigma(T) = \theta(\sigma(A))$ *where* θ *is the Cayley transform* $\theta(\mu) = \frac{\mu+1}{\mu-1}$.

Proof. The result is a consequence of the identity

$$\mu - A = [(\mu-1)T - (\mu+1)]\,(T-1)^{-1} = \frac{1}{\mu-1}(T - \theta(\mu))(T-1)^{-1}.$$

Indeed, it is readily seen that if $(T-\lambda)^{-1}$ exists, then

$$S := \frac{1}{\mu-1}(T-1)(T-\lambda)^{-1} \quad (\mu := \theta^{-1}(\lambda))$$

is an inverse for $\mu - A$.

Conversely if $(\mu - A)^{-1}$ exists, then

$$\frac{1}{\theta(\mu)-1}(I-A)((\mu-A)^{-1}$$

is an inverse for $\theta(\mu) - T$. □

Proposition 2.11. *Let* A *be the infinitesimal generator of the normalized semigroup* $\{e^{-sk}C^\lambda_{\phi_a}|_{H_0^2}\}$, *and let* $k := ln||\frac{1}{(1+a)^{1/2}}||$. *Then* $\sigma(A) = \{z|Re(z) \leq 0\}$.

The proof is an adaptation of an argument given by A. G. Siskakis in [17, p. 238].

Proof. It is easy to see that $\frac{\partial}{\partial t}(\phi_a^t)|_{t=0}(z) = -lz(1+z)$, where $l := log(1+a)$. Thus the infinitesimal generator, A, of $\{C^t_{\phi_a}|_{H_0^2}\}$, is given by the formula

$$Af(z) = -lz(1+z)f'(z),$$

whenever $f' \in H^2$.

Let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}(\lambda) \leq -1/2$, set

$$P_n(z) := 1 + \sum_{j=1}^{n} \frac{\lambda}{j} \binom{n}{j} z^j,$$

and let $y = f_{n,\lambda}(z) := (1+z)^{\lambda} e^{P_n(z)}$. Then it is readily seen that $y' = \lambda((1+z)^n - 1)$ and that

$$lz(1+z)y' = \lambda l(1+z)^{\lambda+1+n} e^{Pn(z)} = \lambda l f_{n,n+1+\lambda}.$$

That is,

$$(\lambda l - A)y = \lambda l f_{n,n+1+\lambda}.$$

Choose n such that $\mathrm{Re}(n+1+\lambda) > -1/2$. Then $f_{n,n+1+\lambda} \in H^2$.

If $\lambda l - A$ were invertible, then $\lambda l R(A, \lambda l) f_{n,n+1+\lambda} = f_{n,\lambda} \in H^2$, a contradiction, since $\mathrm{Re}(\lambda) \leq -1/2$. Hence

$$\{\mu : \mathrm{Re}(\mu) \leq -l/2\} = \{\lambda l : \mathrm{Re}(\lambda) \leq -1/2\} \subseteq \sigma(A). \tag{2.7}$$

On the other hand, by (1.3)

$$\begin{aligned} \sigma(C^t_{\phi_a}) = \sigma(C_{\phi_{(1+a)^t - 1}}) &= \{z \in \mathbb{C} : |z| \leq \frac{1}{(1+a)^{t/2}}\} \cup \{1\} \\ &= \{z \in \mathbb{C} : |z| \leq e^{-\frac{t}{2}l}\} \cup \{1\}. \end{aligned} \tag{2.8}$$

Therefore from (2.7) and Proposition 2.8(i) we have that

$$\sigma(A) = \{z : \mathrm{Re} z \leq -l/2\} \cup \{0\},$$

and consequently

$$\sigma(A|_{H^2_0}) = \{z : \mathrm{Re} z \leq -l/2\}.$$

Now consider the normalized semigroup $\{e^{-kt} C^t_{\phi_a}|_{H^2_0}\}$, where $k = log||C_{\phi_a}||$. Its generator $\tilde{A}$ is given by

$$\begin{aligned} \tilde{A} = \frac{\partial}{\partial t}(e^{-kt} C^t_{\phi_a})|_{t=0} &= -k e^{-kt} C^t_{\phi_a}|_{t=0} + e^{-kt} \frac{\partial}{dt}(C^t_{\phi_a})|_{t=0} \\ &= -k + A \\ &= \frac{1}{2} l + A. \end{aligned}$$

Consequently

$$\mu - \frac{1}{2} l - A \text{ is invertible } \iff \mathrm{Re}(\mu - \frac{1}{2} l) > -\frac{1}{2} l \iff \mathrm{Re}(\mu) > 0.$$

Therefore

$$\sigma(\tilde{A}) = \{z : \mathrm{Re} z \leq 0\}.$$

□

Corollary 2.12. *Let T be the co-generator of the semigroup $\{e^{-sk} C^{\lambda}_{\phi_a}|_{H^2_0}\}$. Then $\sigma(T) = \overline{\Delta}$.*

Proof. This is an immediate consequence of Propositions 2.11 and 2.10. □

Next we cite a result, due to R. Curto and L. Fialkow [8], which is crucial to attaining our goal.

Theorem 2.13 (R. Curto - L. Fialkow). *Let T be a c.n.u. contraction and let u be such that $u(\Delta) \subset \overline{u(\Delta \cap \sigma(T))}$. Then $\sigma_T(u(T)) = \overline{u(\Delta)}$.*

By virtue of the preceding discussion, the following proposition is a corollary of the previous theorem.

Theorem 2.14. *Let $\psi \equiv z/(z+2)$, and let $C_{\mathbf{\Psi}} := (C_\psi{}^{\lambda_1}, C_\psi{}^{\lambda_2}, \ldots, C_\psi{}^{\lambda_n})$. Then*

$$\sigma_T(C_{\mathbf{\Psi}}) = \overline{u(\Delta)} \cup \{(1,1,\ldots,1)\}$$

where $u \in H^\infty$ is the function defined by

$$u(z) = (e^{k\lambda_1} e_{\lambda_1}(z), e^{k\lambda_2} e_{\lambda_2}(z), \ldots, e^{k\lambda_n} e_{\lambda_n}(z)), \tag{2.9}$$

with $e_{\lambda_i}(z) = \exp\left[\lambda_i \frac{z+1}{z-1}\right]$ and $k := ln\|C_\psi|_{H_0^2}\|$ $(= \frac{-ln2}{2})$.

Remarks. • The function ψ in Theorem 2.14 is equal to ϕ_a with $a = 1$.

• By (1.3) $\sigma(C_\psi) = \overline{\{e^k e_1(z) : z \in \Delta\}} \cup \{1\}$.

• Putting $w := e^k e_1(z)$, $z \in \Delta$, then

$$\sigma_T(C_{\mathbf{\Psi}}) = \overline{\{(w^{\lambda_1}, w^{\lambda_2}, \ldots, w^{\lambda_n})\}} \cup \{(1,1,\ldots,1)\}.$$

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