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GENUS ONE KNOTS WHICH ADMIT (1,1)-DECOMPOSITIONS

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Abstract. We determine the knot types of genus one knots which admit genus one, one bridge decompositions.

1. Introduction

A properly embedded arc t in a solid torus V is *trivial* if there is an embedded disc C in V such that $t \subset \partial C$ and $C \cap \partial V = c\ell(\partial C - t)$. This disc C is called a *cancelling disc* of t. Let M be the 3-sphere S^3 or a lens space (not homeomorphic to $S^1 \times S^2$). A torus H embedded in M is a *genus one Heegaard splitting surface* of M if H splits M into two solid tori V_1 and V_2. A knot K in M is said to be in *genus one* 1*-bridge position* with respect to H if K intersects H transversely in two points and $K \cap V_i$ is a trivial arc in V_i for $i = 1$ and 2. We also say that K is a (1,1)-knot, and that K admits a $(1, 1)$-decomposition. See [3] for a general definition.

A knot K in S^3 is *satellite* if the exterior $E(K) = S^3 - int\, N(K)$ of K in S^3 contains an essential (that is, incompressible and not boundary parallel) torus. A non-trivial knot K in S^3 is *tunnel number one* if there exists an arc τ embedded in S^3 such that $\tau \cap K = \partial\tau$ and $S^3 - int\, N(K \cup \tau)$ is a genus two handlebody. This arc τ is called an *unknotting tunnel* for K. It is known that $(1, 1)$-knots in S^3 are tunnel number one. Morimoto, Sakuma and Yokota [8] showed that there are tunnel number one knots which do not admit $(1, 1)$-decompositions.

Morimoto and Sakuma [7] determined the knot types of satellite tunnel number one knots in S^3. These knots are constructed as follows. Let K_0 be a torus knot of type (p, q) in S^3 with $p \neq 1$ and $q \neq 1$, and let $L = K_1 \cup K_2$ be a 2-bridge link of type (α, β) in S^3 with $\alpha \geq 4$. Note that K_0 is a non-trivial knot, and L is neither a trivial link nor a Hopf link. Since K_2 is the trivial knot in S^3, there is an orientation preserving homeomorphism $f : E(K_2) \to N(K_0)$ which takes a meridian $m_2 \subset \partial E(K_2)$ of K_2 to a fiber $h \subset \partial N(K_0) = \partial E(K_0)$ of the unique Seifert fibration of $E(K_0)$. The knot $f(K_1) \subset N(K_0) \subset S^3$ is denoted by the symbol $K(\alpha, \beta; p, q)$. Every satellite knot of tunnel number one has the form $K(\alpha, \beta; p, q)$ for some integers α, β, p and q. Eudave-Muñoz [4] obtained another description of these knots. These knots are known to admit (1,1)-decompositions.

A non-trivial knot K in S^3 is *free genus one* if K bounds a genus one Seifert surface F such that $S^3 - int\, N(F)$ is a genus two handlebody. Goda and Teragaito [5] showed that the set of satellite knots in S^3 of genus one and tunnel number one is

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the same as the set of $K(8m, 4m+1; p, q)$, where $m \neq 0$. These knots are known to be free genus one. Ozawa [10] showed that the set of satellite knots in S^3 of free genus one is also the same as the set of $K(8m, 4m+1; p, q)$, where $m \neq 0$. In this paper, we prove the following theorem.

Theorem 1.1. *Let K be a genus one knot in S^3 which admits a $(1,1)$-decomposition. Then K is either a 2-bridge knot $K(4\alpha\beta - 1, 2\alpha)$ or a tunnel number one knot $K(8m, 4m+1; p, q)$, where α, β, p, q and m are non-zero integers, $p \neq 1$ and $q \neq 1$.*

Goda and Teragaito raised the following conjecture in [5].

Conjecture 1.2. *Non-satellite genus one, tunnel number one knots in S^3 are 2-bridge.*

By Theorem 1.1, we obtain the following corollary.

Corollary 1.3. *$(1,1)$-knots in S^3 satisfy Conjecture* 1.2.

Theorem 1.1 implies that Conjecture 1.2 is equivalent to the following conjecture.

Conjecture 1.4. *Genus one, tunnel number one knots in S^3 admit $(1,1)$-decompositions.*

2. Preliminary theorems

In this section, we review some results of Hayashi [6] which we use in §3.

A $(1,1)$-decomposition $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ of K in M is said to be *K-reducible* if there are a meridian disc D_i of V_i and a cancelling disc C_j of t_j in V_j such that $D_i \cap t_i = \emptyset$ and $\partial D_i \cap \partial C_j = \emptyset$ on H for $(i, j) = (1, 2)$ or $(2, 1)$. A knot K in M is *trivial* if K bounds an embedded disc in M. The following theorem is Theorem B in [6].

Theorem 2.1. *Let $(M, K) = (V_1, t_1) \cup (V_2, t_2)$ be a $(1,1)$-decomposition of a knot K in M. Then K is trivial if and only if the decomposition is K-reducible.*

A knot K in M is *splittable* if M contains a 2-sphere S which decomposes M into a 3-ball containing K in its interior and a punctured lens space. Note that every knot in S^3 is not splittable. Let F_1 and F_2 be surfaces properly embedded in $M - int\, N(K)$. The surfaces F_1 and F_2 are said to be *roughly isotopic* if either (1) they are isotopic in $M - int\, N(K)$, or (2) the complement of $N(K) \cup F_1 \cup F_2$ contains a once-punctured lens space X, and F_1 and F_2 are isotopic after X is replaced by a 3-ball. If K is not splittable in M, then roughly isotopic surfaces are isotopic in $M - int\, N(K)$.

Let F_0 be an orientable, incompressible and ∂-incompressible surface embedded in $M - int\, N(K)$ such that ∂F_0 consists of p non-meridional loops on $\partial N(K)$. Every component of ∂F_0 winds around K longitudinally q times for some positive integer q. For ease of description, we extend F_0 naturally in M to obtain a surface F such that $\partial F \subset K$, $int\, F$ does not have self-intersections, and every point of K is a pq multiple point of F. The following theorem is Theorem A$'$ in [6].

Theorem 2.2. *Let $(M, K) = (V_1, t_1) \cup (V_2, t_2)$ be a $(1,1)$-decomposition of a knot K in M, and F be a surface as above. Then one of the following holds:*

(1) *The decomposition is K-reducible.*

(2) *The surface F can be roughly isotoped so that $F \cap V_1$ consists of pq cancelling discs of t_1.*

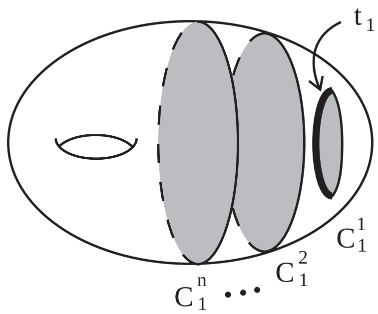

FIGURE 1.

(3) *We can roughly isotope* F *so that* $F \cap V_1$ *consists of two kinds of surfaces: pq cancelling discs of* t_1 *which are mutually isotopic fixing* t_1 *in* V_1*, and a non-empty set of parallel peripheral discs, each of which cuts off a* 3*-ball containing* t_1*.*

3. PROOF OF THEOREM 1.1

Let K be a genus one knot in S^3 which admits a $(1,1)$-decomposition $(S^3, K) = (V_1, t_1) \cup_H (V_2, t_2)$. Let F denote a genus one Seifert surface of K. Note that K is not the trivial knot, and $F \cap E(K)$ is incompressible and ∂-incompressible in $E(K)$. The following theorem follows from Theorems 2.1 and 2.2.

Theorem 3.1. *Let* K *and* F *be as above. Then* F *can be isotoped so that* $F \cap V_1$ *consists of:*

(1) *one cancelling disc* C_1^1 *of* t_1*, and*

(2) $n-1$ *parallel peripheral discs* $C_1^2, \cdots, C_1^n$*, each of which cuts off a* 3*-ball containing* t_1 *in* V_1 *as illustrated in Figure* 1*.*

Note that if $n = 1$, then condition (2) of Theorem 3.1 is vacuous. We say that F is in *standard position* if F satisfies the conditions of Theorem 3.1.

Theorem 3.2. *Let* K *be a non-torus genus one knot in* S^3*. Suppose* K *admits a* $(1,1)$*-decomposition* $(S^3, K) = (V_1, t_1) \cup_H (V_2, t_2)$*. Then* K *is either a* 2*-bridge knot* $K(4\alpha\beta - 1, 2\alpha)$ *or a tunnel number one knot* $K(8m, 4m+1; p, q)$*, where* α, β, p, q *and* m *are non-zero integers,* $p \neq 1$ *and* $q \neq 1$*.*

Since the genus of a torus knot of type (p, q) is $\frac{(|p|-1)(|q|-1)}{2}$ (see, for example, Theorem 7.5.2 in [9]), the genus one torus knots are the right-handed and left-handed trefoil knots. These trefoil knots are represented as the 2-bridge knots $K(3, 2)$ and $K(3, -2)$. Since the trefoil knots are fibered knots, the number of genus one Seifert surfaces of the right-handed and left-handed trefoil knots is one, up to isotopy. See, for example, [1, p. 241] or [11, p. 373]. Then Theorem 1.1 follows from Theorem 3.2.

Proof. Let F be a genus one Seifert surface of K. Suppose F is in standard position, that is, $F \cap V_1$ consists of one cancelling disc C_1^1 of the arc t_1 and $n-1$ parallel peripheral discs $C_1^2, \cdots, C_1^n$. If $n = 1$, then $F \cap V_1$ consists only of one cancelling disc C_1^1 of t_1. We assume that the 3-ball cut off by C_1^k $(k = 2, 3, \cdots, n)$ in V_1 contains C_1^1 and C_1^j for $j = k - 1$. See Figure 1. Let C_2 be a cancelling disc of t_2

in V_2. If $(int\,(C_1^1 \cap H)) \cap (int\,(C_2 \cap H)) = \emptyset$, then K can be isotoped onto H, and K is a torus knot. Hence we may assume $(int\,(C_1^1 \cap H)) \cap (int\,(C_2 \cap H)) \neq \emptyset$.

Isotope C_2 so that $N(t_2; C_2) \cap F = \emptyset$, where $N(t_2; C_2)$ denotes a neighborhood of t_2 in C_2. The *complexity* $c(F)$ for a cancelling disc C_2 of t_2 and a Seifert surface F in standard position is defined as the pair $(n, |C_2 \cap F|)$. We assign the standard lexicographic ordering to this complexity function $c(F)$. We suppose that $c(F)$ is minimal among all cancelling discs of t_2 and all genus one Seifert surfaces of K which are isotopic to F. We may assume, by the minimality of $c(F)$ and the incompressibility of $F \cap E(K)$ in $E(K)$, that each component s of $C_2 \cap F$ in C_2 is a properly embedded arc such that both endpoints of s lie on $C_2 \cap H$. Since $(int\,(C_1^1 \cap H)) \cap (int\,(C_2 \cap H)) \neq \emptyset$, $|C_2 \cap F| \neq 0$.

Let δ be an outermost disc separated by $C_2 \cap F$ in C_2 such that $\partial\delta$ is disjoint from t_2. Let γ denote the arc $\delta \cap H$. An isotopy of F along δ produces a band in V_1 which connects components of $F \cap V_1$. The core of the band is γ. Let $F^{(1)}$ denote the image of F after this isotopy.

First suppose γ connects C_1^1 and C_1^2. Then $F^{(1)}$ is in standard position with $n-2$ parallel peripheral discs, and $c(F^{(1)}) < c(F)$. Next suppose γ connects C_1^i and C_1^{i+1} $(i = 2, 3, \cdots, n-1)$. Then $F^{(1)}$ can be isotoped to F' which is in standard position with $n-3$ parallel peripheral discs, and $c(F') < c(F)$. Next suppose both endpoints of γ lie on $C_1^1 \cap H$, and that a subarc of $C_1^1 \cap H$ together with γ cobounds a disc on H. The disc C_1^1 is deformed to an annulus by an isotopy of F to $F^{(1)}$. Note that $|C_2 \cap F^{(1)}| = |C_2 \cap F| - 1$. There is a compressing disc Δ in V_1 for this annulus such that Δ is disjoint from C_1^j for $j = 2, 3, \cdots, n$. Since $F^{(1)} \cap E(K)$ is incompressible in $E(K)$, $\partial\Delta$ bounds a disc Δ' on $F^{(1)}$. The 2-sphere $\Delta \cup \Delta'$ bounds a 3-ball whose interior is disjoint from $F^{(1)}$ and K. Isotope $F^{(1)}$ to F' so that Δ' is isotoped along the 3-ball to Δ. Then F' is in standard position with $n-1$ or less parallel peripheral discs, and $|C_2 \cap F'| \leq |C_2 \cap F^{(1)}|$. Therefore $c(F') < c(F)$. Next suppose both endpoints of γ lie on $C_1^i \cap H$ $(i = 2, 3, \cdots, n)$, and a subarc of $C_1^i \cap H$ together with γ cobounds a disc on H. Similar arguments as above show that we can isotope $F^{(1)}$ to F' so that F' is in standard position and $c(F') < c(F)$. Hence both endpoints of γ lie on $C_1^n \cap H$, and a subarc of $C_1^n \cap H$ together with γ does not cobound a disc on H, that is, forms a torus knot on H.

Let D_1^n be the component of $F^{(1)} \cap V_1$ which contains the image of C_1^n. Note that D_1^n is an annulus. If $n = 1$, then we proceed to the paragraph just before Lemma 3.3. Suppose $n \geq 2$. The annulus D_1^n is properly embedded in V_1. Let d be a disc in V_1 such that $d \cap D_1^n$ is an essential arc in D_1^n, $d \cap H$ is an arc on H, and that F is obtained by an isotopy of $F^{(1)}$ along d. Let $\delta^{(1)}$ be an outermost disc separated by $C_2 \cap F^{(1)}$ in C_2 such that $\partial\delta^{(1)}$ is disjoint from t_2. Let $\gamma^{(1)}$ denote the arc $\delta^{(1)} \cap H$.

First suppose $\gamma^{(1)}$ connects C_1^{n-1} and D_1^n. We may assume, after an isotopy of d, that d is disjoint from $\gamma^{(1)}$ on H. Isotope $F^{(1)}$ to F' along d and $\delta^{(1)}$. Then F' can be further isotoped to F'' which is in standard position with $n-3$ parallel peripheral discs (respectively without parallel peripheral discs) if $n \geq 4$ (resp. if $n = 2$ and 3). Therefore $c(F'') < c(F)$. Similar arguments as above show that both endpoints of $\gamma^{(1)}$ lie on either D_1^n or C_1^{n-1}. Next suppose both endpoints of $\gamma^{(1)}$ lie on D_1^n. The torus H is separated by ∂D_1^n into two annuli. We may assume, by similar arguments as above, that $\gamma^{(1)}$ is an essential arc of one of the two annuli on H. Let $\widetilde{F^{(1)}}$ be the image of $F^{(1)}$ after an isotopy along $\delta^{(1)}$, and $\widetilde{D_1^n}$

be the component of $\widetilde{F^{(1)}} \cap V_1$ which contains the image of D_1^n. Then $\widetilde{D_1^n}$ is a once-punctured torus, and there is an annulus on the Seifert surface $\widetilde{F^{(1)}}$ cobounded by K and $\partial\widetilde{D_1^n}$. This shows that K is isotopic to the circle $\partial\widetilde{D_1^n}$, and K is the trivial knot. Next suppose that both endpoints of $\gamma^{(1)}$ lie on C_1^{n-1}. Similar arguments as above show that $\gamma^{(1)}$ and a subarc of $C_1^{n-1} \cap H$ form a torus knot on H which is isotopic to a component of ∂D_1^n.

Let $F^{(k)}$ $(k = 2, 3, \cdots, n)$ denote the image of $F^{(k-1)}$ after an isotopy of $F^{(k-1)}$ along an outermost disc $\delta^{(k-1)}$ of $C_2 \cap F^{(k-1)}$ in C_2 which is disjoint from t_2. Similar arguments as above show that both endpoints of the arc $\gamma^{(k)} = \delta^{(k)} \cap H$ $(k = 2, 3, \cdots, n-1)$ lie on C_1^{n-k}, and that the arc $\gamma^{(k)}$ and a subarc of $C_1^{n-k} \cap H$ form a torus knot on H which is isotopic to a component of ∂D_1^n. Since K is not a torus knot, $C_2 \cap F^{(k)} \neq \emptyset$ for $k < n$. The intersection $F^{(k)} \cap V_1$ consists of k annuli and $n-k$ discs for $k = 1, \cdots, n$. Let D_1^j $(j = 1, \cdots, n-1)$ denote the component of $F^{(n)} \cap V_1$ which contains the image of C_1^j.

Suppose $n \geq 1$. Let $G^1, \cdots, G^m$ be components of $F^{(n)} \cap V_2$. Since $F^{(n)} \cap H$ consists of one arc and $2n-1$ loops, $\chi(F^{(n)} \cap H) = 1$. A calculation of Euler characteristics shows that

$$-1 = \chi(F^{(n)}) = (\chi(D_1^1) + \cdots + \chi(D_1^n)) + (\chi(G^1) + \cdots + \chi(G^m)) - \chi(F^{(n)} \cap H),$$

so $\chi(G^1) + \cdots + \chi(G^m) = 0$.

Lemma 3.3. *Every component of $F^{(n)} \cap V_2$ is an annulus.*

Proof. Suppose there is a disc component G of $F^{(n)} \cap V_2$. The component $G \cap H$ is either an arc component of $D_1^1 \cap H$ or a loop component of $D_1^j \cap H$ $(j = 1, \cdots, n)$.

First suppose $G \cap H$ is an arc component of $D_1^1 \cap H$. Note that t_2 is contained in ∂G. The construction of $F^{(n)}$ shows that there is a cancelling disc C_1 of t_1 with $C_1 \cap F^{(n)} = t_1$. The union of discs C_1 and G shows that $K = t_1 \cup t_2$ is isotopic to a torus knot on H.

Next suppose $G \cap H$ is a loop component of $D_1^j \cap H$. Suppose $C_2 \cap G \neq \emptyset$. Each component of $C_2 \cap G$ is a properly embedded arc in C_2 and G. Let γ_G denote an outermost arc of $C_2 \cap G$ in G, and δ_G denote the corresponding outermost disc in G. Let γ_C be the arc of $C_2 \cap G$ in C_2 which corresponds to γ_G, and δ_C be the disc in C_2 cobounded by γ_C and a subarc of $C_2 \cap H$. Let C_2' be the disc $(C_2 - \delta_C) \cup \delta_G$ isotoped slightly off δ_G. This disc C_2' is a cancelling disc of t_2 with $|C_2' \cap G| < |C_2 \cap G|$. So we may suppose $C_2 \cap G = \emptyset$. Then the arc t_2 is a trivial arc in the 3-ball $c\ell(V_2 - N(G))$. Let C_1 be a cancelling disc of t_1 with $C_1 \cap F^{(n)} = t_1$. The arc t_1 can be isotoped into $H - G$ along C_1, and t_1 may be regarded as a trivial arc in the 3-ball $V_1 \cup N(G)$. Therefore $K = t_1 \cup t_2$ is the trivial knot in $S^3 = (V_1 \cup N(G)) \cup (c\ell(V_2 - N(G)))$.

Since $\chi(G^1) + \cdots + \chi(G^m) = 0$ and no component of $F^{(n)} \cap V_2$ is a disc, $\chi(G^1) = \cdots = \chi(G^m) = 0$ and every component of $F^{(n)} \cap V_2$ is an annulus. □

Note that $F^{(n)} \cap H$ consists of one arc and $2n-1$ loops which are essential in $F^{(n)}$. Since $F^{(n)} \cap E(K)$ is incompressible in $E(K)$, each component of $F^{(n)} \cap V_1$ (respectively $F^{(n)} \cap V_2$) is incompressible in V_1 (resp. V_2). Let D_2^1 denote the component of $F^{(n)} \cap V_2$ such that t_2 is contained in ∂D_2^1. Let $D_2^2, \cdots, D_2^n$ denote the other components of $F^{(n)} \cap V_2$. Since D_2^j $(j = 2, 3, \cdots, n)$ is an annulus properly embedded in V_2, D_2^j is boundary parallel in V_2.

First suppose $(int\, C_2) \cap D_2^1 = \emptyset$. Since D_2^1 is incompressible in V_2, the union $C_2 \cup D_2^1$ is an annulus which is boundary parallel in V_2. Then we can find a disc

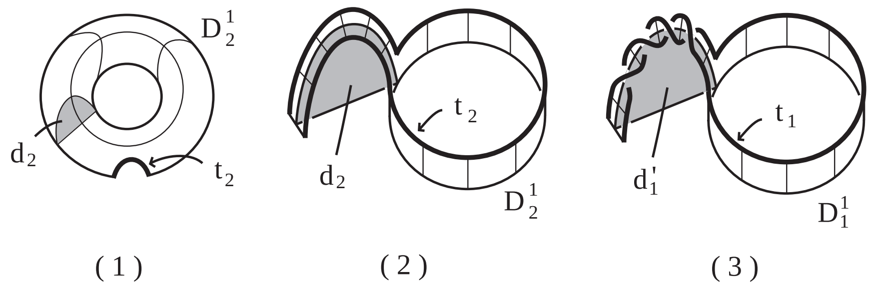

FIGURE 2.

d_2 in V_2 such that $d_2 \cap D_2^1$ is an essential arc on D_2^1, $d_2 \cap H$ is an arc on H, and $(d_2 \cap D_2^1) \cup (d_2 \cap H) = \partial d_2$. See Figure 2 (1).

Next suppose $(int\, C_2) \cap D_2^1 \neq \emptyset$. Let γ be an outermost arc of $C_2 \cap D_2^1$ in C_2, and d_2 be the corresponding outermost disc in C_2 such that ∂d_2 is disjoint from t_2. Similar arguments as in the proof of Lemma 3.3 show that γ corresponds to an essential arc on D_2^1. Then $d_2 \cap D_2^1$ is an essential arc on D_2^1, $d_2 \cap H$ is an arc on H, and $(d_2 \cap D_2^1) \cup (d_2 \cap H) = \partial d_2$. See Figure 2 (1).

Suppose $n \geq 2$. If $n = 1$, we skip this paragraph. Suppose the solid torus of a parallelism between an annulus D_2^i $(i = 2, 3, \cdots, n)$ and an annulus on H does not contain D_2^1. By retaking D_2^i as an outermost annulus, if necessary, we may assume that the interior of the solid torus contains no component of $F^{(n)} \cap V_2$. Suppose D_2^i connects two boundary components of D_1^n. Then the union $D_1^n \cup D_2^i$ is a torus. This contradicts the fact that $F^{(n)}$ is a Seifert surface. Suppose D_2^i connects loop components of ∂D_1^j and ∂D_1^{j+1} $(j = 1, \cdots, n-1)$. Isotope D_2^i into V_1 along the solid torus of the parallelism; then $F^{(n)}$ is isotoped to F' so that $F' \cap H$ consists of one arc and $2n-3$ loops which are essential in F', and that each component of $F' \cap V_1$ (resp. $F' \cap V_2$) is an annulus in V_1 (resp. V_2). Therefore F' satisfies Lemma 3.3 and $|F' \cap H| = |F^{(n)} \cap H| - 2$. So we may suppose the interior of the solid torus W of a parallelism in V_2 between an annulus D_2^i and an annulus on H contains D_2^1 and no D_2^j $(j = 2, 3, \cdots, n)$. Assume $d_2 \cap D_2^i \neq \emptyset$. We may suppose that each component of $d_2 \cap D_2^i$ is an essential arc in D_2^i. Let δ be an outermost disc separated by $d_2 \cap D_2^i$ in d_2 such that $\partial \delta$ is disjoint from $d_2 \cap D_2^1$. Note that each component of ∂D_2^i is parallel to the loop component of D_2^1 on H. Then δ is a meridian disc of the solid torus $W' = c\ell(V_2 - W)$. This shows that each component of ∂D_2^i winds around ∂V_2 once longitudinally, and the solid torus W' realizes a parallelism between D_2^i and an annulus on H such that W' does not contain D_2^1. Hence we may assume $d_2 \cap D_2^i = \emptyset$.

Suppose $n \geq 1$. Let ℓ_2 be the loop component of $D_2^1 \cap H$. Isotope $N(D_2^1; F^{(n)})$ so that the union of $N(\ell_2; D_2^1)$ and $N(d_2 \cap D_2^1; D_2^1)$ is D_2^1. See Figure 2 (2). The loop components of $F^{(n)} \cap H$ separate H into $2n-1$ annuli. Let A be the component of these annuli on H such that A contains the arc component of $F^{(n)} \cap H$. Let c be an essential arc in A such that the arc $d_2 \cap H$ is contained in c. We can take a meridian disc Q of V_2 so that d_2 is contained in Q. Note that c is contained in ∂Q.

The construction of $F^{(n)}$ shows that there is a disc d_1 in V_1 such that $d_1 \cap D_1^j = \emptyset$ $(j = 2, 3, \cdots, n)$, $d_1 \cap D_1^1$ is an essential arc in D_1^1, $d_1 \cap H$ is an arc on A, and $(d_1 \cap D_1^1) \cup (d_1 \cap H) = \partial d_1$. Isotope d_1 so that the two points $\partial(d_1 \cap H)$ are contained in c, and that there is an essential arc a of A such that a is disjoint from c and $d_1 \cap H$. Then the arc $d_1 \cap H$ intersects the arc $d_2 \cap H$ transversely on H. We regard the arc $d_1 \cap H$ as being very short, and we take a neighborhood $N(c; A)$ so that it contains the arc $d_1 \cap H$. Let ℓ_1 be the loop component of $D_1^1 \cap H$. We may regard D_1^1 as the union of $N(\ell_1; D_1^1)$ and $N(d_1 \cap D_1^1; D_1^1)$. The arc $d_1 \cap D_1^1$ together with a subarc of c cobounds a disc d_1' in V_1. See Figure 2 (3). Note that t_1 may intersect d_1'.

First suppose $d_1' \cap H = d_2 \cap H$. Then the union $D_1^1 \cup D_2^1$ is a Seifert surface of K, and $\ell_1 = \ell_2$ on H. This Seifert surface is said to be of type I. The disc which is a union of d_1' and d_2 shows that the Seifert surface of type I is constructed as a plumbing of the two annuli

$$N(\ell_1; D_1^1) \cup N(\ell_2; D_2^1) \quad \text{and} \quad N(\partial d_1' \cap D_1^1; D_1^1) \cup N(d_2 \cap D_2^1; D_2^1).$$

Since the Seifert surface $D_1^1 \cup D_2^1$ is orientable, a sub-disc of $N(d_1' \cap H; A)$ together with the disc $N(\partial d_1' \cap D_1^1; D_1^1)$ forms an annulus P such that $\partial d_1'$ is a core of P. Let k_1 and k_2 be the boundary components of P, and let β denote the linking number of k_1 with k_2, where the orientations of k_1 and k_2 are induced from an orientation of P. If $\beta = 0$, then the disc $d_1' \cup d_2$ is a compressing disc of the Seifert surface $D_1^1 \cup D_2^1$, and K is the trivial knot. First suppose ℓ_1 is a torus knot of type $(1, \alpha)$ on ∂V_1. The loop ℓ_1 is the trivial knot in S^3. If $\alpha = 0$, that is, ℓ_1 is the boundary of a meridian disc of V_1, then the meridian disc is a compressing disc of the Seifert surface $D_1^1 \cup D_2^1$, and K is the trivial knot. The construction of the Seifert surface of type I shows that K is a 2-bridge knot $K(4\alpha\beta - 1, 2\alpha)$, where $\alpha \neq 0$ and $\beta \neq 0$. Note that every genus one 2-bridge knot is represented as $K(4\alpha\beta - 1, 2\alpha)$ for some non-zero integers α and β. See, for example, Proposition 12.25 in [2]. Next suppose ℓ_1 is a torus knot of type $(\alpha, 1)$ on ∂V_1. The same arguments as above show, after interchanging the roles of V_1 and V_2, that K is a 2-bridge knot $K(4\alpha\beta - 1, 2\alpha)$. Next suppose ℓ_1 is a torus knot of type (p, q) on ∂V_1, where $p \neq 1$ and $q \neq 1$. Let H_W be a genus one Heegaard splitting surface of S^3, and W_1, W_2 be the solid tori in S^3 with $W_1 \cap W_2 = \partial W_1 = \partial W_2 = H_W$. Let h be a loop on H_W which is the boundary of a meridian disc of W_2, and let w_2 be a loop in W_2 which is isotopic to a core of W_2. Let U be the solid torus which is a union of $N(D_1^1 \cup D_2^1)$ and $N(d_1' \cup d_2)$. We assume ∂U intersects H in two loops. Let u be a component of $\partial U \cap H$. Let f be a homeomorphism from U to W_1 such that $f(u) = h$. The image $f(K)$ is a knot in W_1. Figure 3 illustrates a pair $(W_1, f(K))$. This pair is uniquely determined, up to isotopy of $f(K)$ in W_1. Then the union $f(K) \cup w_2$ in $S^3 = W_1 \cup W_2$ is a 2-bridge link $K(8\beta, 4\beta + 1)$. This shows that K is a tunnel number one knot $K(8\beta, 4\beta + 1; p, q)$. Since the union of the arcs $\partial d_1' \cap D_1^1$, $\partial d_2 \cap D_2^1$ and the loop ℓ_1 is a deformation retract of $D_1^1 \cup D_2^1$, $S^3 - int\, N(D_1^1 \cup D_2^1)$ is not a handlebody.

Next suppose the union of the two arcs $d_1' \cap H$ and $d_2 \cap H$ is the arc c. Suppose $n \geq 2$. Then D_1^1 connects D_2^1 and D_2^k, and D_2^1 connects D_1^1 and D_1^j $(j, k \neq 1)$. Isotope D_1^1 (resp. D_2^1) into V_2 (resp. V_1) so that the disc $N(\partial d_1' \cap D_1^1; D_1^1)$ (resp. $N(d_2 \cap D_2^1; D_2^1)$) is isotoped along the disc d_1' (resp. d_2). The union of D_1^1 and D_2^k (resp. D_2^1 and D_1^j) is isotoped to an annulus in V_2 (resp. V_1). See Figure 4 for a sketch of this isotopy. Figure 4 illustrates an intersection of the surface $D_1^1 \cup D_2^k \cup D_2^1 \cup D_1^j$ with a plane in S^3 which contains Q. Let F' denote the image of $F^{(n)}$ after this isotopy. Then $F' \cap H$ consists of one arc and $2n - 3$ loops which are essential in F', and every component of $F' \cap V_1$ (resp. $F' \cap V_2$) is an annulus

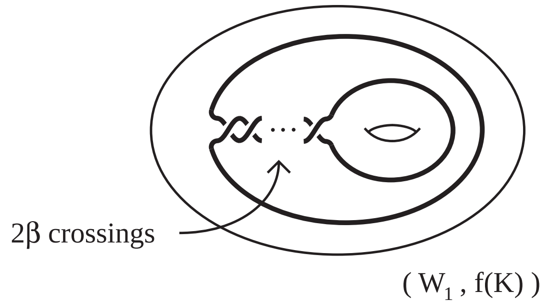

FIGURE 3.

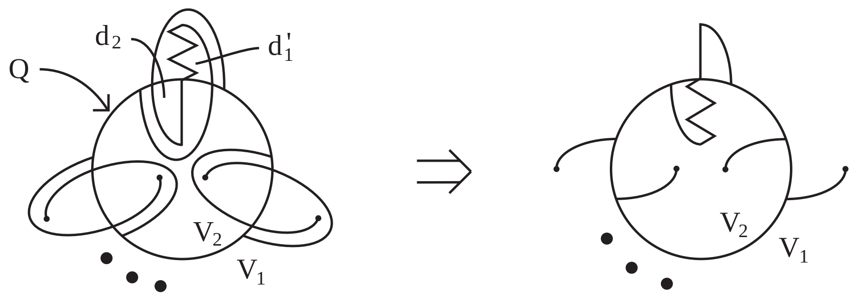

FIGURE 4.

in V_1 (resp. V_2). So we may suppose $n = 1$. The union $D_1^1 \cup D_2^1$ is a Seifert surface of K, and $\ell_1 = \ell_2$ on H. This Seifert surface is said to be of type II. Let P be the same annulus as in the previous paragraph, and let β be the linking number as above. First suppose $\ell_1 = \ell_2$ is a torus knot of type $(\alpha, 1)$ on the torus ∂V_2. The same arguments as in the previous paragraph show $\alpha \neq 0$. The arcs $d_2 \cap D_2^1$ and $c\ell(\partial Q - d_2 \cap H)$ cobound a disc d_2' in Q. The discs d_1' and d_2' show that the Seifert surface $D_1^1 \cup D_2^1$ is of type I. Therefore K is a 2-bridge knot $K(4\alpha\beta - 1, 2\alpha)$, and the Seifert surface of type II is isotopic to that of type I. Next suppose $\ell_1 = \ell_2$ is a torus knot of type $(1, \alpha)$ on ∂V_2. The same arguments as above show, after interchanging the roles of V_1 and V_2, that K is a 2-bridge knot $K(4\alpha\beta - 1, 2\alpha)$, and the Seifert surface of type II is isotopic to that of type I. Next suppose $\ell_1 = \ell_2$ is a torus knot of type (p, q) on ∂V_2, where $p \neq 1$ and $q \neq 1$. Let R_0 be the annulus $H \cap (S^3 - int\, N(\ell_1))$. The annuli D_1^1, D_2^1 and the torus H separate the torus $\partial N(\ell_1)$ into four annuli R_1, R_2, R_3 and R_4. Assume $R_1 \cap d_1' \neq \emptyset$ and $R_2 \cap d_2 \neq \emptyset$. Let R' be the union of the surfaces $(D_1^1 \cup D_2^1) \cap (S^3 - int\, N(\ell_1))$, R_0, R_1 and R_2. Let U be the solid torus which is the union of $N(R')$, $N(d_1')$ and $N(d_2)$. The same arguments as in the previous paragraph show that K is a tunnel number one knot $K(8\beta, 4\beta + 1; p, q)$. The union of the arcs $\partial d_1' \cap D_1^1$, $\partial d_2 \cap D_2^1$ and the loop ℓ_1 is a deformation retract of $D_1^1 \cup D_2^1$. Let τ be the arc which is a union of the arcs $\partial d_1' \cap D_1^1$ and $\partial d_2 \cap D_2^1$. The arc τ may be regarded as an unknotting tunnel for the torus knot ℓ_1, so $S^3 - int\, N(D_1^1 \cup D_2^1)$ is a genus two handlebody. Hence the Seifert surface of type II is not isotopic to that of type I for $K(8\beta, 4\beta + 1; p, q)$. □

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Added in proof

After the author had completed this work, Martin Scharlemann proved Conjecture 1.2 by using Theorem 1.1 in his preprint "There are no unexpected tunnel number one knots of genus one".

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