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A PRIME CURVE IS DETERMINED BY ITS *VF*-MATRIX

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ABSTRACT. We show that a prime generic immersion $S^1 \to S^2$ is determined up to ambient isotopy by its vertex-face matrix, and give an algorithm for obtaining the curve's Gauss code directly from that matrix.

1. THE *VF*-MATRIX OF A CURVE

The *VF*-matrix of a generic curve $S^1 \to S^2$ was introduced in [4], where a connection was given between that matrix and the trip matrix (and thus the Jones polynomial) of an alternating knot diagram constructed from the curve. We begin by recalling the definition of the *VF*-matrix of a generic curve.

By a *curve*, we mean a smooth immersion $k\colon S^1 \to S^2$. We assume that S^1 and S^2 are oriented, and that a basepoint $*$ has been chosen for S^1. By *generic*, we mean that the following conditions are satisfied:

- $k^{-1}(k(*)) = \{*\}$.
- $\{\theta \in S^1\colon \#(k^{-1}(k(\theta))) > 2\} = \emptyset$.
- If $k(\theta_1) = k(\theta_2)$ and $\theta_1 \neq \theta_2$, then $k'(\theta_1)$ and $k'(\theta_2)$ are linearly independent.

Thus the image of k is a based and oriented "smooth curve" in S^2, with a finite number of transverse double points and no other self-intersections. Let n denote the number of double points in $k(S^1)$. Note that $k(*)$ is not a double point.

To each double point, assign a standard basis vector e_i of $(\mathbb{Z}/2\mathbb{Z})^n$ as follows: Start at $k(*)$ and trace along $k(S^1)$ in the positive direction, as determined by the orientation on S^1. To the first double point encountered, assign the vector e_1. Continue to assign vectors in this manner until a bijection has been established between the set of double points and the set of standard basis vectors.

By Euler's Formula, $S^2 - k(S^1)$ consists of $n+2$ regions. Each region has one or more corners, where by a *corner* we mean a location at which a region approaches a double point. Note that a region can have two distinct corners that are adjacent to the same double point. To each corner, assign the vector associated with the double point adjacent to that corner. To each region, assign a vector in $(\mathbb{Z}/2\mathbb{Z})^n$ by summing the vectors from the region's corners.

Now partition the set of regions into a set of black regions and a set of white regions as follows: Rotate the tangent vector $k'(*)$ in the positive sense—as determined by the orientation on S^2—through $\pi/2$ radians. If necessary, shorten this

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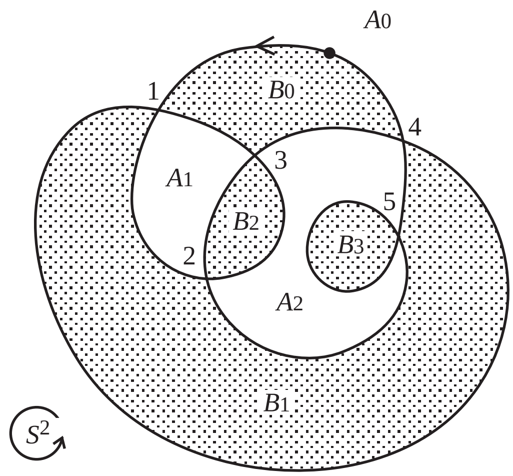

$$VF = \left[\begin{array}{ccccc} 1&1&1&0&0 \\ 0&1&1&1&0 \\ \hline 1&1&0&1&1 \\ 0&1&1&0&0 \\ 0&0&0&0&1 \end{array}\right]$$

FIGURE 1. A curve and its matrix

new vector so it lies entirely in a single region. Label this region B_0 and shade it black. Extend the shading to a checkerboard shading of S^2. Let A_0 denote the white region directly across $k(S^1)$ from B_0 at $k(*)$. Label the remaining white regions $A_1, \dots, A_\alpha$ as follows: Start at $k(*)$ and trace along $k(S^1)$ in the positive direction. To the first unlabeled white region encountered, assign the label A_1. Continue to assign labels in this manner until each white region has received a single label. Use an analogous procedure to label the remaining black regions $B_1, \dots, B_\beta$. Note that $\alpha + \beta = n$.

Recall that each region has been assigned a vector in $(\mathbb{Z}/2\mathbb{Z})^n$. For $i = 0, \dots, \alpha$, let a_i denote the vector assigned to region A_i. For $j = 0, \dots, \beta$, let b_j denote the vector assigned to region B_j. Note that $a_0 + \dots + a_\alpha = 0 = b_0 + \dots + b_\beta$. Let A be the $\alpha \times n$ matrix whose rows are the vectors $a_1, \dots, a_\alpha$, and let $\mathcal{A}$ denote the row space of A. Let $\tilde{A}$ be the $(\alpha+1) \times n$ matrix whose rows are the vectors $a_0, \dots, a_\alpha$. Let B be the $\beta \times n$ matrix whose rows are $b_1, \dots, b_\beta$, and let $\mathcal{B}$ denote the row space of B. Let $\tilde{B}$ be the $(\beta+1) \times n$ matrix whose rows are the vectors $b_0, \dots, b_\beta$. The partitioned $n \times n$ matrix obtained by placing A atop B is called the *vertex-face matrix* of the curve k, and denoted VF_k. See Figure 1 for an example. The partitioned $(n+2) \times n$ matrix obtained by placing $\tilde{A}$ atop $\tilde{B}$ will be called the *extended vertex-face matrix* of k and denoted $\widetilde{VF}_k$. Note that $\widetilde{VF}_k$ is determined by VF_k, since $a_0 = a_1 + \dots + a_\alpha$ and $b_0 = b_1 + \dots + b_\beta$.

By construction, VF_k is invariant under orientation-preserving automorphisms of S^2. That is, let $h\colon S^2 \to S^2$ be an orientation-preserving diffeomorphism and let $\hat{k}\colon S^1 \to S^2$ denote the composition $h \circ k$. Then, keeping the basepoint and orientations fixed, $VF_k = VF_{\hat{k}}$ as partitioned matrices.

The VF-matrix is not a complete invariant of generic curves $S^1 \to S^2$. For example, the curves depicted in Figure 2 share the same VF-matrix, but the images of the curves are not even abstractly homeomorphic. In Section 4 we prove that the VF-matrix *is* a complete invariant of prime generic curves $S^1 \to S^2$, and in Section 5 we give an algorithm for obtaining the Gauss code of such a curve directly from the curve's matrix. Prime curves are discussed in the next section.

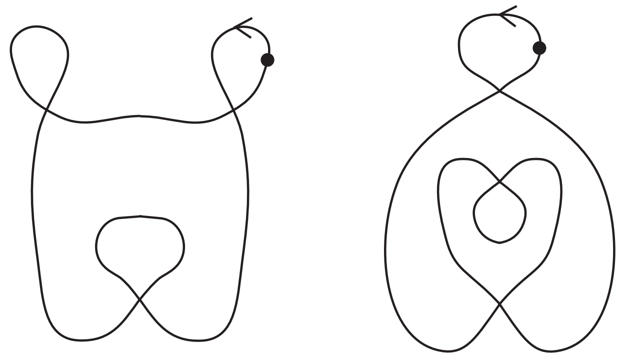

FIGURE 2. Inequivalent curves

2. Prime curves, ordinary and exceptional

Let $k\colon S^1 \to S^2$ be a generic curve. We will say that k is *composite* if there exists a smoothly embedded circle C in S^2 with the following properties:

- C intersects $k(S^1)$ transversely in two points, neither of which is a double point of k.
- Each component of $S^2 - C$ contains at least one double point of k.

If k is not composite, then k will be called *prime*. Note that a curve with fewer than two double points is prime by definition, and that a curve with two double points must be composite. It is also simple to verify

Lemma 2.1. *For a prime curve with three or more double points:*

(a) *Each double point is adjacent to four distinct regions.*
(b) *Each edge joins distinct double points, and is the only edge common to its neighboring black and white regions.*
(c) *Each region has two or more corners, and no two of these are adjacent to the same double point.*

A prime curve k will be called *ordinary* if no two of its regions share the same set of double points, or equivalently, if no two rows of $\widetilde{VF}_k$ are identical. Otherwise k will be called *exceptional*. It is not difficult to verify that an exceptional curve is equivalent to a member of the infinite family of curves suggested in Figure 3.

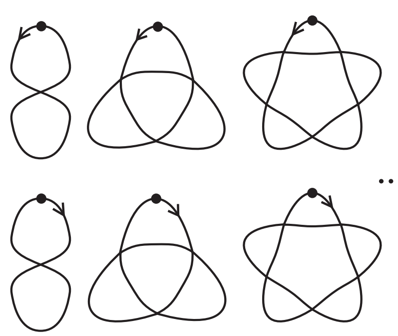

FIGURE 3. Exceptional curves

3. The handle structure determined by a curve

In Section 1 we gave an algorithm that assigns a partitioned square matrix over $\mathbb{Z}/2\mathbb{Z}$ to each generic curve in S^2. In this section we show that each such curve determines a handle structure on S^2, and we describe certain labelings of the handles in this decomposition. We continue with the notation from above, so $k\colon S^1 \to S^2$ is a generic smooth immersion—not necessarily prime—with n double points.

The curve k determines a handle structure $\mathcal{H}$ on S^2 in which there are n 0-handles, $2n$ 1-handles and $n+2$ 2-handles. Each 0-handle will be considered a square, with a double point of $k(S^1)$ as its core and a corner in each of the four local regions adjacent to the double point. See Figure 4. When the interiors of the 0-handles are removed from S^2, what remains of $k(S^1)$ is the disjoint union of $2n$ arcs. Each such arc will form the core of a 1-handle, as suggested in Figure 5. The union of the 0-handles and 1-handles of $\mathcal{H}$ is a regular neighborhood N of $k(S^1)$ in S^2. Each component of $S^2 - \mathrm{int}(N)$ is a disk, which will be considered a 2-handle of $\mathcal{H}$, as in Figure 6.

To each 0-handle, assign the element of $\{1, \ldots, n\}$ that indexes the standard basis vector that corresponds to the double point at the core of the handle. To each corner of each 0-handle, assign the vector in $(\mathbb{Z}/2\mathbb{Z})^n$ associated with the region that contains the corner. To each side of each 0-handle, assign the ordered pair that consists of the vectors associated with the endpoints of the side, with the vector from $\mathcal{A}$ preceding the vector from $\mathcal{B}$. This ordered pair will be called the *label* of the side. To each 1-handle, assign the ordered pair that consists of the vectors associated with the regions adjacent to the core of the handle, with the vector from $\mathcal{A}$ preceding the vector from $\mathcal{B}$. This ordered pair will be called the *label* of the 1-handle. To each 2-handle, assign the vector in $(\mathbb{Z}/2\mathbb{Z})^n$ associated with the region that corresponds to the handle.

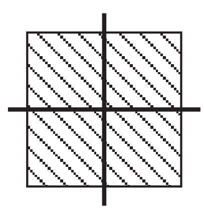

Figure 4. A 0-handle

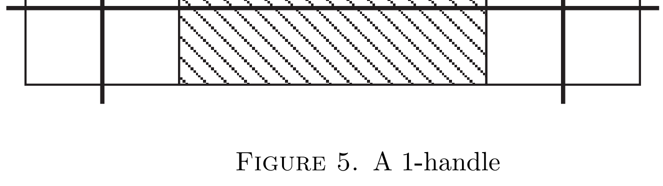

Figure 5. A 1-handle

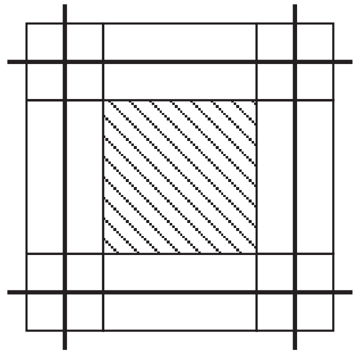

FIGURE 6. A 2-handle

4. The main result

In this section we show that a prime curve is determined by its *VF*-matrix. Specifically, let S^1 be based and oriented, and let S^2 be oriented. Let k_1 and k_2 be generic curves $S^1 \to S^2$.

Theorem 4.1. *If k_1 and k_2 are prime and share the same VF-matrix, then they are equivalent, i.e., there exists an orientation-preserving self-homeomorphism h of S^2 such that $k_2 = h \circ k_1$.*

Remark 4.2. If desired, h can be taken to be a diffeomorphism.

Proof. Since the curves share the same *VF*-matrix, they also share the same extended vertex-face matrix, which we will denote $\widetilde{VF}$, and the same number of double points, which we will denote n. If two rows in $\widetilde{VF}$ are identical, then each of the curves is exceptional, and so each is equivalent to a curve from the family depicted in Figure 3. Since no two distinct curves in that family share the same matrix, it must be the case that k_1 and k_2 are equivalent to the same standard curve, and thus are equivalent. If no two rows in $\widetilde{VF}$ are identical, then each of the curves is ordinary. In this case, we will use the handle structures $\mathcal{H}_1$ and $\mathcal{H}_2$ determined by the curves to construct a suitable automorphism h. The key point here is that the labeling of an ordinary prime curve's handle structure—as described in Section 3—is completely determined by the curve's $\widetilde{VF}$-matrix. So ordinary prime curves that share the same matrix have handle structures that are combinatorially equivalent, which makes the production of a suitable automorphism h rather simple.

Before we begin to construct h, we make two observations about the labels associated with an ordinary prime curve's handle structure. First, no two of the curve's 1-handles share the same label. This follows from Lemma 2.1(b) and the fact that, since the curve is ordinary, no two distinct regions share the same vector. Second, since each of these labels is also the label for each side to which the 1-handle attaches, it follows that the label of each side of each 0-handle is also the label of precisely one side of one other 0-handle. So the sides of 0-handles come in pairs, and we can use the basepoint and the orientation on S^1 to choose a preferred member of each such pair of sides. Specifically, for each set of paired sides, we will use the phrase *initial side* for the side that we encounter first when we trace along the curve's image in the positive sense, starting at the basepoint.

The automorphism h will be constructed in stages. We start by defining h on the 0-handles of $\mathcal{H}_1$. Let $i \in \{1, \dots, n\}$ be arbitrary, and focus on 0-handle i of $\mathcal{H}_1$. By Lemma 2.1(a) and the fact that k_1 is ordinary, the corners of this 0-handle are labeled by four distinct row vectors from $\widetilde{VF}$. Since these are precisely those rows of $\widetilde{VF}$ that have a 1 in column i, these same vectors label the corners of 0-handle i of $\mathcal{H}_2$. Furthermore, since vectors that label adjacent corners must appear on opposite sides of the partition line in $\widetilde{VF}$, it must be the case that each label that appears on a side of 0-handle i of $\mathcal{H}_1$ also appears on some side of 0-handle i of $\mathcal{H}_2$. For each $i \in \{1, \dots, n\}$, define h to take each corner of 0-handle i of $\mathcal{H}_1$ to the corner with the same label in 0-handle i of $\mathcal{H}_2$. Extend h to the boundary of 0-handle i, by mapping each side of the 0-handle onto the corresponding side of 0-handle i of $\mathcal{H}_2$ in a manner that takes the point of intersection of the side and $k_1(S^1)$ to the corresponding point of intersection in $\mathcal{H}_2$. Extend h to all of 0-handle i in a manner that carries the portion of $k_1(S^1)$ within the handle onto the portion of $k_2(S^1)$ within 0-handle i of $\mathcal{H}_2$.

Now extend h to the regular neighborhood N of $k_1(S^1)$, as follows: For each 1-handle of $\mathcal{H}_1$, there is a unique 1-handle of $\mathcal{H}_2$ that has the same label. Because the two 1-handles attach to corresponding sides of corresponding 0-handles, we may extend h over this 1-handle in a manner that carries the portion of $k_1(S^1)$ within the handle onto the portion of $k_2(S^1)$ within the corresponding 1-handle of $\mathcal{H}_2$. Do this for each 1-handle of $\mathcal{H}_1$.

Now extend h to all of S^2 by viewing each 2-handle of $\mathcal{H}_1$ and the corresponding 2-handle of $\mathcal{H}_2$ as cones on their boundaries. Finally, adjust h as needed on $\operatorname{int}(N)$ so that $h(k_1(\theta)) = k_2(\theta)$ for each $\theta \in S^1$. This completes the construction of the homeomorphism h.

To complete the proof, we must show that h is orientation-preserving and that h carries $k_1(S^1)$ onto $k_2(S^1)$ in a manner that preserves the orientations on the curves. For the latter, it suffices to show that, for each set of paired sides in $\mathcal{H}_1$, h carries the initial side in the set to the initial side in the corresponding set of paired sides in $\mathcal{H}_2$. But h has this property by construction, since the initial side in a set of paired sides in $\mathcal{H}_1$ and the initial side in the corresponding set of paired sides in $\mathcal{H}_2$ belong to corresponding 0-handles, as we show in Section 5. To see that h preserves the orientation on S^2, it is now enough to note that, by construction, h takes the black region adjacent to $k_1(*)$ to the black region adjacent to $k_2(*)$. $\square$

5. Extracting the Gauss code

The Gauss code of a generic curve $S^1 \to S^2$ records the sequence of double points we encounter as we trace around the curve's image in the positive direction, starting at the basepoint. For example, the Gauss code of the curve depicted in Figure 1 is $1, 2, 3, 1, 4, 3, 2, 5, 5, 4$. (See [1] for a description of Dowker notation, which is closely related and more compact.) In this section we present an algorithm that extracts the Gauss code of a prime curve directly from the curve's $\widetilde{VF}$-matrix. For a discussion of Gauss codes and their connection to quantum algebras, see [2]. To understand the role of Gauss codes in the theory of virtual knots, see [3]. Throughout this section we use the notation from Section 1, so $k\colon S^1 \to S^2$ is a prime generic immersion with n double points.

If k is exceptional, then n is odd and the Gauss code of k is $1, \dots, n, 1, \dots, n$. If k is ordinary, then the Gauss code can be obtained from the sequence of side labels that is generated as we trace around the curve's image in the positive direction, starting at the basepoint. This sequence of side labels can be obtained from $\widetilde{VF}_k$ as follows: The first label in the sequence is (a_0, b_0) and the first entry in the Gauss code is 1. These are generated as we enter 0-handle 1 along the edge that contains $k(*)$. To determine the label on the side of 0-handle 1 through which we will exit, note that there are precisely four rows in $\widetilde{VF}_k$ that have a 1 in column 1, by Lemma 2.1(a). Two of these row vectors are a_0 and b_0; call the other two a_i and b_j. The second entry in the sequence of side labels is then (a_i, b_j), generated as we exit 0-handle 1. The third entry in the sequence of side labels is also (a_i, b_j), generated as we enter a new 0-handle; call it 0-handle m. To determine m, and thus obtain the next entry in the Gauss code of k, note that there are precisely two indices l for which $a_{il} = 1 = b_{jl}$, by Lemma 2.1(b). One of these is $l = 1$, since 1 labels the 0-handle we just left. The other value of l gives us m. We can now iterate this procedure to generate the complete side label sequence and Gauss code for the curve k.

Remark 5.1. This algorithm can be extended to one that reconstructs an ordinary prime curve's image directly from the curve's $\widetilde{VF}$-matrix.

Remark 5.2. We have shown that the Gauss code of a prime generic curve is determined by the curve's *VF*-matrix. On the set of composite generic curves, the Gauss code and the *VF*-matrix are independent invariants: The curves depicted in Figure 2 share the same *VF*-matrix, but their Gauss codes differ. The curves depicted in Figure 7 have identical Gauss codes, but their matrices are unequal as partitioned matrices.

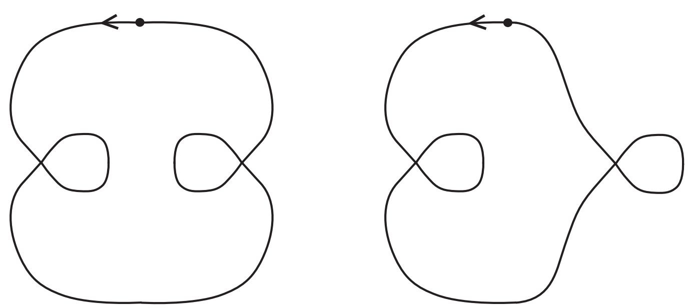

FIGURE 7. Inequivalent curves

Acknowledgement

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