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QUADRATIC INITIAL IDEALS OF ROOT SYSTEMS

HIDEFUMI OHSUGI AND TAKAYUKI HIBI

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ABSTRACT. Let $\Phi \subset \mathbb{Z}^n$ be one of the root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$ and write $\Phi^{(+)}$ for the set of positive roots of Φ together with the origin of $\mathbb{R}^n$. Let $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ denote the Laurent polynomial ring $K[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}, s]$ over a field K and write $\mathcal{R}_K[\Phi^{(+)}]$ for the affine semigroup ring which is generated by those monomials $\mathbf{t}^{\mathbf{a}}s$ with $\mathbf{a} \in \Phi^{(+)}$, where $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_n^{a_n}$ if $\mathbf{a} = (a_1, \dots, a_n)$. Let $K[\Phi^{(+)}] = K[\{x_{\mathbf{a}} ; \mathbf{a} \in \Phi^{(+)}\}]$ denote the polynomial ring over K and write $I_{\Phi^{(+)}}$ $(\subset K[\Phi^{(+)}])$ for the toric ideal of $\Phi^{(+)}$. Thus $I_{\Phi^{(+)}}$ is the kernel of the surjective homomorphism $\pi : K[\Phi^{(+)}] \to \mathcal{R}_K[\Phi^{(+)}]$ defined by setting $\pi(x_{\mathbf{a}}) = \mathbf{t}^{\mathbf{a}}s$ for all $\mathbf{a} \in \Phi^{(+)}$. In their combinatorial study of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov discovered a quadratic initial ideal of the toric ideal $I_{\mathbf{A}_{n-1}^{(+)}}$ of $\mathbf{A}_{n-1}^{(+)}$. The purpose of the present paper is to show the existence of a reverse lexicographic (squarefree) quadratic initial ideal of the toric ideal of each of $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$. It then follows that the convex polytope of the convex hull of each of $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$ possesses a regular unimodular triangulation arising from a flag complex, and that each of the affine semigroup rings $\mathcal{R}_K[\mathbf{B}_n^{(+)}]$, $\mathcal{R}_K[\mathbf{C}_n^{(+)}]$ and $\mathcal{R}_K[\mathbf{D}_n^{(+)}]$ is Koszul.

INTRODUCTION

A *configuration* in $\mathbb{R}^n$ is a finite set $\mathcal{A} \subset \mathbb{Z}^n$. Let $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ denote the Laurent polynomial ring $K[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}, s]$ over a field K. We associate a configuration $\mathcal{A} \subset \mathbb{Z}^n$ with the homogeneous semigroup ring $\mathcal{R}_K[\mathcal{A}] = K[\{\mathbf{t}^{\mathbf{a}}s ; \mathbf{a} \in \mathcal{A}\}]$, the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by all monomials $\mathbf{t}^{\mathbf{a}}s$ with $\mathbf{a} \in \mathcal{A}$, where $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_n^{a_n}$ if $\mathbf{a} = (a_1, \dots, a_n)$. Let $K[\mathcal{A}] = K[\{x_{\mathbf{a}} ; \mathbf{a} \in \mathcal{A}\}]$ be the polynomial ring in the variables $x_{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{A}$ over K. The *toric ideal* $I_{\mathcal{A}}$ of $\mathcal{A}$ is the kernel of the surjective homomorphism $\pi : K[\mathcal{A}] \to \mathcal{R}_K[\mathcal{A}]$ defined by setting $\pi(x_{\mathbf{a}}) = \mathbf{t}^{\mathbf{a}}s$ for all $\mathbf{a} \in \mathcal{A}$. It is known [10, Lemma 4.1] that the toric ideal $I_{\mathcal{A}}$ is generated by the binomials $u - v$, where u and v are monomials of $K[\mathcal{A}]$, with $\pi(u) = \pi(v)$.

Before discussing the details of the present paper, we recall fundamental material on initial ideals of toric ideals. Let $\mathcal{M}(K[\mathcal{A}])$ denote the set of monomials belonging to $K[\mathcal{A}]$. Thus, in particular, $1 \in \mathcal{M}(K[\mathcal{A}])$. Fix a *monomial order* $<$ on $K[\mathcal{A}]$; thus $<$ is a total order on $\mathcal{M}(K[\mathcal{A}])$ such that (i) $1 < u$ if $1 \neq u \in \mathcal{M}(K[\mathcal{A}])$ and (ii) for $u, v, w \in \mathcal{M}(K[\mathcal{A}])$, if $u < v$ then $uw < vw$. The *initial monomial* $in_<(f)$ of $0 \neq f \in I_{\mathcal{A}}$ with respect to $<$ is the biggest monomial appearing in f with respect to $<$. The *initial ideal* of $I_{\mathcal{A}}$ with respect to $<$ is the ideal $in_<(I_{\mathcal{A}})$ of $K[\mathcal{A}]$ generated

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by all initial monomials $in_<(f)$ with $0 \neq f \in I_{\mathcal{A}}$. One of the most fundamental facts on the initial ideal $in_<(I_{\mathcal{A}})$ is that $\{\pi(u)\,;\, u \in \mathcal{M}(K[\mathcal{A}]), u \notin in_<(I_{\mathcal{A}})\}$ is a K-basis of $\mathcal{R}_K[\mathcal{A}]$. An initial ideal $in_<(I_{\mathcal{A}})$ is called *quadratic* (resp. *squarefree*) if $in_<(I_{\mathcal{A}})$ is generated by quadratic (resp. squarefree) monomials. Let, in general, $\mathcal{G}$ be a finite subset of $I_{\mathcal{A}}$ and write $in_<(\mathcal{G})$ for the ideal $(in_<(g)\,;\, g \in \mathcal{G})$ of $K[\mathcal{A}]$. A finite set $\mathcal{G}$ of $I_{\mathcal{A}}$ is said to be a *Gröbner basis* of $I_{\mathcal{A}}$ with respect to $<$ if $in_<(\mathcal{G}) = in_<(I_{\mathcal{A}})$. Dickson's Lemma [4, p. 69], which says that any nonempty subset of $\mathcal{M}(K[\mathcal{A}])$ (in particular, $in_<(I_{\mathcal{A}}) \cap \mathcal{M}(K[\mathcal{A}])$) has only finitely many minimal elements in the partial order by divisibility, guarantees that a Gröbner basis of $I_{\mathcal{A}}$ with respect to $<$ always exists. Moreover, if $\mathcal{G}$ is a Gröbner basis of $I_{\mathcal{A}}$, then $I_{\mathcal{A}}$ is generated by $\mathcal{G}$.

Even though the following fact (0.1) on Gröbner bases is simple and well-known (e.g., [1, Lemma 1.1]) and, in fact, can be easily proved, this technique will play important roles throughout the present paper.

(0.1) *A finite set $\mathcal{G}$ of $I_{\mathcal{A}}$ is a Gröbner basis of $I_{\mathcal{A}}$ with respect to $<$ if and only if $\{\pi(u)\,;\, u \in \mathcal{M}(K[\mathcal{A}]), u \notin in_<(\mathcal{G})\}$ is linearly independent over K; in other words, if and only if $\pi(u) \neq \pi(v)$ for all $u \notin in_<(\mathcal{G})$ and $v \notin in_<(\mathcal{G})$ with $u \neq v$.*

Consult, e.g., [4] and [5] for detailed information about Gröbner bases and related topics on commutative algebra, algebraic geometry and combinatorics.

Let $\Phi \subset \mathbb{Z}^n$ be one of the classical irreducible root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$ ([8, pp. 64 – 65]) and write $\Phi^{(+)}$ for the configuration consisting of the origin of $\mathbb{R}^n$ together with all positive roots of Φ. More explicitly,

$$\begin{aligned}
\mathbf{A}_{n-1}^{(+)} &= \{\mathbf{O}\} \cup \{\mathbf{e}_i - \mathbf{e}_j\,;\, 1 \leq i < j \leq n\},\\
\mathbf{B}_{n}^{(+)} &= \mathbf{A}_{n-1}^{(+)} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \cup \{\mathbf{e}_i + \mathbf{e}_j\,;\, 1 \leq i < j \leq n\},\\
\mathbf{C}_{n}^{(+)} &= \mathbf{A}_{n-1}^{(+)} \cup \{2\mathbf{e}_1, \ldots, 2\mathbf{e}_n\} \cup \{\mathbf{e}_i + \mathbf{e}_j\,;\, 1 \leq i < j \leq n\},\\
\mathbf{D}_{n}^{(+)} &= \mathbf{A}_{n-1}^{(+)} \cup \{\mathbf{e}_i + \mathbf{e}_j\,;\, 1 \leq i < j \leq n\}.
\end{aligned}$$

Here $\mathbf{e}_i$ is the ith unit coordinate vector of $\mathbb{R}^n$ and $\mathbf{O}$ is the origin of $\mathbb{R}^n$.

In the study of combinatorics of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov [7] discovered a squarefree quadratic initial ideal of the toric ideal of the configuration $\mathbf{A}_{n-1}^{(+)}$. Let

$$K[\mathbf{A}_{n-1}^{(+)}] = K[\{x\} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]$$

denote the polynomial ring in $n(n-1)/2+1$ variables over K. The affine semigroup ring $\mathcal{R}_K[\mathbf{A}_{n-1}^{(+)}]$ is the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by s together with the $n(n-1)/2$ monomials $t_i t_j^{-1} s$ with $1 \leq i < j \leq n$. The toric ideal $I_{\mathbf{A}_{n-1}^{(+)}}$ is the kernel of the surjective homomorphism $\pi : K[\mathbf{A}_{n-1}^{(+)}] \to \mathcal{R}_K[\mathbf{A}_{n-1}^{(+)}]$ defined by setting $\pi(x) = s$ and $\pi(f_{i,j}) = t_i t_j^{-1} s$. Write $<_{rev}^{a}$ for the reverse lexicographic order on $K[\mathbf{A}_{n-1}^{(+)}]$ induced by the ordering of the variables satisfying (i) $x < f_{i,j}$ for all $1 \leq i < j \leq n$ and (ii) $f_{i,j} < f_{i',j'}$ if either (a) $i < i'$, or (b) $i = i'$ and $j > j'$. Then [7, Theorem 6.3] guarantees the following:

(0.2) *The set of binomials*

$$\begin{aligned}
f_{i,k} f_{j,\ell} - f_{i,\ell} f_{j,k}, &\qquad i < j < k < \ell,\\
f_{i,j} f_{j,k} - x f_{i,k}, &\qquad i < j < k,
\end{aligned}$$

is a Gröbner basis of $I_{\mathbf{A}_{n-1}^{(+)}}$ with respect to $<_{rev}^{a}$.

Since $f_{i,\ell}f_{j,k} <^a_{rev} f_{i,k}f_{j,\ell}$ if $i < j < k < \ell$, it follows that $in_{<^a_{rev}}(I_{\mathbf{A}^{(+)}_{n-1}})$ is generated by the squarefree quadratic monomials $f_{i,k}f_{j,\ell}$ with $i < j < k < \ell$ and $f_{i,j}f_{j,k}$ with $i < j < k$. (In [7, Theorem 6.3], instead of using the notion of Gröbner bases and initial ideals, Gelfand, Graev and Postnikov state and prove their result in terms of triangulations of the convex hull of the configuration $\mathbf{A}^{(+)}_{n-1}$. A simple and quick proof of the above fact (0.2) is obtained by applying the technique (0.1). Consult, e.g., (Case I) with $r = p = 0$ in the proof of Theorem 1.1.)

It turns out that the question whether the toric ideal of each of the configurations $\mathbf{B}^{(+)}_n$, $\mathbf{C}^{(+)}_n$ and $\mathbf{D}^{(+)}_n$ possesses a squarefree quadratic initial ideal is reasonable. In fact, our goal of the present paper is to show the existence of a reverse lexicographic squarefree quadratic initial ideal of the toric ideal of each of the configurations $\mathbf{B}^{(+)}_n$, $\mathbf{C}^{(+)}_n$ and $\mathbf{D}^{(+)}_n$. On the other hand, in her dissertation [6] Wungkum Fong studies combinatorial aspects of reverse lexicographic initial ideals of the toric ideals of $\mathbf{B}^{(+)}_n$,$\mathbf{C}^{(+)}_n$ and $\mathbf{D}^{(+)}_n$. In our Appendix we briefly discuss the combinatorial and algebraic significance of the existence of squarefree quadratic initial ideals of toric ideals.

We conclude by remarking that the role of the origin of $\mathbb{R}^n$ is essential in our discussions. In fact, the toric ideal of the configuration consisting of all positive roots of the root system $\Phi \subset \mathbb{Z}^n$, where Φ is one of the root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$, is *not* generated by quadratic binomials if $n \geq 6$.

1. The root system $\mathbf{B}_n$

First of all, the existence of a reverse lexicographic squarefree quadratic initial ideal of the toric ideal of the configuration $\mathbf{B}^{(+)}_n$ will be proved. Let

$$K[\mathbf{B}^{(+)}_n] = K[\{x\} \cup \{y_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}]$$

be the polynomial ring in $n^2 + 1$ variables over K. The affine semigroup ring $\mathcal{R}_K[\mathbf{B}^{(+)}_n]$ is the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by the $n^2 + 1$ monomials

$$s, \quad t_i s \ (1 \leq i \leq n), \quad t_i t_j s \ (1 \leq i < j \leq n), \quad t_i t_j^{-1} s \ (1 \leq i < j \leq n).$$

The toric ideal $I_{\mathbf{B}^{(+)}_n}$ is the kernel of the surjective homomorphism $\pi : K[\mathbf{B}^{(+)}_n] \to \mathcal{R}_K[\mathbf{B}^{(+)}_n]$ defined by setting

$$\pi(x) = s, \quad \pi(y_i) = t_i s, \quad \pi(e_{i,j}) = t_i t_j s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

We now introduce the reverse lexicographic order $<^b_{rev}$ on $K[\mathbf{B}^{(+)}_n]$ induced by the ordering of the variables

$$\begin{array}{ll} & y_1 < y_2 < \cdots < y_n \\ < & f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < f_{2,n-1} < \cdots < f_{2,3} \\ < & \cdots < f_{n-2,n} < f_{n-2,n-1} < f_{n-1,n} \\ < & e_{1,n} < e_{1,n-1} < \cdots < e_{1,2} < e_{2,n} < e_{2,n-1} < \cdots < e_{2,3} \\ < & \cdots < e_{n-2,n} < e_{n-2,n-1} < e_{n-1,n} < x. \end{array}$$

To simplify the notation below, we understand $e_{j,i} = e_{i,j}$ in case of $i < j$.

Theorem 1.1. *Under the above conditions, the set of the binomials*

$$
\begin{array}{rl}
e_{i,j}e_{k,\ell} - e_{i,\ell}e_{j,k}, & i<j<k<\ell, \\
e_{i,k}e_{j,\ell} - e_{i,\ell}e_{j,k}, & i<j<k<\ell, \\
f_{i,k}f_{j,\ell} - f_{i,\ell}f_{j,k}, & i<j<k<\ell, \\
f_{i,j}f_{j,k} - xf_{i,k}, & i<j<k, \\
e_{i,j}f_{k,\ell} - f_{i,\ell}e_{j,k}, & i<j<k<\ell, \\
e_{i,k}f_{j,\ell} - f_{i,\ell}e_{j,k}, & i<j<k<\ell, \\
e_{i,\ell}f_{j,k} - f_{i,k}e_{j,\ell}, & i<j<k<\ell, \\
f_{i,j}e_{j,k} - y_iy_k, & i<j, \;\; j \neq k, \\
e_{i,j}y_k - y_ie_{j,k}, & i<j<k, \\
e_{i,k}y_j - y_ie_{j,k}, & i<j<k, \\
f_{i,k}y_j - y_if_{j,k}, & i<j<k, \\
f_{i,j}y_j - y_ix, & i<j, \\
xe_{i,j} - y_iy_j, & i<j,
\end{array}
$$

is a Gröbner basis of $I_{\mathbf{B}_n^{(+)}}$ *with respect to the reverse lexicographic order* $<^b_{rev}$.

Proof. Each binomial $g = u - v$ listed above belongs to $I_{\mathbf{B}_n^{(+)}}$ with u of its initial monomial $in_{<^b_{rev}}(g)$. Let $\mathcal{G}$ denote the set of binomials listed above and $in_{<^b_{rev}}(\mathcal{G}) = (in_{<^b_{rev}}(g)\,;\, g \in \mathcal{G})$. To see why the finite set $\mathcal{G}$ is a Gröbner basis of $I_{\mathbf{B}_n^{(+)}}$, we rely on the technique (0.1), and our work is to show that

$$\{\pi(u)\,;\, u \in \mathcal{M}(K[\mathbf{B}_n^{(+)}]), u \notin in_{<^b_{rev}}(\mathcal{G})\}$$

is linearly independent over K.

Let

$$
\begin{aligned}
u &= x^{\alpha} y_{k_1} \cdots y_{k_r} e_{a_1,b_1} \cdots e_{a_p,b_p} f_{i_1,j_1} \cdots f_{i_q,j_q}, \\
u' &= x^{\alpha'} y_{k'_1} \cdots y_{k'_{r'}} e_{a'_1,b'_1} \cdots e_{a'_{p'},b'_{p'}} f_{i'_1,j'_1} \cdots f_{i'_{q'},j'_{q'}}
\end{aligned}
$$

belong to $\mathcal{M}(K[\mathbf{B}_n^{(+)}])$ with $u \notin in_{<^b_{rev}}(\mathcal{G})$ and $u' \notin in_{<^b_{rev}}(\mathcal{G})$, where

$$
\begin{aligned}
&y_{k_1} \leq^b_{rev} \cdots \leq^b_{rev} y_{k_r}, \;\; e_{a_1,b_1} \leq^b_{rev} \cdots \leq^b_{rev} e_{a_p,b_p}, \;\; f_{i_1,j_1} \leq^b_{rev} \cdots \leq^b_{rev} f_{i_q,j_q}, \\
&y_{k'_1} \leq^b_{rev} \cdots <^b_{rev} y_{k'_{r'}}, \;\; e_{a'_1,b'_1} \leq^b_{rev} \cdots \leq^b_{rev} e_{a'_{p'},b'_{p'}}, \;\; f_{i'_1,j'_1} \leq^b_{rev} \cdots \leq^b_{rev} f_{i'_{q'},j'_{q'}}.
\end{aligned}
$$

In what follows, it will be proved that if $\pi(u) = \pi(u')$, then at least one variable of $K[\mathbf{B}_n^{(+)}]$ appears in both u and u'.

To begin with, the equalities $\alpha = \alpha'$, $r = r'$, $p = p'$ and $q = q'$ will be proved. Since $f_{i,j}f_{j,k} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k$, $f_{i,j}e_{j,k} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j$ with $j \neq k$ and $f_{i,j}y_j \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j$, one has $q = q'$, $\alpha + r + p = \alpha' + r' + p'$ and $r + 2p = r' + 2p'$. If $\alpha = \alpha' = 0$, then $p = p'$ and $r = r'$. If $\alpha \geq \alpha' > 0$, then $p = p' = 0$ since $xe_{i,j} \in in_{<^b_{rev}}(\mathcal{G})$ for all $i < j$; thus $r = r'$ and $\alpha = \alpha'$. If $\alpha = 0$ and $\alpha' > 0$, then $r + p = \alpha' + r'$ and $r + 2p = r'$; thus $\alpha' + p = 0$, a contradiction.

(Case I) *Let* $\alpha = \alpha' = 0$ *and* $q = q' > 0$.

We will prove $f_{i_q,j_q} = f_{i'_q,j'_q}$. Let $i'_q < i_q$. One has $i'_{s'} \leq i'_q < i_q < j_q = j'_{s'}$ for some $1 \leq s' \leq q$. Since $f_{i,k}y_j \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k$, there is no $k'_{\mu'}$ with $i_q \leq k'_{\mu'} < j_q$. Thus $p = p' > 0$ and $a'_1 \leq i_q$. Note, in particular, that $i'_q = i_q$ if $p = p' = 0$.

(i) Let $a_1 < i_q$. Since $e_{i,j}f_{k,\ell} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, $e_{i,k}f_{j,\ell} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k < \ell$ and $e_{i,\ell}f_{j,k} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, one has $b_1 = i_q$. Since $e_{i,j}e_{k,\ell} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, it follows that $a_\xi \leq b_1 = i_q$ for all $1 \leq \xi \leq p$. If $a_\xi < i_q$, then $b_\xi = i_q$. Hence, for each $1 \leq \xi \leq p$, one has either $a_\xi = i_q$ or $b_\xi = i_q$. Thus the total number of the variable t_{i_q} appearing in $\pi(u)$ is at least $p+1$. However, since $k'_{\mu'} = i_q$ for no $1 \leq \mu \leq r$, the total number of the variable t_{i_q} appearing in $\pi(u')$ is at most p, a contradiction.

(ii) Let $i_q \leq a_1$. If either $r = r' = 0$ or $r = r' > 0$ with $k'_r < i_q$, then the total number of variables t_δ with $\delta \geq i_q$ appearing in $\pi(u)$ (resp. $\pi(u')$) is at least $2p+1$ (resp. at most $2p$).

If $r = r' > 0$ with $(i'_{s'} <)\, i_q \leq k'_r$, then $(j'_{s'} =)\, j_q < k'_r$. Since $e_{i,j}y_k \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k$ and $e_{i,k}y_j \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k$, it follows from $a'_1 \leq i_q$ that one has $b'_1 = k'_r$. In addition, if $k'_{\mu'} < k'_r$, then $k'_{\mu'} \leq i'_{s'}$ since $k'_{\mu'} \leq a'_1 < j'_{s'}$. Again, since $e_{i,j}e_{k,\ell} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, there is no $1 \leq \xi' \leq p$ with $(b'_1 =)$ $k'_r < a'_{\xi'}$. Thus, for each $1 \leq \xi' \leq p$, one has either $a'_{\xi'} = k'_r$ or $b'_{\xi'} = k'_r$. Hence the total number of variables t_δ with $k'_r \neq \delta \geq i_q$ appearing in $\pi(u')$ is at most p. Since either $i_q \leq a_\xi \neq k'_r$ or $i_q \leq b_\xi \neq k'_r$ for each $1 \leq \xi \leq p$, the total number of variables t_δ with $k'_r \neq \delta \geq i_q$ appearing in $\pi(u)$ is at least $p+1$.

This completes the proof of $i_q = i'_q$. Let $j'_q < j_q$. Since $f_{i,k}f_{j,\ell} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, there is no $1 \leq \eta \leq q$ with $i_\eta < i_q = i'_q < j'_q = j_\eta < j_q$. Hence $j_q = j'_q$, as desired.

(Case II) *Let $\alpha = \alpha' = 0$, $r = r' > 0$, $p = p' > 0$ and $q = q' = 0$.*

If $k_1 \leq a_1$ and $k'_1 \leq a'_1$, then $k_1 = k'_1$. If $a_1 < k_1$, then $b_1 = k_\mu$ for all $1 \leq \mu \leq r$. One has $a_\xi \leq b_1$ for all $1 \leq \xi \leq p$ and, moreover, $b_\xi = k_1$ if $a_\xi < b_1 = k_1$. Hence the total number of the variable t_{k_1} appearing in $\pi(u)$ is $r+p$. It then follows that $k'_{\mu'} = k_1$ for all $1 \leq \mu' \leq r$ and that each $e_{a'_{\xi'},b'_{\xi'}}$ satisfies either $a'_{\xi'} = k_1$ or $b'_{\xi'} = k_1$.

(Case III) *Let $\alpha = \alpha' = 0$, $r = r' = 0$, $p = p' > 0$ and $q = q' = 0$.*

Even though the desired result follows from [9, Theorem 1.4], we give its quick proof here for the sake of completeness. Let $a'_p < a_p$. Since $e_{i,j}e_{k,\ell} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, each b_ξ satisfies $a_p \leq b_\xi$. In addition, each $a'_{\xi'}$ satisfies $a'_{\xi'} \leq a'_p < a_p$. Hence the total number of variables t_δ with $\delta \geq a_p$ appearing in $\pi(u)$ (resp. $\pi(u')$) is at least $p+1$ (resp. at most p), a contradiction. Thus $a'_p = a_p$. Let $b'_p < b_p$. Since $e_{i,k}e_{j,\ell} \in in_{<^b_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, there is no b_ξ with $b_\xi = b'_p$. Thus $b'_p = b_p$, as required. ☐

2. The root system $\mathbf{C}_n$

Second, the existence of a reverse lexicographic squarefree quadratic initial ideal of the toric ideal of the configuration $\mathbf{C}_n^{(+)}$ will be proved. Let

$$K[\mathbf{C}_n^{(+)}] = K[\{x\} \cup \{e_{i,j}\}_{1 \leq i \leq j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]$$

be the polynomial ring in n^2+1 variables over K. The affine semigroup ring $\mathcal{R}_K[\mathbf{C}_n^{(+)}]$ is the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by the n^2+1 monomials

$$s, \quad t_i t_j s \;\; (1 \leq i \leq j \leq n), \quad t_i t_j^{-1} s \;\; (1 \leq i < j \leq n).$$

The toric ideal $I_{\mathbf{C}_n^{(+)}}$ is the kernel of the surjective homomorphism $\pi : K[\mathbf{C}_n^{(+)}] \to \mathcal{R}_K[\mathbf{C}_n^{(+)}]$ defined by setting

$$\pi(x) = s, \quad \pi(e_{i,j}) = t_i t_j s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

We now introduce the reverse lexicographic order $<_{rev}^c$ on $K[\mathbf{C}_n^{(+)}]$ induced by the ordering of the variables

$$\begin{aligned} x &< f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < f_{2,n-1} < \cdots < f_{2,3} \\ &< \cdots < f_{n-2,n} < f_{n-2,n-1} < f_{n-1,n} \\ &< e_{1,n} < e_{1,n-1} < \cdots < e_{1,2} < e_{1,1} < e_{2,n} < e_{2,n-1} < \cdots < e_{2,3} < e_{2,2} \\ &< \cdots < e_{n-2,n} < e_{n-2,n-1} < e_{n-2,n-2} < e_{n-1,n} < e_{n-1,n-1} < e_{n,n}. \end{aligned}$$

To simplify the notation below, we understand $e_{j,i} = e_{i,j}$ in case of $i < j$.

Theorem 2.1. *The set of the binomials*

$$\begin{array}{rl} e_{i,j}e_{k,\ell} - e_{i,\ell}e_{j,k}, & i \leq j < k \leq \ell, \\ e_{i,k}e_{j,\ell} - e_{i,\ell}e_{j,k}, & i < j < k < \ell, \\ e_{i,j}e_{j,k} - e_{i,k}e_{j,j}, & i < j < k, \\ f_{i,k}f_{j,\ell} - f_{i,\ell}f_{j,k}, & i < j < k < \ell, \\ f_{i,j}f_{j,k} - xf_{i,k}, & i < j < k, \\ e_{i,j}f_{k,\ell} - f_{i,\ell}e_{j,k}, & i \leq j < k < \ell, \\ e_{i,k}f_{j,\ell} - f_{i,\ell}e_{j,k}, & i < j < k < \ell, \\ e_{i,j}f_{j,k} - f_{i,k}e_{j,j}, & i < j < k, \\ e_{i,\ell}f_{j,k} - f_{i,k}e_{j,\ell}, & i < j < k < \ell, \\ f_{i,j}e_{j,k} - xe_{i,k}, & i < j, \end{array}$$

is a Gröbner basis of $I_{\mathbf{C}_n^{(+)}}$ with respect to the reverse lexicographic order $<_{rev}^c$.

Proof. Each binomial $g = u - v$ listed above belongs to $I_{\mathbf{C}_n^{(+)}}$ with $u = in_{<_{rev}^c}(g)$. Let $\mathcal{G}$ denote the set of binomials listed above and $in_{<_{rev}^c}(\mathcal{G}) = (in_{<_{rev}^c}(g)\,;\, g \in \mathcal{G})$. Our proof will proceed along the proof of Theorem 1.1, and the goal is to show that

$$\{\pi(u)\,;\, u \in \mathcal{M}(K[\mathbf{C}_n^{(+)}]), u \notin in_{<_{rev}^c}(\mathcal{G})\}$$

is linearly independent over K.

Let

$$u = x^{\alpha}e_{a_1,b_1}\cdots e_{a_p,b_p}f_{i_1,j_1}\cdots f_{i_q,j_q},$$
$$u' = x^{\alpha'}e_{a'_1,b'_1}\cdots e_{a'_{p'},b'_{p'}}f_{i'_1,j'_1}\cdots f_{i'_{q'},j'_{q'}}$$

belong to $\mathcal{M}(K[\mathbf{C}_n^{(+)}])$ with $u \notin in_{<_{rev}^c}(\mathcal{G})$ and $u' \notin in_{<_{rev}^c}(\mathcal{G})$, where

$$e_{a_1,b_1} \leq_{rev}^c \cdots \leq_{rev}^c e_{a_p,b_p}, \;\; f_{i_1,j_1} \leq_{rev}^c \cdots \leq_{rev}^c f_{i_q,j_q},$$
$$e_{a'_1,b'_1} \leq_{rev}^c \cdots \leq_{rev}^c e_{a'_{p'},b'_{p'}}, \;\; f_{i'_1,j'_1} \leq_{rev}^c \cdots \leq_{rev}^c f_{i'_{q'},j'_{q'}}.$$

In what follows, it will be proved that if $\pi(u) = \pi(u')$, then at least one variable of $K[\mathbf{C}_n^{(+)}]$ appears in both u and u'. Since $f_{i,j}f_{j,k} \in in_{<_{rev}^c}(\mathcal{G})$ if $i < j < k$ and $f_{i,j}e_{j,k} \in in_{<_{rev}^c}(\mathcal{G})$ if $i < j$, one has $\alpha = \alpha'$, $p = p'$ and $q = q'$.

(Case I) *Let $\alpha = \alpha' = 0$ and $q = q' > 0$.*

We claim $f_{i_q,j_q} = f_{i'_q,j'_q}$. Let $i'_q < i_q$. Since the monomials $e_{i,j}f_{k,\ell}$ $(i \leq j < k < \ell)$, $e_{i,k}f_{j,\ell}$ $(i < j < k < \ell)$, $e_{i,j}f_{j,k}$ $(i < j < k)$ and $e_{i,\ell}f_{j,k}$ $(i < j < k < \ell)$ belong to $in_{<_{rev}^c}(\mathcal{G})$, there is no a_ξ with $a_\xi < i_q$. Hence the total number of variables t_δ with $\delta \geq i_q$ appearing in $\pi(u)$ (resp. $\pi(u')$) is at least $2p+1$ (resp. at most $2p$), a contradiction. Thus $i'_q = i_q$. Since $f_{i,k}f_{j,\ell} \in in_{<^c_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, it follows that $j_q = j'_q$, as desired.

(Case II) *Let* $\alpha = \alpha' = 0$ *and* $q = q' = 0$.

A slight modification of Case III in the proof of Theorem 1.1 is valid. Let $a'_p < a_p$. Since $e_{i,j}e_{k,\ell} \in in_{<^c_{rev}}(\mathcal{G})$ if $i \leq j < k \leq \ell$, each b_ξ satisfies $a_p \leq b_\xi$. In addition, each $a'_{\xi'}$ satisfies $a'_{\xi'} \leq a'_p < a_p$. Hence the total number of variables t_δ with $\delta \geq a_p$ appearing in $\pi(u)$ (resp. $\pi(u')$) is at least $p+1$ (resp. at most p), a contradiction. Thus $a'_p = a_p$. Let $b'_p < b_p$. Since $e_{i,k}e_{j,\ell} \in in_{<^c_{rev}}(\mathcal{G})$ if $i < j < k < \ell$ and $e_{i,j}e_{j,k} \in in_{<^c_{rev}}(\mathcal{G})$ if $i < j < k$, each b_ξ satisfies $b_p \leq b_\xi$. Moreover, each $a'_{\xi'}$ satisfies $a'_{\xi'} \leq a'_p \leq b'_p < b_p$. Hence the total number of variables t_δ with $\delta \geq b_p$ appearing in $\pi(u)$ (resp. $\pi(u')$) is at least p (resp. at most $p-1$), a contradiction. Thus $b'_p = b_p$, as required. ☐

3. The root system $\mathbf{D}_n$

Finally, the existence of a reverse lexicographic squarefree quadratic initial ideal of the toric ideal of the configuration $\mathbf{D}_n^{(+)}$ will be proved. Let

$$K[\mathbf{D}_n^{(+)}] = K[\{x\} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]$$

be the polynomial ring in $n^2 - n + 1$ variables over K. The affine semigroup ring $\mathcal{R}_K[\mathbf{D}_n^{(+)}]$ is the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by the $n^2 - n + 1$ monomials

$$s, \quad t_i t_j s \ \ (1 \leq i < j \leq n), \quad t_i t_j^{-1} s \ \ (1 \leq i < j \leq n).$$

The toric ideal $I_{\mathbf{D}_n^{(+)}}$ is the kernel of the surjective homomorphism $\pi : K[\mathbf{D}_n^{(+)}] \to \mathcal{R}_K[\mathbf{D}_n^{(+)}]$ defined by setting

$$\pi(x) = s, \quad \pi(e_{i,j}) = t_i t_j s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

We now introduce the reverse lexicographic order $<^d_{rev}$ on $K[\mathbf{D}_n^{(+)}]$ induced by the ordering of the variables

$$\begin{aligned}
& f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < f_{2,n-1} < \cdots < f_{2,3} \\
< \ & \cdots < f_{n-2,n} < f_{n-2,n-1} < f_{n-1,n} \\
< \ & e_{1,n} < e_{1,n-1} < \cdots < e_{1,2} < e_{2,n} < e_{2,n-1} < \cdots < e_{2,3} \\
< \ & \cdots < e_{n-2,n} < e_{n-2,n-1} < e_{n-1,n} < x.
\end{aligned}$$

Theorem 3.1. *The set of the binomials*

$$\begin{array}{rl}
e_{i,j}e_{k,\ell} - e_{i,\ell}e_{j,k}, & i < j < k < \ell, \\
e_{i,k}e_{j,\ell} - e_{i,\ell}e_{j,k}, & i < j < k < \ell, \\
f_{i,k}f_{j,\ell} - f_{i,\ell}f_{j,k}, & i < j < k < \ell, \\
f_{i,j}f_{j,k} - x f_{i,k}, & i < j < k, \\
e_{i,j}f_{k,\ell} - f_{i,\ell}e_{j,k}, & i < j < k < \ell, \\
e_{i,k}f_{j,\ell} - f_{i,\ell}e_{j,k}, & i < j < k < \ell, \\
e_{i,\ell}f_{j,k} - f_{i,k}e_{j,\ell}, & i < j < k < \ell, \\
f_{i,k}e_{j,k} - f_{i,n}e_{j,n} & i \leq j < k < n, \\
e_{i,k}f_{j,k} - f_{i,n}e_{j,n} & i < j < k \leq n, \\
f_{i,j}e_{j,k} - f_{i,n}e_{k,n} & i < j < k < n,
\end{array}$$

$$
\begin{array}{rl}
f_{i,j}e_{j,n} - f_{i,n-1}e_{n-1,n} & i < j < n-1, \\
xe_{i,j} - f_{i,n}e_{j,n} & i < j < n, \\
xe_{i,n} - f_{i,n-1}e_{n-1,n} & i < n-1,
\end{array}
$$

is a Gröbner basis of $I_{\mathbf{D}_n^{(+)}}$ *with respect to the reverse lexicographic order* $<_{rev}^{d}$.

Proof. Each binomial $g = u - v$ listed above belongs to $I_{\mathbf{D}_n^{(+)}}$ with $u = in_{<_{rev}^{d}}(g)$. Write $\mathcal{G}$ for the set of binomials listed above and $in_{<_{rev}^{d}}(\mathcal{G}) = (in_{<_{rev}^{d}}(g)\,;\, g \in \mathcal{G})$. Again, our work is to show that

$$\{\pi(u)\,;\, u \in \mathcal{M}(K[\mathbf{D}_n^{(+)}]), u \notin in_{<_{rev}^{d}}(\mathcal{G})\}$$

is linearly independent over K.

Let

$$
\begin{aligned}
u &= x^{\alpha} f_{i_1,j_1} \cdots f_{i_q,j_q} e_{a_1,b_1} \cdots e_{a_p,b_p}, \\
u' &= x^{\alpha'} f_{i'_1,j'_1} \cdots f_{i'_{q'},j'_{q'}} e_{a'_1,b'_1} \cdots e_{a'_{p'},b'_{p'}}
\end{aligned}
$$

belong to $\mathcal{M}(K[\mathbf{D}_n^{(+)}])$ with $u \notin in_{<_{rev}^{d}}(\mathcal{G})$ and $u' \notin in_{<_{rev}^{d}}(\mathcal{G})$, where

$$
\begin{gathered}
f_{i_1,j_1} \leq_{rev}^{d} \cdots \leq_{rev}^{d} f_{i_q,j_q}, \;\; e_{a_1,b_1} \leq_{rev}^{d} \cdots \leq_{rev}^{d} e_{a_p,b_p}, \\
f_{i'_1,j'_1} \leq_{rev}^{d} \cdots \leq_{rev}^{d} f_{i'_{q'},j'_{q'}}, \;\; e_{a'_1,b'_1} \leq_{rev}^{d} \cdots \leq_{rev}^{d} e_{a'_{p'},b'_{p'}}.
\end{gathered}
$$

Write p_1 (resp. p'_1) for the number of e_{a_ξ,b_ξ} (resp. $e_{a'_{\xi'},b'_{\xi'}}$) with $b_\xi = n$ (resp. $b'_{\xi'} = n$) and write p_2 (resp. p'_2) for the number of $e_{n-1,n}$ appearing in u (resp. u'). Write q_1 (resp. q'_1) for the number of f_{i_η,j_η} (resp. $f_{i'_{\eta'},j'_{\eta'}}$) with $j_\eta = n$ (resp. $j'_{\eta'} = n$) and write q_2 (resp. q'_2) for the number of f_{i_η,j_η} (resp. $f_{i'_{\eta'},j'_{\eta'}}$) with $j_\eta = n-1$ (resp. $j'_{\eta'} = n-1$). Let

$$r = \min\{p_1, q_1\} + \min\{p_2, q_2\}, \quad r' = \min\{p'_1, q'_1\} + \min\{p'_2, q'_2\}.$$

If $\pi(u) = \pi(u')$, then

$$p = p', \quad \alpha + q = \alpha' + q', \quad \alpha + r = \alpha' + r', \quad p_1 - q_1 = p'_1 - q'_1.$$

Now, in what follows, it will be proved that a contradiction arises if $\pi(u) = \pi(u')$ and if none of the variables of $K[\mathbf{D}_n^{(+)}]$ appears in both u and u'.

(Case I) *Let* $p = 0$.

Since $f_{i,j}f_{j,k} \in in_{<_{rev}^{d}}(\mathcal{G})$ if $i < j < k$, one has $i_q = i'_q$. Since $f_{i,k}f_{j,\ell} \in in_{<_{rev}^{d}}(\mathcal{G})$ if $i < j < k < \ell$, it follows that $j_q = j'_q$, as required.

(Case II) *Let* $p > 0$, $\alpha > 0$ *and* $\alpha' = 0$.

Since $xe_{i,j} \in in_{<_{rev}}(\mathcal{G})$ unless $i = n-1$ and $j = n$, the monomial u is of the form

$$u = x^{\alpha} f_{i_1,j_1} \cdots f_{i_q,j_q} e_{n-1,n}^{p}.$$

Since $p > 0$, one has $p'_2 = 0$. Let $q'_1 \leq p'_1$. Then $r \geq q_1$ and $r' = q'_1$. Since $p'_1 - q'_1 = p - q_1$ and $q'_1 \geq q_1 + \alpha$, one has $p'_1 > p$, a contradiction. Let $q'_1 > p'_1$. Then $r \geq p$ and $r' = p'_1$. This contradicts $\alpha + r = r'$.

(Case III) *Let* $p > 0$, $\alpha = \alpha' = 0$ *and* $q = 0$.

Since $e_{i,j}e_{k,\ell} \in in_{<_{rev}^{d}}(\mathcal{G})$ if $i < j < k < \ell$ and $e_{i,k}e_{j,\ell} \in in_{<_{rev}^{d}}(\mathcal{G})$ if $i < j < k < \ell$, the required argument coincides with Case III in the proof of Theorem 1.1.

(Case IV) *Let* $\alpha = \alpha' = 0$, $p_2 > 0$ *and* $q_2 > 0$.

Since $p_2 > 0$, one has $p'_2 = 0$. Since $e_{i,j}e_{k,\ell} \in in_{<_{rev}^{d}}(\mathcal{G})$ if $i < j < k < \ell$ and since $p_2 > 0$, it follows that each e_{a_ξ,b_ξ} appearing in u satisfies either $b_\xi = n-1$ or

$b_\xi = n$. However, since $q_2 > 0$ and since $f_{i,n-1}e_{j,n-1} \in in_{<^d_{rev}}(\mathcal{G})$ if $i < n-1$ and $j < n-1$, each e_{a_ξ,b_ξ} appearing in u must satisfy $b_\xi = n$. Thus $p = p_1$.

Let $q'_1 \leq p'_1$. Then $p - q_1 = p'_1 - q'_1$, $r' = q'_1$ and $r > q_1$. Since $p \geq p'_1$ and $q_1 < q'_1$, a contradiction arises. Let $q'_1 > p'_1$. Then $r' = p'_1$ and $r > p$, a contradiction.

(Case V) *Let $\alpha = \alpha' = 0$, $p > 0$, $q > 0$, $p_2q_2 = 0$ and $p'_2q'_2 = 0$.*

Since $r = r'$ and $p_1 - q_1 = p'_1 - q'_1$, one has $p_1 = p'_1$ and $q_1 = q'_1$. Let $\pi(u)^+ = \prod_{\xi=1}^{p} t_{a_\xi}t_{b_\xi}\prod_{\eta=1}^{q} t_{i_\eta}$, $\pi(u')^+ = \prod_{\xi'=1}^{p} t_{a'_{\xi'}}t_{b'_{\xi'}}\prod_{\eta'=1}^{q} t_{i'_{\eta'}}$, $\pi(u)^- = \prod_{\eta=1}^{q} t_{j_\eta}^{-1}$ and $\pi(u')^- = \prod_{\eta'=1}^{q} t_{j'_{\eta'}}^{-1}$. Then $\pi(u)^+ = \pi(u')^+$ and $\pi(u)^- = \pi(u')^-$.

(i) Let $i'_q < i_q$ and $a_1 < i_q$. Since $e_{i,j}f_{k,\ell}$, $e_{i,k}f_{j,\ell}$ and $e_{i,\ell}f_{j,k}$ belong to $in_{<^d_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, it follows that if $a_\xi < i_q$ then $b_\xi = i_q$. In particular, $b_1 = i_q$. Since $e_{i,j}e_{k,\ell} \in in_{<^d_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, one has either $a_\xi = i_q$ or $b_\xi = i_q$. Hence the variable t_{i_q} appears in $\pi(u)^+$ at least $p+1$ times. However, t_{i_q} appears in $\pi(u')^+$ at most p times, a contradiction.

(ii) Let $i'_q < i_q$ and $a_1 \geq i_q$. Then each a_ξ satisfies $a_\xi \geq i_q$. Hence the variables t_δ with $\delta \geq i_q$ appear in $\pi(u)^+$ at least $2p+1$ times. However, t_δ with $\delta \geq i_q$ appear in $\pi(u')^+$ at most $2p$ times, a contradiction.

(iii) Let $i'_q = i_q$ and $j'_q < j_q$. Since $f_{i,k}f_{j,\ell} \in in_{<^d_{rev}}(\mathcal{G})$ if $i < j < k < \ell$, it follows that $t_{j'_q}^{-1}$ cannot appear in $\pi(u)^-$, a contradiction. $\square$

Appendix

We briefly discuss the combinatorial and algebraic significance of the existence of (squarefree) quadratic initial ideals of toric ideals. We work with the same notation $\mathcal{A} \subset \mathbb{Z}^n$, $\mathcal{R}_K[\mathcal{A}]$, $K[\mathcal{A}]$ and $I_\mathcal{A}$ as in the Introduction.

First, a fundamental question in commutative algebra is to determine whether $\mathcal{R}_K[\mathcal{A}]$ is Koszul [2]. Even though it is difficult to prove that $\mathcal{R}_K[\mathcal{A}]$ is Koszul, the hierarchy (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is known, e.g., [3], among the following properties:

(i) $I_\mathcal{A}$ possesses a quadratic initial ideal;

(ii) $\mathcal{R}_K[\mathcal{A}]$ is Koszul;

(iii) $I_\mathcal{A}$ is generated by quadratic binomials.

Thus, in particular, each of the affine semigroup rings $\mathcal{R}_K[\mathbf{A}_{n-1}^{(+)}]$, $\mathcal{R}_K[\mathbf{B}_n^{(+)}]$, $\mathcal{R}_K[\mathbf{C}_n^{(+)}]$ and $\mathcal{R}_K[\mathbf{D}_n^{(+)}]$ is Koszul.

Second, to construct a triangulation of the convex polytope $\mathrm{conv}(\mathcal{A}) \subset \mathbb{R}^n$, the convex hull of $\mathcal{A}$, is one of the most traditional topics in discrete geometry and combinatorics. Let $<$ be any monomial order on $K[\mathcal{A}]$ and $\sqrt{in_<(I_\mathcal{A})}$ the radical of the initial ideal $in_<(I_\mathcal{A})$. Write $\Delta(in_<(I_\mathcal{A}))$ for the (abstract) simplicial complex on the vertex set $\mathcal{A}$ whose faces are those subsets $\mathcal{A}' \subset \mathcal{A}$ with $\prod_{\mathbf{a}\in\mathcal{A}'} x_\mathbf{a} \notin \sqrt{in_<(I_\mathcal{A})}$. It is known [10, Theorem 8.3] that $|\Delta(in_<(I_\mathcal{A}))| = \{\mathrm{conv}(\mathcal{A}')\,;\,\mathcal{A}' \in \Delta(in_<(I_\mathcal{A}))\}$ is a triangulation of $\mathrm{conv}(\mathcal{A})$. Such a triangulation is called *regular*. Moreover, $|\Delta(in_<(I_\mathcal{A}))|$ is *unimodular* (i.e., the normalized volume of each simplex $\mathrm{conv}(\mathcal{A}')$ with $\mathcal{A}' \in \Delta(in_<(I_\mathcal{A}))$ is equal to 1) if and only if $in_<(I_\mathcal{A})$ is squarefree.

When $in_<(I_\mathcal{A})$ is squarefree and quadratic, the simplicial complex $\Delta(in_<(I_\mathcal{A}))$ is a flag complex, i.e., every minimal nonface of $\Delta(in_<(I_\mathcal{A}))$ is a two-element subset of $\mathcal{A}$. Thus the combinatorics on $|\Delta(in_<(I_\mathcal{A}))|$ can be discussed in terms of the skeleton $\Delta(in_<(I_\mathcal{A}))^{(1)}$ of $\Delta(in_<(I_\mathcal{A}))$, the finite graph on the vertex set $\mathcal{A}$ whose edge set consists of all two-element faces of $\Delta(in_<(I_\mathcal{A}))$. For example, the normalized volume of $\mathrm{conv}(\mathcal{A})$ coincides with the number of maximal complete subgraphs of $\Delta(in_<(I_\mathcal{A}))^{(1)}$. Note that the normalized volume of the convex polytope

$\mathrm{conv}(\mathbf{A}_{n-1}^{(+)})$ is computed in [7, Theorem 6.4] and the normalized volume of each of the convex polytopes $\mathrm{conv}(\mathbf{B}_n^{(+)})$, $\mathrm{conv}(\mathbf{C}_n^{(+)})$ and $\mathrm{conv}(\mathbf{D}_n^{(+)})$ is computed in Fong [6].

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560–0043, Japan

E-mail address: ohsugi@math.sci.osaka-u.ac.jp

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560–0043, Japan

E-mail address: hibi@math.sci.osaka-u.ac.jp