PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 7, Pages 1975–1980
S 0002-9939(02)06357-8
Article electronically published on February 27, 2002

ON THE HARTOGS–BOCHNER PHENOMENON FOR CR FUNCTIONS IN $P_2(\mathbb{C})$

ROMAN DWILEWICZ AND JOËL MERKER

(Communicated by Steven R. Bell)

Abstract. Let M be a compact, connected, $\mathcal{C}^2$-smooth and *globally minimal* hypersurface M in $P_2(\mathbb{C})$ which divides the projective space into two connected parts U^+ and U^-. We prove that there exists a side, U^- or U^+, such that every continuous CR function on M extends holomorphically to this side. Our proof of this theorem is a simplification of a result originally due to F. Sarkis.

1. Introduction

Historically, one of the amazing theorems in the theory of holomorphic functions of several complex variables is the theorem of Hartogs of 1906 about extension of functions from a neighborhood of the boundary of a domain to the inside of the domain. This theorem was generalized by Bochner in 1943, namely that it is enough to consider $\mathcal{C}^1$-smooth functions defined just on the boundary of the domain that satisfy the tangential Cauchy-Riemann equations. Since that time many versions and generalizations of the theorem appeared; see, for instance, Kohn-Rossi [12], Ehrenpreis [3], Ivashkovich [10], Harvey [7], Harvey-Lawson [8], Laurent-Thiébaut [13], Dolbeault-Henkin [2], Sarkis [16], [17], and others — for a review see [13].

Recently, solving the boundary problem in the sense of Harvey and Lawson for the graph of CR functions defined on boundaries of domains in disc-convex Kähler manifolds, Sarkis [16] proved the extension of CR-meromorphic mappings with values in $P_2(\mathbb{C})$ (for $P_n(\mathbb{C})$, $n \geq 3$, see [8], part II). As an application, he obtained Theorem 1.1 below. Our contribution to this subject essentially lies in a simplification of his proof, as will appear below.

Recall that a CR manifold M (locally embeddable or not) is called *globally minimal* if any two points of it can be joined by a piece-wise smooth curve running in complex tangential directions ([20]). The result is the following:

Theorem 1.1. *Let M be a compact connected $\mathcal{C}^2$-smooth real hypersurface in $P_2(\mathbb{C})$ that divides the projective space into two open parts U^- and U^+. If M is globally minimal, then*

Received by the editors December 13, 2000.
2000 *Mathematics Subject Classification.* Primary 32V25; Secondary 32V10, 32V15, 32D15.
Key words and phrases. Smooth hypersurfaces of the complex projective space, holomorphic extension of CR functions, jump formula, global minimality, one-sided neighborhood.
This research was partially supported by a grant of the Polish Committee for Scientific Research KBN 2 PO3A 044 15 and by a grant from the French-Polish program "Polonium 1999".

(1) *There exists a side,* U^- or U^+*, to which every continuous CR function on* M *extends holomorphically.*
(2) *All holomorphic functions on the other side of* M *which are continuous up to* M *are constant.*

Remarks. Theorem 1.1 also holds true (cf. [16], [17]) if, instead of a globally minimal M as above and instead of CR functions on M, we consider an arbitrary $\mathcal{C}^2$-smooth hypersurface as above and *holomorphic functions in a neighborhood* $\mathcal{V}$ *of* M *in* $P_2(\mathbb{C})$. Our proof is valid in $P_n(\mathbb{C})$ for $n \geq 2$. Of course, the implication **(1)** $\Rightarrow$ **(2)** is trivial.

Summary of the proof. Let f be a continuous CR function on M. Using global minimality, we shall extend f to a one-sided neighborhood $\mathcal{V}^{\pm}(M)$ of M in $P_2(\mathbb{C})$. By deforming M into $\mathcal{V}^{\pm}(M)$, we shall argue that we can suppose that M and f are $\mathcal{C}^\infty$-smooth in the assumptions of Theorem 1.1, and even that f is holomorphic in a neighborhood of M. As the Dolbeault cohomology group $H^{0,1}(P_n(\mathbb{C}))$ vanishes for $n \geq 2$ (see [6], [9]), every $\mathcal{C}^\infty$ CR function f on M can be decomposed as a jump $f = f^+ - f^-$ of some functions $f^{\pm}$, holomorphic on $U^{\pm}$ and $\mathcal{C}^\infty$ on $\overline{U^{\pm}}$. This decomposition property (which holds without assuming global minimality) easily implies that points **(1)** and **(2)** from Theorem 1.1 are equivalent in the $\mathcal{C}^\infty$ category (see Lemma 3.1). Afterwards, using a theorem of Takeuchi [19], we may embed a strip neighborhood of M in $\mathbb{C}^N$ which then bounds a complex manifold to which f^- *and* f^+ *both extend holomorphically* (see [7] and [8], part I), and we easily deduce using the maximum principle that either f^- or f^+ is constant.

Open question. In Theorem 1.1, the question arises of whether the assumption that M is globally minimal can be removed. This question appears to have deep relations to some well-known conjectures of foliation theory, especially the (non-)existence of Levi flat hypersurfaces in projective spaces. Indeed, every compact CR manifold $M \subset P_2(\mathbb{C})$ can be decomposed as a disjoint union of CR orbits. Some of them are of dimension 3, and the others are of dimension 2, i.e., Riemann surfaces, which cannot be closedly embedded in M. The closure of each such Riemann surface defines a non-trivial lamination of $P_2(\mathbb{C})$ and every other Riemann surface orbit, having common points with the closure, is dense in this lamination. *The open question is whether such laminations exist* (cf. [5]). For $n \geq 3$ (only), this question has been negatively fixed for laminations arising from global foliations of $P_n(\mathbb{C})$, for real analytic Levi-flat hypersurfaces (see Cerveau [1], Lins Neto [14]) and for smooth Levi-flat hypersurfaces (see Siu [18]). Thus, as a part of the folklore, we conjecture that every smooth hypersurface $M \subset P_n(\mathbb{C})$, $n \geq 2$, is globally minimal.

2. Decomposition of CR functions

As we are essentially interested in the geometric approach and not to the best suited regularity assumptions, we shall restrict our attention to the $\mathcal{C}^\infty$-smooth category. By a deformation argument, we shall see in §3 below that the case where M is $\mathcal{C}^2$ and the CR function f on M is only continuous can be reduced to the case where both M and f are $\mathcal{C}^\infty$. We recall that $H^{0,1}(P_2(\mathbb{C})) = 0$ (for $\mathcal{C}^\infty$-smooth forms). It is easy to deduce:

Lemma 2.1. *Let* M *be a* $\mathcal{C}^\infty$ *real hypersurface in* $P_2(\mathbb{C})$ *dividing it into two open parts* U^- *and* U^+*. Every* $\mathcal{C}^\infty$*-smooth CR function* f *on* M *decomposes as* $f = f^+ - f^-$*, where* $f^{\pm} \in \mathcal{O}(U^{\pm}) \cap \mathcal{C}^\infty(\overline{U^{\pm}})$.

Proof. We can extend f to a $\mathcal{C}^\infty$-smooth function F over $P_2(\mathbb{C})$ in such a way that $\operatorname{supp} F$ lies in an arbitrarily small neighborhood of M and $\overline{\partial} F|_M$ vanishes to infinite order. We define $\omega = \overline{\partial} F$ on U^+ and $\omega = 0$ on U^-. The form ω is a $\mathcal{C}^\infty$-smooth $(0,1)$-form. As $H^{0,1}(P_2(\mathbb{C})) = 0$, we can solve the equation $\overline{\partial} u = \omega$ with u of class $\mathcal{C}^\infty$. So we have

$$\begin{aligned} &\overline{\partial}(F-u) = 0 \quad \text{on} \quad U^+ \quad \text{i.e., } F-u \text{ is holomorphic on } U^+, \\ &\overline{\partial} u = 0 \qquad\qquad \text{on} \quad U^- \quad \text{i.e., } u \text{ is holomorphic on } U^-, \end{aligned}$$

and furthermore

$$F = (F-u) - (-u), \qquad F|_M = f.$$

Obviously the components of the decomposition are of class $\mathcal{C}^\infty$. □

3. Embedding of a strip neighborhood of M in $\mathbb{C}^N$

Let M be a $\mathcal{C}^\infty$-smooth hypersurface in $P_2(\mathbb{C})$ bounding U^- and U^+ as above. As a preliminary to the proof of Theorem 1.1, we may establish an interesting equivalence between **(1)** and **(2)** in the $\mathcal{C}^\infty$ category.

Lemma 3.1. *Let M be a $\mathcal{C}^\infty$ real hypersurface in $P_2(\mathbb{C})$ dividing it into two open parts U^- and U^+. Then the following properties are equivalent:*

- **(1′)** *There exists a side, U^- or U^+, to which every $\mathcal{C}^\infty$ CR function on M extends holomorphically.*
- **(2′)** *All holomorphic functions on the other side of M and $\mathcal{C}^\infty$ up to M are constant.*

Proof. Without loss of generality, we can fix U^+ to be this side. Assume therefore that every $\mathcal{C}^\infty$-smooth CR function on M extends holomorphically to U^+. Let $g \in \mathcal{O}(U^-) \cap \mathcal{C}^\infty(\overline{U^-})$. Then the trace $g|_M$ extends holomorphically to U^+. So g extends holomorphically to $P_2(\mathbb{C})$, whence it is constant; this is **(2′)**.

Conversely, suppose that $\mathcal{O}(U^-) \cap \mathcal{C}^\infty(\overline{U^-})$ consists of constant functions only. Let f be a $\mathcal{C}^\infty$-smooth CR function on $M = \partial U^-$. From Lemma 2.1 we obtain that $f = f^+ - f^-$, where f^+ and f^- belong to the evident spaces. By hypothesis, f^- is constant. Consequently, the CR function f holomorphically extends to U^+; this is **(1′)**. □

Now let M be an orientable, $\mathcal{C}^2$-smooth, real hypersurface in a complex manifold X. By $\mathcal{V}^\pm(M)$ we denote a *one-sided* (global) *neighborhood* of M, i.e. a connected open set which contains a (local) one-sided neighborhood of M at each point of M and such that $\mathcal{V}^\pm(M) = \operatorname{Int}(\overline{\mathcal{V}^\pm(M)})$, so that $\mathcal{V}^\pm(M)$ contains a neighborhood in X of a point $p \in M$ whenever it contains the two (local) one-sided neighborhoods at p. Such one-sided neighborhoods are usually constructed by gluing analytic discs to M. Then the side where the discs are lying may of course depend on the point $p \in M$ and it may vary effectively if M is pseudoconvex somewhere and pseudoconcave elsewhere. Without convexity assumption, we have:

Theorem 3.2 ([20], [15], [11])**.** *Let M be an oriented globally minimal $\mathcal{C}^\infty$-smooth embedded hypersurface in a complex manifold X. Then there exists a one-sided neighborhood $\mathcal{V}^\pm(M)$, constructed by gluing small analytic discs to M, to which all continuous CR functions on M can be holomorphically extended.*

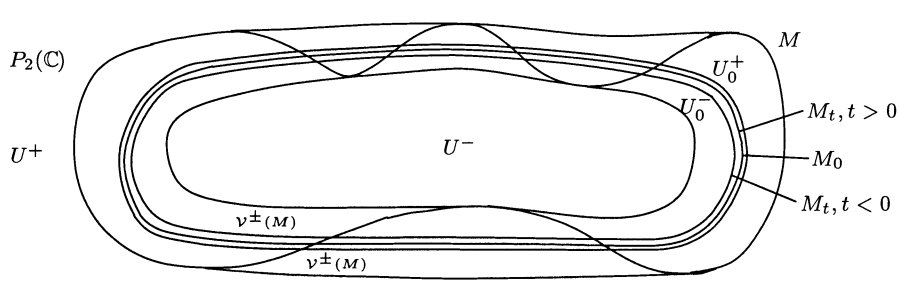
Proof of Theorem 1.1. Recall that **(1)** implies **(2)** trivially. We prove **(1)**. At first, we show that we can assume that M and f are $\mathcal{C}^\infty$. So, let M be a compact connected $\mathcal{C}^2$ real hypersurface in $P_2(\mathbb{C})$ dividing $P_2(\mathbb{C})$ into two connected open components U^+ and U^- and let f be a continuous CR function on M. By the theorem above, f extends holomorphically to a one-sided neighborhood $\mathcal{V}^\pm(M)$ of M. Let f still denote the resulting extension. Let us smoothly deform M into $\mathcal{V}^\pm(M)$. We get a $\mathcal{C}^\infty$ hypersurface $M' \subset\subset \mathcal{V}^\pm(M)$ arbitrarily close to M and the trace $f' := f|_{M'}$ of f on M' defines a $\mathcal{C}^\infty$ CR function (even better, we could assume that M' and f' are real analytic). Notice that as global minimality is a stable property under sufficiently small perturbations, the deformed hypersurface M' can also be assumed to be globally minimal. Again, M' divides $P_2(\mathbb{C})$ into two open parts U'^- and U'^+. Suppose that we can show that there exists a side, say U'^+, to which every $\mathcal{C}^\infty$ CR function f' on M' extends holomorphically. By inspecting the relative geometry of the three open sets U^+, $\mathcal{V}^\pm(M)$ and U'^+, and using the fact that f is holomorphic in the whole of $\mathcal{V}^\pm(M)$, we deduce that every continuous CR function f on M extends holomorphically to U^+. In summary, it suffices for us to prove Theorem 1.1 in the case where both M and f are $\mathcal{C}^\infty$-smooth.

Thus, let M be a compact connected $\mathcal{C}^\infty$ real hypersurface of $P_2(\mathbb{C})$ which divides it into two open parts U^- and U^+. Let f be a $\mathcal{C}^\infty$ CR function on M and let $f^+ \in \mathcal{O}(U^+) \cap \mathcal{C}^\infty(\overline{U^+})$, $f^- \in \mathcal{O}(U^-) \cap \mathcal{C}^\infty(\overline{U^-})$ be given by Lemma 2.1 so that $f = f^+ - f^-$ on M. By virtue of Lemma 3.1, it suffices to show that at least one of the two functions f^+ or f^- is constant. We then proceed by contradiction.

Assume that f^+ *and* f^- are both nonconstant. By global minimality, f, f^+ and f^- can be holomorphically extended to a one-sided neighborhood $\mathcal{V}^\pm(M)$ of M constructed by gluing discs. This open set is in fact a one-sided strip open set, although it is not, in general, a tubular neighborhood of M. However, we can easily deform M into $\mathcal{V}^\pm(M)$ in a $\mathcal{C}^\infty$ fashion, getting a $\mathcal{C}^\infty$ hypersurface $M_0 \subset\subset \mathcal{V}^\pm(M)$. Furthermore, we can include M_0 in a one-parameter family of manifolds M_t, $|t| \leq \varepsilon$, satisfying:

1. $M_t \cap M_{t'} = \emptyset$ for all $t \neq t'$,
2. $P_2(\mathbb{C}) \backslash M_t = U_t^+ \cup U_t^-$,
3. For $t < t'$, we have $M_t \subset\subset U_{t'}^-$ and $U_t^- \subset\subset U_{t'}^-$. Also, $M_{t'} \subset\subset U_t^+$ and $U_{t'}^+ \subset\subset U_t^+$,
4. The strip $S := \bigcup_{|t|<\varepsilon} M_t$ is an open neighborhood of M_0 which is contained in $\mathcal{V}^\pm(M)$.

The schematic picture of the geometric situation is as follows:



Notice that $M_0 \subset\subset U_\varepsilon^-$. We shall apply the following theorem due to Takeuchi over the domain U_ε^-.

Theorem 3.3 ([19]). *Let $U \subset P_2(\mathbb{C})$ be a domain. Then either:*

(i) *holomorphic functions on U are constant, or*
(ii) *holomorphic functions on U separate points.*

Lemma 3.4. *For every δ with $0 < \delta < \varepsilon$, the holomorphic functions from $\mathcal{O}(U_\delta^-)$ separate points and give a local coordinate system in a neighborhood of each point of U_δ^-.*

Proof. According to Theorem 3.3 and since there exists by assumption the nonconstant function $f^-|_{U_\varepsilon^-} \in \mathcal{O}(U_\varepsilon^-)$, then $\mathcal{O}(U_\varepsilon^-)$ separate points. We can therefore find at least two functions $h_j \in \mathcal{O}(U_\varepsilon^-)$, $j = 1, 2$, such that the zero locus of the Jacobian of the mapping $h := (h_1, h_2)$ is a proper, closed, complex analytic subvariety Σ of U_ε^- of complex dimension ≤ 1. Of course, h gives a local coordinate system at each point of U_ε^- not in Σ. Fix δ with $0 < \delta < \varepsilon$. Now let $p \in U_\delta^- \cap \Sigma$ be arbitrary. Then after composing h with an automorphism A of $P_2(\mathbb{C})$ arbitrarily close to the identity, we obtain that $h \circ A$ gives a local coordinate system at p. Such an automorphism A moves U_ε^- a little bit. As A can be chosen to be arbitrarily close to the identity, we can ensure that the domain of definition of $h \circ A$ still contains U_δ^-. This completes the proof. □

Now let η with $0 < \eta < \delta < \varepsilon$. The above lemma implies that the manifold with boundary $M_\eta \cup U_\eta^- \subset\subset U_\delta^-$ is embeddable into some complex euclidean space $\mathbb{C}^N$, $N \in \mathbb{N}$, through an embedding $\Phi : M_\eta \cup U_\eta^- \to \mathbb{C}^N$ whose components are holomorphic functions defined in U_δ^-. The result of the embedding is a complex *manifold* $\Phi(U_\eta^-)$ with boundary the three-dimensional maximally complex CR manifold $\Phi(M_\eta)$. Notice that Φ also embeds into $\mathbb{C}^N$ the closed strip $S' := \bigcup_{|t| \leq \eta} M_t$ which provides a tubular neighborhood of M_0. By construction, for all $|t| \leq \eta$, the $\Phi(M_t)$ are oriented maximally complex three-dimensional CR manifolds in $\mathbb{C}^N$. The orientation on $\Phi(M_t)$ is simply the push-forward of the orientation on M_t. Further, $\Phi(M_t)$ bounds the complex manifold $\Phi(U_t^-)$.

Finally, the holomorphic function f^+ induces a nonconstant holomorphic function on the strip $\Phi(S')$, say $g^+ := f^+ \circ \Phi^{-1}|_{\Phi(S')}$. Applying a generalization of Bochner's theorem (see [7]), we deduce that the CR function $g^+|_{\Phi(M_t)}$ extends holomorphically to the complex submanifold $\Phi(U_t^-)$ for all $|t| \leq \eta$ (the smoothness of $\Phi(U_t^-)$ is needed, otherwise we could only say that the *graph* of $g^+|_{\Phi(M_t)}$ extends as a complex analytic set by Harvey-Lawson's theorem). Thanks to the maximum principle, we may now easily derive the desired contradiction.

Recall that by assumption, f^+ is nonconstant. Therefore, by the maximum principle applied to f^+ in the projective space over U^+, we have for all t with $-\eta \leq t < 0$ the strict inequality

$$\sup_{z \in M_t} |f^+(z)| > \sup_{z \in M_0} |f^+(z)|,$$

because $M_0 \subset\subset U_t^+$ and f^+ is nonconstant. But on the other hand, by the maximum principle applied in the complex euclidean space $\mathbb{C}^N$ to the nonconstant holomorphic function $g^+|_{\Phi(S')}$ which extends holomorphically to $\Phi(U_\eta^-)$, we have for all t with $-\eta \leq t < 0$ the reverse strict inequality

$$\sup_{z \in M_t} |f^+(z)| = \sup_{z \in \Phi(M_t)} |g^+(z)| < \sup_{z \in \Phi(M_0)} |g^+(z)| = \sup_{z \in M_0} |f^+(z)|.$$

This gives the desired contradiction, which completes the proof of Theorem 1.1. □

Acknowledgement

We are grateful to the referee for his clever support in fixing some incorrections in the manuscript and for having pointed out to us substantial improvements in the proofs.

References

1. D. Cerveau, *Minimaux des feuilletages algébriques de $P_n(\mathbb{C})$.* Ann. Inst. Fourier (Grenoble) **43** (1993), 1535–1543. MR **95m:**32046
2. P. Dolbeault and G.M. Henkin, *Chaînes holomorphes de bord donné dans $P_n(\mathbb{C})$.* Bull. Soc. Math. France **125** (1997), 383–446. MR **98m:**32014
3. L. Ehrenpreis, *A new proof and an extension of Hartogs' theorem.* Bull. Amer. Math. Soc. **67** (1961), 507–509. MR **24:**A1511
4. B. Fabre, *Sur l'intersection d'une surface de Riemann avec des hypersurfaces algébriques.* C.R. Acad. Sci. Paris Sér. I Mathématiques **322** (1996), 371–376. MR **97a:**32008
5. É. Ghys, *Laminations par des surfaces de Riemann.* Dynamique et géométrie complexe (Lyon, 1997), ix, xi, 49–95, Panor. Synthèses, 8, Soc. Math. France, Paris, 1999. MR **2001g:**37068
6. P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* Wiley Classics Library, John Wiley and Sons, 1994. MR **95d:**14001
7. R. Harvey, *Holomorphic chains and their boundaries.* Proceedings of Symposia in Pure Mathematics, **XXX** (1977), vol. 1, AMS, Providence, RI, 309–382. MR **56:**5929
8. R. Harvey and B. Lawson, *On boundaries of complex analytic varieties.* Ann. of Math., I: **102** (1975), 233 - 290; II: **106** (1977), 213–238. MR **54:**13130; MR **58:**17186
9. G.M. Henkin and J. Leiterer, *Theory of functions on complex manifolds.* Monographs in Mathematics, **79**, Birkhäuser, Basel-Boston, Mass., 1984, 226 pp. MR **86i:**32004
10. S. Ivashkovich, *The Hartogs-type extension theorem for meromorphic maps into compact Kähler manifolds.* Invent. Math. **109** (1992), 47–54. MR **93g:**32016
11. B. Jöricke, *Some remarks concerning holomorphically convex hulls and envelopes of holomorphy.* Math. Z. **218** (1995), 143–157. MR **96b:**32014
12. J.J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold.* Ann. of Math. **81** (1965), 451–472. MR **31:**1399
13. C. Laurent-Thiébaut, *Phénomène de Hartogs-Bochner dans les variétés CR.* Topics in Complex Analysis, Banach Center Publications, Warszawa **31** (1995), 233–247. MR **96h:**32010
14. A. Lins Neto, *A note on projective Levi flats and minimal sets of algebraic foliations.* Ann. Inst. Fourier (Grenoble) **49** (1999), 1369–1385. MR **2000h:**32047
15. J. Merker, *Global minimality of generic manifolds and holomorphic extendibility of CR functions.* Int. Math. Res. Not. (1994), no. 8, 329–343. MR **95i:**32015
16. F. Sarkis, *CR meromorphic extension and the non embedding of the Andreotti-Rossi CR structure in the projective space.* Int. J. Math. **10** (1999), 897–915. MR **2001a:**32042
17. F. Sarkis, *Problème de Plateau complexe dans les variétés kahlériennes.* Preprint, 1999.
18. Y.T. Siu, *Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension ≥ 3.* Ann. Math. (2) **151** (2000), no. 3, 1217–1243. CMP 2000:17
19. A. Takeuchi, *Domaines psudoconvexes infinis et la métrique riemannienne dans un espace projectif.* J. Math. Soc. Japan **16** (1964), 159–181. MR **30:**3997
20. J.-M. Trépreau, *Sur la propagation des singularités dans les variétés CR.* Bull. Soc. Math. France **118** (1990), 403–450. MR **92b:**58229

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, P.O. Box 137, 00-950 Warsaw, Poland

E-mail address: rd@impan.gov.pl

Laboratoire d'Analyse, Topologie et Probabilités, Centre de Mathématiques et Informatique, UMR 6632, 39 rue Joliot Curie, F-13453 Marseille Cedex 13, France

E-mail address: merker@cmi.univ-mrs.fr