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ON ZARISKI'S MULTIPLICITY PROBLEM

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ABSTRACT. We show that to answer affirmatively Zariski's question concerning the topological invariance of the multiplicity of complex analytic hypersurfaces at isolated singular points, it suffices to prove two combined statements, each of which may be obtained separately.

A celebrated question of Zariski, posed in 1971 [Za], concerns the topological invariance of the multiplicity of complex hypersurfaces. This was part of the vast equisingularity programme initiated by Zariski in the sixties, aiming at understanding the relations between the invariants of different nature (topological, differential, numerical and algebraic) of a singular germ. From this point of view, the question of the behaviour of the multiplicity under homeomorphism is natural, since the multiplicity is the simplest analytic invariant of a complex germ.

We first recall Zariski's question. Let $f, g : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ be two germs at 0 of holomorphic functions, and let $f^{-1}(0)$ and $g^{-1}(0)$ be the germs at 0 of the hypersurfaces they define in $\mathbf{C}^n$. We denote by $e(f^{-1}(0), 0)$ the local multiplicity at 0 of $f^{-1}(0)$; thus $e(f^{-1}(0), 0)$ is the number of points of intersection near 0 of the hypersurface $f^{-1}(0)$ with a generic complex line passing close to 0, but not through 0. Suppose that there exists a germ of a homeomorphism $\phi : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^n, 0)$ sending $f^{-1}(0)$ onto $g^{-1}(0)$. In this case we say that the two hypersurfaces have the *same topological type* or are *topologically V-equivalent*. Zariski asked: is it then the case that $e(f^{-1}(0), 0) = e(g^{-1}(0), 0)$? Zariski wrote in 1971 that he would be "greatly disappointed" if topologists were unable to answer this question "in a relatively short order" ([Za], page 483).

Almost thirty years later, the problem is still unresolved, even for analytic families of complex hypersurfaces with isolated singularities and constant topological type in $\mathbf{C}^3$, although the question is periodically asked (see for instance [Te]). Surveys describing some of the many partial results obtained can be found in [Tr], [Ka] and [Ri-Tr] and [Co]. Quite recently the first author has shown that local bilipschitz triviality along a smooth analytic subspace implies equimultiplicity along the subspace for complex analytic sets of any codimension [Co] (while the Zariski question concerning the topological equivalence makes sense only for hypersurfaces). In particular the simple argument of [Co] gives equimultiplicity along strata of a lipschitz stratification (in the sense of [Mo]). J.-J. Risler and the third author proved

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in [Ri-Tr] that if f and g are right-left bilipschitz equivalent germs at 0 of complex analytic functions on $\mathbf{C}^n$, then $e(f^{-1}(0),0) = e(g^{-1}(0),0)$. It is this last result that we improve here, at least for isolated hypersurface singularities, since in what follows we assume that the origin is an isolated critical point of f.

First we observe that in 1989 it was established that if $f^{-1}(0)$ and $g^{-1}(0)$ have the same topological type, then there exists locally near 0 a homeomorphism $\varphi_1 : (\mathbf{C}^n,0) \longrightarrow (\mathbf{C}^n,0)$, sending $f^{-1}(0)$ onto $g^{-1}(0)$ and such that $|\varphi_1(z)| = |z|$ for all z in some neighbourhood of the origin. This follows from very deep work of Osamu Saeki [Sa]: he used major results in cobordism theory of Kirby and Siebenmann [KS1], [KS2], Perron [Pe], Quinn [Q] and Wall [W], to prove that in all dimensions the two hypersurfaces $f^{-1}(0)$ and $g^{-1}(0)$ have the same topological type if and only if there exists a small sphere S_ϵ^{2n-1} centred at the origin in $\mathbf{C}^n$ and a homeomorphism sending $(S_\epsilon^{2n-1}, S_\epsilon^{2n-1} \cap f^{-1}(0))$ onto $(S_\epsilon^{2n-1}, S_\epsilon^{2n-1} \cap g^{-1}(0))$. One says that $f^{-1}(0)$ and $g^{-1}(0)$ are *link-equivalent*. Clearly, when $f^{-1}(0)$ and $g^{-1}(0)$ are link-equivalent, the proof of Milnor [Mi, Theorem 2.10] on the conical structure of isolated singularities then implies the existence of φ_1, sending $f^{-1}(0)$ onto $g^{-1}(0)$ and such that $|\varphi_1(z)| = |z|$ for all z in some neighbourhood of the origin.

Now it is also known that if $f^{-1}(0)$ and $g^{-1}(0)$ have the same topological type, there exists a homeomorphism $\varphi_2 : (\mathbf{C}^n,0) \longrightarrow (\mathbf{C}^n,0)$, sending $f^{-1}(0)$ onto $g^{-1}(0)$ and such that $|(g\circ\varphi_2)(z)| = |f(z)|$ for all z near 0. This result is a consequence of theorems due to H. King [Ki] when $n \neq 3$ and to Perron [Pe] when $n = 3$, stating that topological V-equivalence is the same as topological right-left-equivalence, together with another theorem of King [Ki] stating that topological right-left equivalence of f and g implies that g is topologically right-equivalent either to f or to $\bar{f}$, the conjugate of f, which has the same multiplicity as f.

After these two remarks, we formulate our result:

Proposition. *With the notations above, if there exists a homeomorphism-germ* $\varphi : (\mathbf{C}^n,0) \longrightarrow (\mathbf{C}^n,0)$ *having the properties of both* φ_1 *and* φ_2*, then* $e(f^{-1}(0),0) = e(g^{-1}(0),0)$.

In fact it suffices to assume there are positive constants A, B, C *and* D *such that:*

(1) $A|z| \leq |\varphi(z)| \leq B|z|$*, for all* z *near* 0*, and*

(2) $C|f(z)| \leq |(g\circ\varphi)(z)| \leq D|f(z)|$*, for all* z *near* 0.

Comments. We have seen that one can obtain homeomorphisms φ having *either* the property of φ_1 *or* the property of φ_2, but it is apparently not known whether topological equivalence implies the existence of a homeomorphism having *simultaneously* the properties of φ_1 and of φ_2.

The hypothesis of [Ri-Tr], that there exists a germ of a homeomorphism $\varphi_3 : (\mathbf{C}^n,0) \longrightarrow (\mathbf{C}^n,0)$ and positive constants A and B such that

$$A|z-w| \leq |\varphi_3(z) - \varphi_3(w)| \leq B|z-w|$$

for all w and z near 0 with $w \in f^{-1}(0)$, is weakened here to the hypothesis (1) of the Proposition above, by using the characterization of multiplicity in the following lemma (cf. [Pl]) which differs from the characterization of [Ri-Tr]. Note that it is still not known whether topological equivalence implies the existence of φ_3 as in [Ri-Tr], whereas the existence of φ_1 follows from [Sa] as we described above. It is in this sense that the present note should be of interest because it suggests a plausible

strengthened topological statement for topologists to attack which would answer Zariski's question affirmatively.

Write $\delta(f) = \sup\{\delta : \dfrac{|f(z)|}{|z|^\delta}$ is bounded near $0\}$.

Lemma. *Let* $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ *be the germ of an irreducible holomorphic function. Then* $e(f^{-1}(0), 0) = \delta(f)$.

Proof. The multiplicity $e(f^{-1}(0), 0)$ is just the lowest degree of the homogeneous polynomials in the expression of f as a convergent power series near 0 (we assume that f and g are irreducible).

First we show that $e(f^{-1}(0), 0) \leq \delta(f)$. Suppose that $\delta < m = e(f^{-1}(0), 0)$. We must show that $\lim\limits_{z\to 0} \dfrac{|f(z)|}{|z|^\delta} = 0$. Now $f(z) = f_0(z) + f_1(z)$ where f_0 is a non-zero homogeneous polynomial of degree m, and f_1 has only terms of degree larger than m . Write

$$f_0(z) = \sum_{|\alpha|=m} {z_1}^{\alpha_1} \dots {z_n}^{\alpha_n}.$$

By the triangle inequality, it suffices to show that each term

$$\frac{{z_1}^{\alpha_1} \dots {z_n}^{\alpha_n}}{|z|^\delta}$$

tends to 0. But

$$\frac{|{z_1}^{\alpha_1} \dots {z_n}^{\alpha_n}|}{|z|^\delta} \leq \frac{|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n}}{|z|^\delta} \leq \frac{(sup\{|z_i| : 1 \leq i \leq n\})^m}{|z|^\delta}$$

which tends to 0 as $|z|$ tends to 0, because $|z|$ is equivalent to $sup\{|z_i|^m : 1 \leq i \leq n\}$ (equivalence of norms), and δ is assumed to be less than m.

To prove that $\delta(f) = e(f^{-1}(0), 0)$ it is enough to show that $\delta(f) \leq m$. So suppose now that $\delta > m$. Consider $z = (z_1, \dots, z_n)$ such that $z_1 = \dots = z_n$. We can suppose that z is not in the tangent cone of $f^{-1}(0)$, after a suitable change of coordinates. Then $|f(z)|/|z|^\delta$ is equivalent to $|z_1|^{m-\delta}$, which is unbounded as $|z_1|$ tends to 0. It follows that $\delta > \delta(f)$, and hence that $\delta(f) \leq m$. This completes the proof that $\delta(f) = e(f^{-1}(0), 0)$. □

Proof of the Proposition. Let $\delta(g) = \delta$ and $z \neq 0$ in $\mathbf{C}^n$. The ratio

$$\frac{|f(z)|}{|z|^\delta} = \frac{|f(z)|}{|\varphi(z)|^\delta} \frac{|\varphi(z)|^\delta}{|z|^\delta} \leq \frac{|g \circ \varphi(z)|}{C|\varphi(z)|^\delta} B^\delta = \frac{|g(z')|}{C|z'|^\delta} B^\delta$$

is uniformly bounded near 0, using the hypotheses (1) and (2) and the definition of $\delta(g)$. Again by definition of $\delta(g)$ and $\delta(f)$, this implies that $\delta(g) \leq \delta(f)$. By symmetry, $\delta(g) = \delta(f)$. The Proposition then follows from the characterisation of the Lemma. □

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