

LEFSCHETZ INDEX FOR ORIENTATION REVERSING PLANAR HOMEOMORPHISMS

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ABSTRACT. We prove that an isolated fixed point of an orientation reversing homeomorphism of the plane always has Lefschetz index 0 or ± 1 .

1. INTRODUCTION

If U is an open subset of \mathbb{R}^2 and $p \in U$ an isolated fixed point of a continuous map $h : U \rightarrow \mathbb{R}^2$, the Lefschetz index of h at p , denoted by $\text{Ind}(h, p)$, is the winding number of the vector field $h(z) - z$ on any simple closed curve surrounding p and close enough to p . There is plentiful literature where this fixed-point index plays an important role; for example, to detect other fixed points, or to obtain local and/or global dynamical properties for surfaces homeomorphisms.

Throughout this paper, the map h is a (local) homeomorphism and we will focus on the set of all the possible values for the Lefschetz index. It is well-known that, for every integer $n \in \mathbb{Z}$, there exists an orientation preserving planar homeomorphism h_n which has the origin o as an isolated fixed point and such that $\text{Ind}(h_n, o) = n$ (see for example [1]). Surprisingly, M. Brown announced in the same paper that for an orientation reversing homeomorphism of the plane, the Lefschetz index is one of the three values $-1, 0$ or $+1$, but no proof of this result has been given until now. This motivated the writing of this paper, where we will show precisely the following:

Theorem. *Let V, W be two connected open subsets of \mathbb{R}^2 containing the origin o and let $h : V \rightarrow W = h(V)$ be an orientation reversing homeomorphism which possesses o as an isolated fixed point.*

Then $\text{Ind}(h, o) \in \{-1, 0, +1\}$.

2. INDEX ON A JORDAN CURVE

For completeness, we recall briefly in this section some classical results and definitions.

If C is a Jordan curve (i.e. a simple closed curve in \mathbb{R}^2 , i.e. a subset of \mathbb{R}^2 homeomorphic to the unit circle \mathbb{S}^1), then $\mathbb{R}^2 \setminus C$ has exactly two connected components and C is their common frontier. The bounded one (resp. the unbounded one) is named *the interior domain* (resp. *the exterior domain*) of C and is denoted by

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$\text{int}(C)$ (resp. by $\text{ext}(C)$). A subset of \mathbb{R}^2 which is the interior domain of a Jordan curve is said to be a *Jordan domain*. In this paper, a Jordan curve C is always counter-clockwise oriented. This yields an ordering relation (defined up to circular permutation) on C . If x, y are two points on C , $[x, y]_C$ (resp. $(x, y)_C$) denotes the closed arc (resp. the open arc) on C from x to y for this orientation of C .

Let $u : \mathbb{S}^1 \rightarrow C$ be a homeomorphism which endows the Jordan curve $C = u(\mathbb{S}^1)$ with its counter-clockwise orientation. If X is a subset of \mathbb{R}^2 containing C and $f : X \rightarrow \mathbb{R}^2$ a continuous map without fixed point on C , the degree of the map $z \mapsto (f(u(z)) - u(z)) / \|f(u(z)) - u(z)\|$ ($z \in \mathbb{S}^1$) does not depend on the choice of u . It is named *the index of f on C* and is denoted by $\text{Ind}(f, C)$. Now, if X is open and $p \in X$ is an isolated fixed point of f , choose a disk neighborhood D of p , so small that $D \subset X$ and f has no other fixed point in D . Then the index of f is the same on any Jordan curve C lying in D and such that $p \in \text{int}(C)$, and this common value defines *the Lefschetz index* $\text{Ind}(f, p)$ of f at the point p .

3. PROOF OF THE THEOREM

In the following, we will write respectively \overline{X} and ∂X for the closure and the frontier of a set $X \subset \mathbb{R}^2$.

Choose once and for all a circle C around o , so that the closed disk $D = \overline{\text{int}(C)}$ is contained in $V \cap W$ and o is the only fixed point of h in D . First, observe there are two cases where the index can be easily computed:

- if $h(D) \subset D$, then $\text{Ind}(h, o) = 1$,
- if $D \subset h(D)$, then, since h reverses the orientation, $\text{Ind}(h, o) = -\text{Ind}(h^{-1}, o) = -1$.

From now on, we exclude these two simple situations, that is we suppose $h(\text{int}(C)) \not\subset \text{int}(C)$ and $\text{int}(C) \not\subset h(\text{int}(C))$.

It is well known since Kerékjártó that every connected component of the intersection of two Jordan domains is again a Jordan domain. We need a more precise description, which can be stated as follows. All the assertions in Proposition 3.1 and their proofs are contained in the first section of [2].

Proposition 3.1. *Let U, U' be two Jordan domains containing the origin o , such that $U \not\subset U'$ and $U' \not\subset U$. Denote by $U \wedge U'$ the connected component of $U \cap U'$ which contains o and by $\partial U \wedge \partial U'$ the frontier of $U \wedge U'$.*

(1) *We have a partition*

$$(P) \quad \partial U \wedge \partial U' = ((\partial U \wedge \partial U') \cap \partial U \cap \partial U') \bigcup_{i \in I} \alpha_i \bigcup_{j \in J} \beta_j$$

where

- I, J are nonempty and at most countable sets,
 - for every $i \in I$, $\alpha_i = (a_i, b_i)_{\partial U}$ is a connected component of $\partial U \cap U'$,
 - for every $j \in J$, $\beta_j = (c_j, d_j)_{\partial U'}$ is a connected component of $\partial U' \cap U$.
- (2) *For every $j \in J$, $U \wedge U'$ is contained in the Jordan domain bounded by $\beta_j \cup [d_j, c_j]_{\partial U}$.*
- (3) *$\partial U \wedge \partial U'$ is homeomorphic to ∂U . In particular, it is a Jordan curve.*
- (4) *Three points a, b, c of $(\partial U \wedge \partial U') \cap \partial U$ (resp. of $(\partial U \wedge \partial U') \cap \partial U'$) are met in this order on ∂U (resp on $\partial U'$) if and only if they are met in the same order on $\partial U \wedge \partial U'$.*

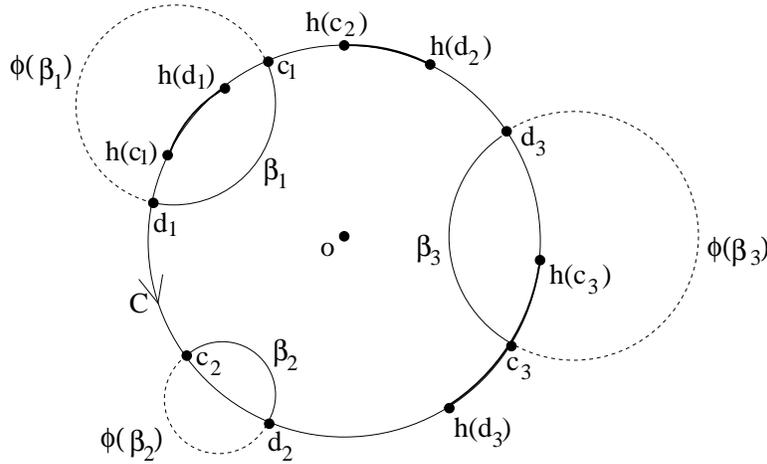


FIGURE 1. The curves Γ and $\phi(\Gamma)$

Let us consider the partition (P_h) and the arcs α_i and β_j that we obtain when we apply Proposition 3.1 with $U = \text{int}(C)$ and $U' = \text{int}(h^{-1}(C)) = h^{-1}(U)$.

Let ϕ be the inversion in the circle C and $\Gamma = C \wedge h^{-1}(C)$ the Jordan curve given by Proposition 3.1. Since ϕ is a homeomorphism of $\mathbb{R}^2 \setminus \{o\}$, $\phi(\Gamma)$ is also a Jordan curve.

Now, define a map H from $\Gamma \cup \phi(\Gamma)$ to \mathbb{R}^2 by setting

$$H(z) = \begin{cases} h(z) & \text{if } z \in \Gamma, \\ h(\phi(z)) & \text{if } z \in \phi(\Gamma). \end{cases}$$

Clearly, H is well-defined, continuous and fixed-point free. Since $H(\phi(\Gamma)) = h(\Gamma) \subset D \subset \text{int}(\phi(\Gamma))$, we have $\text{Ind}(H, \phi(\Gamma)) = 1$. We have also $H = h$ on Γ and consequently $\text{Ind}(H, \Gamma) = \text{Ind}(h, \Gamma) = \text{Ind}(h, o)$.

Observe that, for every $j \in J$, the set $C_j = \overline{\beta_j} \cup \phi(\overline{\beta_j})$ is a Jordan curve such that $(c_j, d_j)_C \subset \text{int}(C_j)$ and $(d_j, c_j)_C \subset \text{ext}(C_j)$. The orientations induced on β_j by C_j and Γ are opposite (see Figure 1). Furthermore, the β_j are pairwise disjoint open arcs in $h^{-1}(C)$ and then, for a given $\epsilon > 0$, there is only a finite number of indexes $j \in J$ such that the diameter of β_j is superior to ϵ . It follows that there are only finitely many indexes $j \in J$ such that $\text{Ind}(H, C_j) \neq 0$.

Bringing together the above remarks, we obtain the formula

$$(*) \quad 1 = \text{Ind}(H, \phi(\Gamma)) = \text{Ind}(h, o) + \sum_{j \in J} \text{Ind}(H, C_j).$$

Keeping in mind that, for every $j \in J$, we have $H(C_j) = h(\overline{\beta_j}) = [h(d_j), h(c_j)]_C$, we now state three basic lemmas:

Lemma 3.2. *If $[h(d_j), h(c_j)]_C \cap [c_j, d_j]_C = \emptyset$, then $\text{Ind}(H, C_j) = 0$.*

Proof. It is enough to remark that $H(C_j) = [h(d_j), h(c_j)]_C \subset (d_j, c_j)_C \subset \text{ext}(C_j)$.

Lemma 3.3. *If the set $[h(d_j), h(c_j)]_C \cap [c_j, d_j]_C$ is nonempty and connected, then $\text{Ind}(H, C_j) = 1$.*

Proof. This is clear if $[h(d_j), h(c_j)]_C \subset [c_j, d_j]_C$ because we have then $H(C_j) \subset \text{int}(C_j)$. The following argument allows us to be reduced to this case: Denote $[z_1, z_2]_C = [h(d_j), h(c_j)]_C \cap [c_j, d_j]_C$. Let $(r_t)_{0 \leq t \leq 1}$ be a strong retracting deformation of the arc $[h(d_j), h(c_j)]_C$ onto its subarc $[z_1, z_2]_C$. The maps $r_t \circ H|_{C_j} : C_j \rightarrow \mathbb{R}^2$ ($0 \leq t \leq 1$) are fixed-point free because $C_j \cap r_t(H(C_j)) \subset \{c_j, d_j\}$ and, if $H(c_j) = h(c_j) \notin [z_1, z_2]_C$ (resp. if $H(d_j) = h(d_j) \notin [z_1, z_2]_C$), then $[z_2, h(c_j)]_C = [d_j, h(c_j)]_C$ does not contain the point c_j (resp. $[h(d_j), z_1]_C = [h(d_j), c_j]_C$ does not contain the point d_j). Since the map $t \mapsto \text{Ind}(r_t \circ H|_{C_j}, C_j)$ is continuous and $r_1(H(C_j)) = [z_1, z_2]_C \subset [c_j, d_j]_C \subset \text{int}(C_j)$, we obtain $\text{Ind}(H, C_j) = \text{Ind}(r_1 \circ H|_{C_j}, C_j) = 1$.

Lemma 3.4. *There are at most two indexes $j \in J$ such that $[h(d_j), h(c_j)]_C \cap [c_j, d_j]_C$ is nonempty.*

Proof. Suppose that we can find three distinct arcs β_{j_k} ($k \in \{1, 2, 3\}$) among the β_j such that $[h(d_{j_k}), h(c_{j_k})]_C \cap [c_{j_k}, d_{j_k}]_C \neq \emptyset$. Since h has no fixed point on C , it is the same to write $(h(d_{j_k}), h(c_{j_k}))_C \cap (c_{j_k}, d_{j_k})_C \neq \emptyset$ and then one can choose a point $x_k \in \beta_{j_k}$ such that $h(x_k) \in (c_{j_k}, d_{j_k})_C$. Renaming the indexes j_k if necessary, one can suppose that $\beta_{j_1}, \beta_{j_2}, \beta_{j_3}$ are met in this order on $h^{-1}(C)$.

On one hand, since $x_k \in \beta_{j_k}$ and h reverses the orientation, we must have $h(x_3), h(x_2), h(x_1)$ in this order on C . On the other hand, we know from Proposition 3.1 (4) that the points $c_{j_1}, d_{j_1}, c_{j_2}, d_{j_2}, c_{j_3}, d_{j_3}$ are met in this order on $h^{-1}(C)$, on Γ and on C . With $h(x_k) \in (c_{j_k}, d_{j_k})_C$, we obtain now that $h(x_1), h(x_2), h(x_3)$ are in this order on C . This is not possible because the points $h(x_k)$ ($k \in \{1, 2, 3\}$) are pairwise distinct.

We can now complete the proof of the Theorem: If, for every $j \in J$, the set $[h(d_j), h(c_j)]_C \cap [c_j, d_j]_C$ is either empty or connected, the result is an obvious consequence of formula (*) and of Lemmas (3.2)–(3.4).

Otherwise, there exists an arc $\beta = (c, d)_{h^{-1}(C)}$ among the β_j such that $c, h(c), h(d), d$ are met in this order on C . According to Proposition 3.1 (2) and (4), the points d, a_i, b_i, c are met in this order on C and on $h^{-1}(C)$ for every $i \in I$. It follows that $c, h(c), h(b_i), h(a_i), h(d), d$ are in this order on C . Thus, we obtain $\overline{\alpha_i} \cap [h(b_i), h(a_i)]_C = \emptyset$. For convenience, let us define now $g = h^{-1}$. Of course, one can use Proposition 3.1 with $U = \text{int}(C)$ and $U' = \text{int}(g^{-1}(C))$ and then obtain a corresponding partition (P_g) for $C \wedge g^{-1}(C)$. But $C \wedge g^{-1}(C)$ is nothing but $h(C \wedge h^{-1}(C))$ and therefore the partition (P_g) can be written using the arcs α_i and β_j of (P_h) . Precisely:

$$(P_g) \quad C \wedge g^{-1}(C) = ((C \wedge g^{-1}(C)) \cap C \cap g^{-1}(C)) \bigcup_{j \in J} h(\beta_j) \bigcup_{i \in I} h(\alpha_i).$$

Observe that the arcs $h(\alpha_i) = (h(b_i), h(a_i))_{g^{-1}(C)}$ play the same role in (P_g) as the arcs β_j in (P_h) . Since $\overline{\alpha_i} = g(h(\overline{\alpha_i}))$ is disjoint from $[h(b_i), h(a_i)]_C$ for every $i \in I$, we see that we are reduced to the previous situation if we replace h by g . Then we obtain $\text{Ind}(h, o) = -\text{Ind}(g, o) = -1$. The Theorem is proved.

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REFERENCES

- [1] M. Brown, *On the fixed points index of iterates of planar homeomorphisms*, Proc. Amer. Math. Soc. **108** (1990), 1109-1114. MR **90g**:54036
- [2] P. Le Calvez and J.C. Yoccoz, *Un théorème d'indice pour les homéomorphismes du plan au voisinage d'un point fixe*, Annals of Mathematics **146** (1997), 241-293. MR **99a**:58129

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