ON THE REGULARITY OF SOLUTIONS TO FULLY NONLINEAR ELLIPTIC EQUATIONS VIA THE LIOUVILLE PROPERTY

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Abstract. We show that any $C^{1,1}$ solution to the uniformly elliptic equation $F(D^2u) = 0$ must belong to $C^{2,\alpha}$, if the equation has the Liouville property.

§1. Introduction

In this paper, we consider the interior regularity of solutions to the following fully nonlinear elliptic equation:

\[ F(D^2u) = 0. \]

We assume that $F$ is uniformly elliptic, i.e., there exist constants $0 < \lambda \leq \Lambda$ such that

\[ \lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\|, \quad \text{for } M, N \in \mathcal{S}, \ N \geq 0, \]

where $\mathcal{S}$ denotes the space of real $n \times n$ symmetric matrices and $\|N\|$ denotes the norm of $N$.

For simplicity, we also assume that $F(0) = 0$.

There have been a number of works concerning equation (1). For instance, see [CC], [GT], [K] and the references cited there. When $F$ is a concave or convex functional, it is well known that the Evans-Krylov estimate

\[ |D^2u|_{C^{2,\alpha}(B_{1/2})} \leq C \|u\|_{C^{1,1}(B_1)} \]

holds, and $C^{1,1}$ viscosity solutions of (1) are $C^{2,\alpha}$ for some $\alpha > 0$.

On the contrary, in the case when $F$ is not concave nor convex, $C^{1,1}$ viscosity solutions of (1) may not be in the $C^2$ class. This has recently been shown by Nadirashvili in [N] in which he found a $C^{1,1}$ viscosity solution $u$ to the equation $F(D^2u) = 0$ where $F$ is smooth, uniformly elliptic and $u$ is not $C^2$. Therefore, it would be interesting to know under what condition a $C^{1,1}$ solution of (1) is actually in the $C^2$ class.

It is our purpose in this paper to show that any $C^{1,1}$ viscosity solution of (1) must be $C^{2,\alpha}$ if the elliptic operator $F$ has the Liouville property.

A continuous function $u(x)$ is said to be a viscosity subsolution (resp., supersolution) of (1) in a domain $\Omega$ if for $x_0 \in \Omega$ and $\phi(x) \in C^2$, $u - \phi$ attains the local

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maximum (resp., minimum) at \( x_0 \), then \( F(D^2\phi(x_0)) \geq 0 \) (resp., \( \leq 0 \)). If \( u \) is both a subsolution and a supersolution, then we say \( u \) is a viscosity solution. We mention that if \( u \in C^{1,1} \), then \( u \) is a viscosity solution of (1) if and only if \( u \) is a strong solution to (1).

Equation (1) or \( F \) is said to satisfy the Liouville property if \( u \in C^{1,1}_{\text{loc}}(\mathbb{R}^n) \) is an entire viscosity solution of (1) with bounded \( D^2u \) in \( \mathbb{R}^n \), \(|D^2u| \leq C\); then \( u \) must be a polynomial of degree at most 2.

Let \( B_r(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < r \} \).

Now we state the main theorem.

**Theorem.** Suppose that \( F \in C^1 \) satisfies (2) and \( F(0) = 0 \). Let \( u \in C^{1,1}(B_1(0)) \) be a viscosity solution of (1) in \( B_1(0) \). If equation (1) satisfies the Liouville property, then for any \( 0 < \alpha < 1 \), \( u \in C^{2,\alpha}(B_{1/2}(0)) \) and \( |D^2u|_{C^{\alpha}(B_{1/2}(0))} \leq C \), where \( C \) depends only on \( n, \lambda, \Lambda, \beta, \|u\|_{C^{1,1}(B_1(0))}, F \), and the modulus of continuity of \( DF \).

§2. **The proof of the Theorem**

We will use the blow-up technique to prove the Theorem. The tool needed to obtain a subsequence of blow-up solutions converging in \( W^{2,2}_{\text{loc}}(\mathbb{R}^n) \) is the \( W^{2,\delta} \) estimate for nondivergent uniform elliptic equations. For the convenience of our readers, let us give a little more preliminary information.

Recall that \( u \in \text{BMO}(\Omega) \) is in \( \text{VMO}(\Omega) \) if

\[
\eta_u(R, \Omega) = \sup_{x_0 \in \Omega} \int_{B_r(x_0) \cap \Omega} |u(x) - u_{x_0,r}| \, dx \to 0, \quad \text{as} \ R \to 0,
\]

where \( \int_A f \, dx \) denotes the average of \( f \) over \( A \) and \( u_{x_0,r} \) the average of \( u \) over \( B_r(x_0) \cap \Omega \). We will call \( \eta_u \) the \( \text{VMO} \) modulus of \( u \) in \( \Omega \).

Now let us recall the class \( S \) of solutions of uniformly elliptic equations. For more details, see [CC]. Let \( A_{\lambda, \Lambda} \) denote all symmetric matrices whose eigenvalues belong to \([\lambda, \Lambda]\). Define Pucci extremal operators \( M^+(M) \) and \( M^-(M) \) by

\[
M^+(M) = \sup_{A \in A_{\lambda, \Lambda}} \text{trace}(AM), \\
M^-(M) = \inf_{A \in A_{\lambda, \Lambda}} \text{trace}(AM),
\]

for \( M \in S \). It is easy to check that \( M^+ \) and \( M^- \) are uniformly elliptic operators. A continuous function \( u \) is in class \( S \) if \( M^- (D^2u) \leq 0 \) and \( M^+ (D^2u) \geq 0 \) in the viscosity sense.

The following result on precompact sets in \( L^p \) is a local variant of Theorem 3.44 in [A].

**Proposition 1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( A \) a bounded subset of \( L^p(\Omega) \), \( 1 \leq p < \infty \). For any domain \( D \subset \subset \Omega \), if

\[
\sup_{u \in A} \int_D |u(x + h) - u(x)|^p \, dx \to 0, \quad \text{as} \ |h| \to 0,
\]

then \( A \) is precompact in \( L^p(D) \).

Now let us prove the following lemma.
Lemma 1. Assume that \( F \) satisfies (2) and \( F(0) = 0 \). Then the following two statements are equivalent:

(i) If \( u \in C^{1,1}(B_1(0)) \) is a viscosity solution of (1) in \( B_1(0) \) and \( |D^2u| \leq M \) in \( B_1(0) \), then \( D^2u \in \text{VMO}(B_{1/2}(0)) \) and \( \eta_{D^2u}(R) \leq \eta(R) \), where \( \eta_{D^2u}(R) \) is the VMO modulus of \( D^2u \) in \( B_{1/2}(0) \), \( \lim_{R \to 0^+} \eta(R) = 0 \), and \( \eta \) depends only on \( n, \lambda, \Lambda, F, \) and \( M \).

(ii) \( F \) satisfies the Liouville property.

Proof. (i) implies (ii). Let \( u \in C^{1,1}_{loc}(\mathbb{R}^n) \) be an entire solution of (1) with \( |D^2u| \leq M \) in \( \mathbb{R}^n \). Consider

\[
v_k(y) = \frac{u(ky) - u(0) - Du(0)ky}{k^2}, \quad k = 1, 2, \ldots.
\]

Obviously \( \|v_k\|_{C^{1,1}(B_1(0))} \leq C_nM \) and

\[
F(D^2v_k) = 0 \quad \text{in } B_1(0).
\]

Therefore by (i), for \( \rho > 0 \) we have

\[
\int_{B_r(0)} |D^2u - (D^2u)_{0,\rho}| \, dx = \int_{B_{\rho}(0)} |D^2v_k - (D^2v_k)_{0,\rho}| \, dy 
\leq \eta_{D^2v_k}(\rho) \leq \eta(\rho) \to 0, \quad \text{as } k \to \infty.
\]

This implies that \( D^2u = \text{const} \) in \( \mathbb{R}^n \) and hence \( u \) is a polynomial of degree at most 2.

Suppose that \( F \) satisfies the Liouville property. We want to show (i). Let

\[
X_M = \{ u \in C^{1,1}(B_1(0)) : F(D^2u) = 0 \text{ and } |D^2u| \leq M \text{ in } B_1(0) \}.
\]

To prove that (i) holds, it suffices to show the following claim:

\[
\sup_{u \in X_M} \int_{B_r(x_0)} |D^2u - (D^2u)_{x_0,r}|^2 \, dx \to 0, \quad \text{as } R \to \infty. (3)
\]

We will show (3) by contradiction. If (3) is false, then there exist \( \varepsilon_0 > 0, r_k \to 0, \) \( x_k \in B_{1/2}(0), u_k \in X_M \) such that for \( k \geq 1 \)

\[
\int_{B_{r_k}(x_k)} |D^2u_k - (D^2u_k)_{x_k,r_k}|^2 \, dx \geq \varepsilon_0.
\]

Let

\[
T_k y = x_k + r_k y, \quad \Omega_k = T_k^{-1} B_1(0); \\
v_k(y) = \frac{u_k(x_k + r_k y) - u_k(x_k) - Du_k(x_k)r_k y}{r_k^2}.
\]

It is easy to check that

\[
F(D^2v_k) = 0, \quad \text{in } \Omega_k. \quad (4)
\]

\[
\int_{B_1(0)} |D^2v_k - (D^2v_k)_{0,1}|^2 \, dy \geq \varepsilon_0. \quad (5)
\]

\[
\|v_k\|_{C^{1,1}(B_{2A}(0))} \leq C_{n,A} M, \quad \text{if } B_{2A}(x_k) \subset B_1(0).
\]
Now we want to show \( \{D^2v_k\} \) is precompact in \( L^2 \). By [CC] (see Prop. 5.5)
\[
\Delta \tau v_k(y) = \frac{v_k(y + \tau e) - v_k(y)}{\tau} \in S, \quad in \ B_{3A/2}(0), \ |e| = 1.
\]
By the \( W^{2,\delta} \) estimate (\( \delta > 0 \)) (see Prop. 7.4, [CC]) for functions in \( S \)
\[
\int_{B_A(0)} |D^2 \Delta \tau v_k|^\delta(y) \, dy \leq C_A \|\Delta \tau v_k\|_{L^\infty(B_{3A/2}(0))}^\delta \leq C_{A,M} M^\delta,
\]
where \( \delta \) and \( C_A \) are independent of \( k \).
Therefore, by (5) we have
\[
\int_{B_A(0)} |D^2v_k(y + \tau e) - D^2v_k(y)|^2 \, dy \leq C \|D^2v_k\|_{L^\infty(B_A(0))}^{2-\delta} \int_{B_A(0)} |D^2v_k(y + \tau e) - D^2v_k(y)|^\delta \, dy \leq C_\tau^\delta.
\]
By Proposition 1, this fact together with
\[
\|D^2v_k\|_{L^2(B_A(0))} \leq C_A
\]
implies that \( \{D^2v_k\} \) is precompact in \( L^2(B_A(0)) \). Since \( v_k(0) = 0 \), \( Dv_k(0) = 0 \), and \( \|v_k\|_{C^{1,1}(B_{2A}(0))} \leq C_A \), we may assume that by the diagonalizing process
\[
v_k \to v, \quad in \ W^{2,2}_{\text{loc}}(\mathbb{R}^n) \cap C^1_{\text{loc}}(\mathbb{R}^n),
\]
\[
D^2v_k \to D^2v, \quad a.e. \text{ in } \mathbb{R}^n.
\]
Therefore, \( F(D^2v) = 0 \) in \( \mathbb{R}^n \). Since \( \|D^2v_k\|_{L^\infty(B_A(0))} \leq \|D^2u_k\|_{L^\infty(B_{1}(0))} \leq M, \ |D^2v| \leq M \) in \( \mathbb{R}^n \). By the Liouville property, \( v \) must be a polynomial of degree at most 2, and hence \( D^2v = \text{const.} \) This contradicts the following:
\[
\int_{B_1(0)} |D^2v - (D^2v)_{0,1}|^2 \, dy = \lim_{k \to \infty} \int_{B_1(0)} |D^2v_k - (D^2v_k)_{0,1}|^2 \, dy \geq \varepsilon_0.
\]
Thus, Lemma 1 follows. \( \square \)

**Lemma 2.** Let \( F \in C^1 \) satisfy (2) and \( F(0) = 0 \). If \( u \) is a viscosity solution of (1) in \( B_1(0) \) and \( D^2u \in \text{VMO}(B_1(0)) \), then for any \( 0 < \alpha < 1 \), \( u \in C^{2,\alpha}(B_{1/2}(0)) \) and \( |D^2u|_{C^{2,\alpha}(B_{1/3,3}(0))} \leq C \), where \( C \) depends on \( n, \alpha, \lambda, \Lambda \), the modulus of continuity of \( DF \), \( \|u\|_{C^1(B_1(0))} \) and the VMO modulus of \( D^2u \).

**Proof.** By differentiating (1), we obtain
\[
a_{ij}(x)D_{ij} \Delta h u(x) = 0, \quad in \ B_{3/4}(0),
\]
where \( \Delta h u(x) = [u(x + h) - u(x)]/h, \ h < \frac{1}{4}, \ |e| = 1 \), and
\[
a_{ij}(x) = \int_0^1 \frac{1}{\partial M_{ij}} ((1 - \theta)D^2u(x) + \theta D^2u(x + he)) \, d\theta.
\]
Let \( u_h(x) = u(x + he) \) and
\[
c_{ij} = \int_0^1 \frac{1}{\partial M_{ij}} ((1 - \theta)(D^2u)_{x_0,r} + \theta (D^2u_h)_{x_0,r}) \, d\theta.
\]
Without loss of generality, we can assume that the continuity modulus of $DF$ denoted by $\omega(R)$ is concave. Obviously by Jensen’s inequality
\[
\int_{B_r(x_0)} |a_{ij}(x) - c_{ij}| \, dx \\
\leq \int_{B_r(x_0)} \int_0^1 \omega[(1 - \theta)(D^2 u - (D^2 u)_{x_0,r}) + \theta(D^2 u_h - (D^2 u_h)_{x_0,r})] \, d\theta \, dx \\
\leq \omega(\eta D^2 u(r)) \rightarrow 0, \quad \text{as } r \rightarrow 0.
\]
Therefore $a_{ij} \in \text{VMO}(B_{3/4}(0))$. Since (2) holds, $\lambda' I \leq (a_{ij}) \leq \Lambda' I$ and (6) is uniformly elliptic. By the $L^p$ estimate in [CFL], we obtain
\[
\|\Delta_{he} u\|_{W^{2,p}(B_{1/2}(0))} \leq C\|\Delta_{he} u\|_{L^\infty(B_{3/4}(0))}.
\]
Thus, we finish the proof of Lemma 2.

The Theorem follows immediately from Lemma 1 and Lemma 2.

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REFERENCES


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