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# ON THE REGULARITY OF SOLUTIONS TO FULLY NONLINEAR ELLIPTIC EQUATIONS VIA THE LIOUVILLE PROPERTY

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ABSTRACT. We show that any $C^{1,1}$ solution to the uniformly elliptic equation $F(D^2u) = 0$ must belong to $C^{2,\alpha}$, if the equation has the Liouville property.

## §1. INTRODUCTION

In this paper, we consider the interior regularity of solutions to the following fully nonlinear elliptic equation:

$$F(D^2u) = 0. \tag{1}$$

We assume that $F$ is uniformly elliptic, i.e., there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\|, \qquad \text{for } M, N \in \mathcal{S},\ N \geq 0, \tag{2}$$

where $\mathcal{S}$ denotes the space of real $n \times n$ symmetric matrices and $\|N\|$ denotes the norm of $N$.

For simplicity, we also assume that $F(0) = 0$.

There have been a number of works concerning equation (1). For instance, see [CC], [GT], [K] and the references cited there. When $F$ is a concave or convex functional, it is well known that the Evans-Krylov estimate

$$[D^2u]_{C^\alpha(B_{1/2})} \leq C\|u\|_{C^{1,1}(B_1)}$$

holds, and $C^{1,1}$ viscosity solutions of (1) are $C^{2,\alpha}$ for some $\alpha > 0$.

On the contrary, in the case when $F$ is not concave nor convex, $C^{1,1}$ viscosity solutions of (1) may not be in the $C^2$ class. This has recently been shown by Nadirashvili in [N] in which he found a $C^{1,1}$ viscosity solution $u$ to the equation $F(D^2u) = 0$ where $F$ is smooth, uniformly elliptic and $u$ is not $C^2$. Therefore, it would be interesting to know under what condition a $C^{1,1}$ solution of (1) is actually in the $C^2$ class.

It is our purpose in this paper to show that any $C^{1,1}$ viscosity solution of (1) must be $C^{2,\alpha}$ if the elliptic operator $F$ has the Liouville property.

A continuous function $u(x)$ is said to be a viscosity subsolution (*resp.*, supersolution) of (1) in a domain $\Omega$ if for $x_0 \in \Omega$ and $\phi(x) \in C^2$, $u - \phi$ attains the local

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maximum (*resp.*, minimum) at $x_0$, then $F(D^2\phi(x_0)) \geq 0$ (*resp.*, $\leq 0$). If $u$ is both a subsolution and a supersolution, then we say $u$ is a viscosity solution. We mention that if $u \in C^{1,1}$, then $u$ is a viscosity solution of (1) if and only if $u$ is a strong solution to (1).

Equation (1) or $F$ is said to satisfy the Liouville property if $u \in C^{1,1}_{loc}(\mathbf{R}^n)$ is an entire viscosity solution of (1) with bounded $D^2u$ in $\mathbf{R}^n$, $|D^2u| \leq C$, then $u$ must be a polynomial of degree at most 2.

Let $B_r(x_0) = \{x \in \mathbf{R}^n : \ |x - x_0| < r\}$.

Now we state the main theorem.

**Theorem.** *Suppose that $F \in C^1$ satisfies* (2) *and $F(0) = 0$. Let $u \in C^{1,1}(B_1(0))$ be a viscosity solution of* (1) *in $B_1(0)$. If equation* (1) *satisfies the Liouville property, then for any $0 < \alpha < 1$, $u \in C^{2,\alpha}(B_{1/2}(0))$ and $[D^2u]_{C^\alpha(B_{1/2}(0))} \leq C$, where $C$ depends only on $n$, $\lambda$, $\Lambda$, $\alpha$, $\|u\|_{C^{1,1}(B_1(0))}$, $F$, and the modulus of continuity of $DF$.*

## §2. The proof of the Theorem

We will use the blow-up technique to prove the Theorem. The tool needed to obtain a subsequence of blow-up solutions converging in $W^{2,2}_{loc}(\mathbf{R}^n)$ is the $W^{2,\delta}$ estimate for nondivergent uniform elliptic equations. For the convenience of our readers, let us give a little more preliminary information.

Recall that $u \in \mathrm{BMO}(\Omega)$ is in $\mathrm{VMO}(\Omega)$ if

$$\eta_u(R, \Omega) = \sup_{\substack{x_0\in\Omega \\ 0<r\leq R}} \fint_{B_r(x_0)\cap\Omega} |u(x) - u_{x_0,r}|\, dx \longrightarrow 0, \qquad \text{as } R \longrightarrow 0,$$

where $\fint_A f\, dx$ denotes the average of $f$ over $A$ and $u_{x_0,r}$ the average of $u$ over $B_r(x_0) \cap \Omega$. We will call $\eta_u$ the VMO modulus of $u$ in $\Omega$.

Now let us recall the class $\mathcal{S}$ of solutions of uniformly elliptic equations. For more details, see [CC]. Let $\mathcal{A}_{\lambda,\Lambda}$ denote all symmetric matrices whose eigenvalues belong to $[\lambda, \Lambda]$. Define Pucci extremal operators $M^+(M)$ and $M^-(M)$ by

$$M^+(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{trace}(AM),$$
$$M^-(M) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{trace}(AM),$$

for $M \in \mathcal{S}$. It is easy to check that $M^+$ and $M^-$ are uniformly elliptic operators. A continuous function $u$ is in class $\mathcal{S}$ if $M^-(D^2u) \leq 0$ and $M^+(D^2u) \geq 0$ in the viscosity sense.

The following result on precompact sets in $L^p$ is a local variant of Theorem 3.44 in [A].

**Proposition 1.** *Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ and $\mathcal{A}$ a bounded subset of $L^p(\Omega)$, $1 \leq p < \infty$. For any domain $D \subset\subset \Omega$, if*

$$\sup_{u\in\mathcal{A}} \int_D |u(x+h) - u(x)|^p dx \to 0, \qquad \text{as } |h| \to 0,$$

*then $\mathcal{A}$ is precompact in $L^p(D)$.*

Now let us prove the following lemma.

**Lemma 1.** *Assume that $F$ satisfies (2) and $F(0) = 0$. Then the following two statements are equivalent:*

(i) *If $u \in C^{1,1}(B_1(0))$ is a viscosity solution of (1) in $B_1(0)$ and $|D^2u| \le M$ in $B_1(0)$, then $D^2u \in VMO(B_{1/2}(0))$ and $\eta_{D^2u}(R) \le \eta(R)$, where $\eta_{D^2u}(R)$ is the VMO modulus of $D^2u$ in $B_{1/2}(0)$, $\lim_{R\to 0^+}\eta(R) = 0$, and $\eta$ depends only on $n$, $\lambda$, $\Lambda$, $F$, and $M$.*

(ii) *$F$ satisfies the Liouville property.*

*Proof.* (i) implies (ii). Let $u \in C^{1,1}_{loc}(\mathbf{R}^n)$ be an entire solution of (1) with $|D^2u| \le M$ in $\mathbf{R}^n$. Consider

$$v_k(y) = \frac{u(ky) - u(0) - Du(0)ky}{k^2}, \qquad k = 1, 2, \cdots.$$

Obviously $\|v_k\|_{C^{1,1}(B_1(0))} \le C_n M$ and

$$F(D^2v_k) = 0 \qquad \text{in } B_1(0).$$

Therefore by (i), for $\rho > 0$ we have

$$\begin{aligned} &\fint_{B_\rho(0)} |D^2u - (D^2u)_{0,\rho}|\,dx \\ =&\fint_{B_{\frac{\rho}{k}}(0)} |D^2v_k - (D^2v_k)_{0,\frac{\rho}{k}}|\,dy \\ \le&\,\eta_{D^2v_k}(\frac{\rho}{k}) \le \eta(\frac{\rho}{k}) \longrightarrow 0, \qquad \text{as } k \longrightarrow \infty. \end{aligned}$$

This implies that $D^2u = \text{const}$ in $\mathbf{R}^n$ and hence $u$ is a polynomial of degree at most 2.

Suppose that $F$ satisfies the Liouville property. We want to show (i). Let

$$X_M = \{u \in C^{1,1}(B_1(0)) : \ F(D^2u) = 0 \text{ and } |D^2u| \le M \text{ in } B_1(0)\}.$$

To prove that (i) holds, it suffices to show the following claim:

$$\sup_{\substack{u\in X_M \\ x_0\in B_{1/2}(0) \\ r\le R}} \fint_{B_r(x_0)} |D^2u - (D^2u)_{x_0,r}|^2\,dx \longrightarrow 0, \qquad \text{as } R \longrightarrow 0. \tag{3}$$

We will show (3) by contradiction. If (3) is false, then there exist $\varepsilon_0 > 0$, $r_k \longrightarrow 0$, $x_k \in B_{1/2}(0)$, $u_k \in X_M$ such that for $k \ge 1$

$$\fint_{B_{r_k}(x_k)} |D^2u_k - (D^2u_k)_{x_k,r_k}|^2\,dx \ge \varepsilon_0.$$

Let

$$T_k y = x_k + r_k y, \qquad \Omega_k = T_k^{-1}B_1(0);$$
$$v_k(y) = \frac{u_k(x_k + r_k y) - u_k(x_k) - Du_k(x_k)r_k y}{r_k^2}.$$

It is easy to check that

$$F(D^2v_k) = 0, \qquad \text{in } \Omega_k.$$

$$\fint_{B_1(0)} |D^2v_k - (D^2v_k)_{0,1}|^2\,dy \ge \varepsilon_0. \tag{4}$$

$$\|v_k\|_{C^{1,1}(B_{2A}(0))} \le C_{n,A}M, \qquad \text{if } B_{2Ar_k}(x_k) \subset B_1(0). \tag{5}$$

Now we want to show $\{D^2v_k\}$ is precompact in $L^2$. By [CC] (see Prop. 5.5)

$$\triangle_{\tau e}v_k(y) = \frac{v_k(y+\tau e) - v_k(y)}{\tau} \in \mathcal{S}, \qquad \text{in } B_{3A/2}(0),\ |e| = 1.$$

By the $W^{2,\delta}$ estimate ($\delta > 0$) (see Prop. 7.4, [CC]) for functions in $\mathcal{S}$

$$\int_{B_A(0)} |D^2\triangle_{\tau e}v_k|^\delta(y)\,dy \le C_A\|\triangle_{\tau e}v_k\|^\delta_{L^\infty(B_{3A/2}(0))} \le C_{A,M}M^\delta,$$

where $\delta$ and $C_A$ are independent of $k$.

Therefore, by (5) we have

$$\begin{aligned}&\int_{B_A(0)} |D^2v_k(y+\tau e) - D^2v_k(y)|^2\,dy\\ \le &C\|D^2v_k\|^{2-\delta}_{L^\infty(B_{2A}(0))}\int_{B_A(0)} |D^2v_k(y+\tau e) - D^2v_k(y)|^\delta\,dy\\ \le &C\tau^\delta.\end{aligned}$$

By Proposition 1, this fact together with

$$\|D^2v_k\|_{L^2(B_{2A}(0))} \le C_A$$

implies that $\{D^2v_k\}$ is precompact in $L^2(B_A(0))$. Since $v_k(0) = 0$, $Dv_k(0) = 0$, and $\|v_k\|_{C^{1,1}(B_{2A}(0))} \le C_A$, we may assume that by the diagonalizing process

$$\begin{aligned}v_k &\longrightarrow v, &&\text{in } W^{2,2}_{loc}(\mathbf{R}^n)\cap C^1_{loc}(\mathbf{R}^n),\\ D^2v_k &\longrightarrow D^2v, &&\text{a.e. in } \mathbf{R}^n.\end{aligned}$$

Therefore, $F(D^2v) = 0$ in $\mathbf{R}^n$. Since $\|D^2v_k\|_{L^\infty(B_A(0))} \le \|D^2u_k\|_{L^\infty(B_1(0))} \le M$, $|D^2v| \le M$ in $\mathbf{R}^n$. By the Liouville property, $v$ must be a polynomial of degree at most 2, and hence $D^2v = \text{const}$. This contradicts the following:

$$⨍_{B_1(0)} |D^2v - (D^2v)_{0,1}|^2\,dy = \lim_{k\to\infty} ⨍_{B_1(0)} |D^2v_k - (D^2v_k)_{0,1}|^2\,dy \ge \varepsilon_0.$$

Thus, Lemma 1 follows. □

**Lemma 2.** *Let $F \in C^1$ satisfy* (2) *and $F(0) = 0$. If $u$ is a viscosity solution of* (1) *in $B_1(0)$ and $D^2u \in VMO(B_1(0))$, then for any $0 < \alpha < 1$, $u \in C^{2,\alpha}(B_{1/2}(0))$ and $[D^2u]_{C^{2,\alpha}(B_{1/2}(0))} \le C$, where $C$ depends on $n$, $\alpha$, $\lambda$, $\Lambda$, the modulus of continuity of $DF$, $\|u\|_{C^1(B_1(0))}$ and the VMO modulus of $D^2u$.*

*Proof.* By differentiating (1), we obtain

$$a_{ij}(x)D_{ij}\triangle_{he}u(x) = 0, \qquad \text{in } B_{3/4}(0), \tag{6}$$

where $\triangle_{he}u(x) = [u(x+he) - u(x)]/h$, $h < \frac{1}{4}$, $|e| = 1$, and

$$a_{ij}(x) = \int_0^1 \frac{\partial F}{\partial M_{ij}}((1-\theta)D^2u(x) + \theta D^2u(x+he))\,d\theta.$$

Let $u_h(x) = u(x+he)$ and

$$c_{ij} = \int_0^1 \frac{\partial F}{\partial M_{ij}}((1-\theta)(D^2u)_{x_0,r} + \theta(D^2u_h)_{x_0,r})\,d\theta.$$

Without loss of generality, we can assume that the continuity modulus of $DF$ denoted by $\omega(R)$ is concave. Obviously by Jensen's inequality

$$\begin{aligned}&\fint_{B_r(x_0)} |a_{ij}(x) - c_{ij}|\,dx\\ \le&\fint_{B_r(x_0)}\int_0^1 \omega[|(1-\theta)(D^2u-(D^2u)_{x_0,r})+\theta(D^2u_h-(D^2u_h)_{x_0,r})|]\,d\theta dx\\ \le&\,\omega(\eta_{D^2u}(r)) \longrightarrow 0, \qquad \text{as } r\longrightarrow 0.\end{aligned}$$

Therefore $a_{ij} \in \mathrm{VMO}(B_{3/4}(0))$. Since (2) holds, $\lambda' I \le (a_{ij}) \le \Lambda' I$ and (6) is uniformly elliptic. By the $L^p$ estimate in [CFL], we obtain

$$\|\triangle_{he}u\|_{W^{2,p}(B_{1/2}(0))} \le C\|\triangle_{he}u\|_{L^\infty(B_{3/4}(0))}.$$

Thus, we finish the proof of Lemma 2. □

The Theorem follows immediately from Lemma 1 and Lemma 2.

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