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TWISTED HIGHER MOMENTS OF KLOOSTERMAN SUMS

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Abstract. Let χ be a nontrivial Dirichlet character modulo an odd prime p. Write

$$S(a) = \sum_{x=1}^{p-1} e(\frac{x + ax^{-1}}{p}) = 2\sqrt{p}\cos\theta(a).$$

We shall prove

$$\sum_{a=1}^{p-1} \chi(a) S(a)^2 = \chi(-1) g(\chi)^2 J(\chi, \bar{\chi}^2)$$

and, for complex χ,

$$|\sum_{a=1}^{p-1} \chi(a) \frac{\sin(k+1)\theta(a)}{\sin\theta(a)}| \leq c(k)\sqrt{p},\ k > 0,$$

where $c(k)$ is a constant depending only on k.

1. Introduction

Let p be an odd prime. Write

$$S(a) = \sum_{x=1}^{p-1} e(\frac{x + ax^{-1}}{p}).$$

It is a Kloosterman sum. It is known that ([I])

$$\sum_{a=1}^{p-1} S(a) = 1,$$

$$\sum_{a=1}^{p-1} S(a)^2 = p^2 - p - 1,$$

$$\sum_{a=1}^{p-1} S(a)^3 = (\frac{-3}{p})p^2 + 2p + 1$$

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and

$$\sum_{a=1}^{p-1} S(a)^4 = 2p^3 - 3p^2 - p - 1.$$

By A. Weil's result ([W]), we may write

$$\cos\theta(a) = \frac{1}{2\sqrt{p}} S(a).$$

Katz ([K]) proved the following equidistribution result:

$$|\sum_{a=1}^{p-1} \frac{\sin(k+1)\theta(a)}{\sin\theta(a)}| \leq \frac{k+1}{2}\sqrt{p},\ \ k \geq 1.$$

We may expect, for any Dirichlet character χ to the modulus p, that

$$|\sum_{a=1}^{p-1} \chi(a)\frac{\sin(k+1)\theta(a)}{\sin\theta(a)}| \leq \frac{k+1}{2}\sqrt{p},\ \ k \geq 1.$$

D. H. and E. Lehmer ([L]) found empirically in 1952 and proved in 1959 that

$$\sum_{a=1}^{p-1} (\frac{a}{p})\frac{\sin 4\theta(a)}{\sin\theta(a)} = 2\sqrt{p}(\frac{-1}{p})(A^2/p - 1)$$

if $p = A^2 + 3B^2, 3|(A+1)$ and vanishes if $3|(p+1)$.

We shall prove

Theorem 1. *If χ is a nonquadratic Dirichlet character to the modulus p, then*

$$|\sum_{a=1}^{p-1} \chi(a)\frac{\sin(k+1)\theta(a)}{\sin\theta(a)}| \leq c(k)\sqrt{p},\ \ k > 0,$$

where $c(k)$ is a constant depending only on k.

If $k = 2$, we can prove more.

Theorem 2.

$$\sum_{a=1}^{p-1} \chi(a)\frac{\sin 3\theta(a)}{\sin\theta(a)} = \frac{1}{p}\chi(-1)g(\chi)^2 J(\chi, \bar{\chi}^2),$$

where χ is a nontrivial Dirichlet character to the modulus p,

$$g(\chi) = \sum_{x=1}^{p-1} \chi(x) e(\frac{x}{p})$$

is a Gauss sum and

$$J(\chi_1, \chi_2) = \sum_{x=1}^{p-1} \chi_1(x)\chi_2(1-x)$$

is a Jacobi sum.

Another form of Theorem 2 is

$$\sum_{a=1}^{p-1}\chi(a)S(a)^2=\chi(-1)g(\chi)^2J(\chi,\bar{\chi}^2),$$

where χ is a nontrivial Dirichlet character to the modulus p. It is equivalent to

$$(p-1)S(a)^2=p^2-p-1+\sum_{\chi}\chi(-1)g(\chi)^2J(\chi,\bar{\chi}^2)\bar{\chi}(a),$$

where χ runs over all nontrivial Dirichlet characters to the modulus p. So

$$(p-1)S(a^2)^2=p^2-p-1+\sum_{\chi}\chi(-1)g(\chi)^2J(\chi,\bar{\chi}^2)\bar{\chi}^2(a),$$

where χ runs over all nontrivial Dirichlet characters to the modulus p. It implies

$$|\sum_{a=1}^{p-1}\chi(a)S(a^2)^2|\leq 2p^{3/2},$$

where χ is a nontrivial Dirichlet character to the modulus p. That is a little sharper than

$$|\sum_{a=1}^{p-1}\chi(a)S(a^2)^2|\leq 4p^{3/2},$$

which was proved for quadratic Dirichlet character χ to the modulus p, and conjectured for general nontrivial Dirichlet character χ in ([C]) by Conrey and Iwaniec.

Remark. Theorems 1 and 2 generalize to the finite field case.

2. The twisted square moment

We now prove Theorem 2. Opening $S(a)^2$, we get

$$\sum_{a=1}^{p-1}\chi(a)S(a)^2=\sum_{a=1}^{p-1}\chi(a)\sum_{x=1}^{p-1}\sum_{y=1}^{p-1}e(\frac{x+y}{p})e(\frac{a(x^{-1}+y^{-1})}{p}).$$

Summing over a first and applying

$$\sum_{a=1}^{p-1}\chi(a)e(\frac{ax}{p})=\overline{\chi}(x)g(\chi),$$

we get

$$\sum_{a=1}^{p-1}\chi(a)S(a)^2=g(\chi)\sum_{x=1}^{p-1}\sum_{y=1}^{p-1}e(\frac{x+y}{p})\overline{\chi}(x^{-1}+y^{-1}).$$

A change of variable yields

$$\sum_{a=1}^{p-1}\chi(a)S(a)^2=g(\chi)\sum_{x=1}^{p-1}\sum_{y=1}^{p-1}e(\frac{x+xy}{p})\overline{\chi}(x^{-1}+x^{-1}y^{-1}).$$

Summing over x first and applying

$$\sum_{a=1}^{p-1}\chi(a)e(\frac{ax}{p})=\overline{\chi}(x)g(\chi)$$

once more, we get

$$\sum_{a=1}^{p-1}\chi(a)S(a)^2 = g(\chi)^2\sum_{y=1}^{p-1}\overline{\chi}(1+y^{-1})\overline{\chi}(1+y)$$
$$= g(\chi)^2\sum_{y=1}^{p-1}\chi(y)\overline{\chi}^2(1+y) = \chi(-1)g(\chi)^2 J(\chi,\bar{\chi}^2).$$

3. The twisted higher moments

We now prove Theorem 1. For $q=p^m$, write

$$S(a;q) = \sum_x e(\frac{tr(x+ax^{-1})}{p}) = 2\sqrt{q}\cos\theta(a;q),$$

where x runs over all nonzero elements in $\mathbb{F}_q$, and tr is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$. Write

$$T(\psi;q) = \sum_a \psi(a)\frac{\sin(k+1)\theta(a;q)}{\sin\theta(a;q)},$$

where a runs over all nonzero elements in $\mathbb{F}_q$, and ψ is a multiplicative character of $\mathbb{F}_q$. Write

$$L(t) = \exp(\sum_{m\geq 1} T(\chi_m;p^m)t^m/m),$$

where χ_m is the lift of χ from $\mathbb{F}_p$ to $\mathbb{F}_{p^m}$. According to B. Dwork ([Dw]), $L(t)$ is a rational function

$$L(t) = \prod_{v\in I}(1-\alpha_v(\chi)t)\prod_{v\in J}(1-\alpha_v(\chi)t)^{-1}.$$

Equivalently,

$$T(\chi_m;p^m) = -\sum_{v\in I}\alpha_v(\chi)^m + \sum_{v\in J}\alpha_v(\chi)^m.$$

According to P. Deligne ([D]),

$$|\alpha_v(\chi)| = p^{l_v/2}$$

and the total number of $\alpha_v(\chi)$ is bounded by a number $c(k)$ depending only on k. We conclude that

$$|\alpha_v(\chi)| \leq \sqrt{p},$$

from which Theorem 2 follows. Otherwise, according to E. Bombieri's arguments ([B]),

$$|T(\chi_m;p^m)| > (1-\varepsilon)p^m$$

for infinitely many m. Then

$$|T(\chi_m;p^m)|^2 + |T(\overline{\chi}_m;p^m)|^2 > 2(1-\varepsilon)^2 p^{2m}$$

for infinitely many m, contradicting the following lemma.

Lemma 3.

$$\frac{1}{q-1}\sum_{\psi}|T(\psi;q)|^2 = q + O(\sqrt{q}),$$

where ψ runs over all multiplicative characters of $\mathbb{F}_q$.

Indeed, we have

$$\frac{1}{q-1}\sum_{\psi}|T(\psi;q)|^2 = \sum_{a}(\frac{\sin(k+1)\theta(a;q)}{\sin\theta(a;q)})^2.$$

As

$$(\frac{\sin(k+1)\theta}{\sin\theta})^2 = 1 + \sum_{1\le l\le 2k} c_l \frac{\sin(l+1)\theta}{\sin\theta},$$

we have

$$\frac{1}{q-1}\sum_{\psi}|T(\psi;q)|^2 = q-1+\sum_{1\le l\le 2k} c_l \sum_{a}\frac{\sin(l+1)\theta(a;q)}{\sin\theta(a;q)}.$$

The lemma now follows from N. Katz's equidistribution result ([K])

$$|\sum_{a}\frac{\sin(l+1)\theta(a;q)}{\sin\theta(a;q)}| \le \frac{l+1}{2}\sqrt{q}.$$

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Added in proof

From Theorem 1 one can deduce that, for every odd integer m,

$$|\sum_{a=1}^{p-1}\frac{\sin(k+1)\theta(a^m)}{\sin\theta(a^m)}| \le (m,p-1)c(k)\sqrt{p},\ k>0,$$

where $c(k)$ is the constant in Theorem 1. That is, for every fixed odd integer m, the angles $\theta(a^m)$, for $1\le a\le p-1$, are equidistributed with respect to the Sato-Tate measure as p goes to infinity.

References

[B] E. Bombieri, *On exponential sums in finite fields II*, Invent. Math. **47** (1978), 29-39. MR **58:**22072

[C] J. Conrey and H. Iwaniec, *The cubic moment of central values of automorphic L-functions*, Ann. Math. **151** (2000), 1175-1216. MR **2001g:**11070

[D] P. Deligne, *La conjecture de Weil I*, Publ. Math. IHES **43** (1974), 273-307. MR **49:**5013

[Dw] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631-648. MR **25:**3914

[I] H. Iwaniec, *Topics in classical automorphic forms, GSM 17*, AMS, Providence, Rhode Island, 1997. MR **98e:**11051

[K] N. Katz, *Gauss Sums, Kloosterman Sums and Monodromy Groups, Ann. Math. Studies 116*, Princeton Univ. Press, 1988. MR **91a:**11028

[L] D. H. and E. Lehmer, *On the cubes of Kloosterman sums*, Acta Arith. **6** (1960), 15-22. MR **22:**6773

[M] L. J. Mordell, *On Lehmer's congruence associated with cubes of Kloostermann sums*, J. London Math. Soc. **36** (1961), 335-339. MR **23:**A3717

[W] A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. USA **34** (1948), 204-207. MR **10:**234e

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