TWISTED HIGHER MOMENTS OF KLOOSTERMAN SUMS

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Abstract. Let \( \chi \) be a nontrivial Dirichlet character modulo an odd prime \( p \).
Write
\[
S(a) = \sum_{x=1}^{p-1} e\left( \frac{x + ax^{-1}}{p} \right) = 2\sqrt{p}\cos \theta(a).
\]

We shall prove
\[
\sum_{a=1}^{p-1} \chi(a)S(a)^2 = \chi(-1)g(\chi)^2 J(\chi, \chi^2)
\]
and, for complex \( \chi \),
\[
\left| \sum_{a=1}^{p-1} \chi(a) \frac{\sin(k + 1)\theta(a)}{\sin \theta(a)} \right| \leq c(k)\sqrt{p}, \quad k > 0,
\]
where \( c(k) \) is a constant depending only on \( k \).

1. Introduction

Let \( p \) be an odd prime. Write
\[
S(a) = \sum_{x=1}^{p-1} e\left( \frac{x + ax^{-1}}{p} \right).
\]

It is a Kloosterman sum. It is known that (\( \square \))
\[
\sum_{a=1}^{p-1} S(a) = 1,
\]
\[
\sum_{a=1}^{p-1} S(a)^2 = p^2 - p - 1,
\]
\[
\sum_{a=1}^{p-1} S(a)^3 = \left( \frac{-3}{p} \right)p^2 + 2p + 1
\]

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and
\[ \sum_{a=1}^{p-1} S(a)^4 = 2p^3 - 3p^2 - p - 1. \]

By A. Weil’s result ([W]), we may write
\[ \cos \theta(a) = \frac{1}{2\sqrt{p}} S(a). \]

Katz ([K]) proved the following equidistribution result:
\[ \left| \sum_{a=1}^{p-1} \sum_{a=1}^{p-1} \frac{\sin(k + 1)\theta(a)}{\sin \theta(a)} \right| \leq \frac{k + 1}{2} \sqrt{p}, \quad k \geq 1. \]

We may expect, for any Dirichlet character \( \chi \) to the modulus \( p \), that
\[ \left| \sum_{a=1}^{p-1} \chi(a) \frac{\sin(k + 1)\theta(a)}{\sin \theta(a)} \right| \leq \frac{k + 1}{2} \sqrt{p}, \quad k \geq 1. \]

D. H. and E. Lehmer ([L]) found empirically in 1952 and proved in 1959 that
\[ \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) \sin 4\theta(a) = 2\sqrt{p} \left( \frac{-1}{p} \right)(A^2/p - 1) \]
if \( p = A^2 + 3B^2, 3|(A + 1) \) and vanishes if \( 3|(p + 1) \).

We shall prove

**Theorem 1.** If \( \chi \) is a nonquadratic Dirichlet character to the modulus \( p \), then
\[ \left| \sum_{a=1}^{p-1} \chi(a) \frac{\sin(k + 1)\theta(a)}{\sin \theta(a)} \right| \leq c(k) \sqrt{p}, \quad k > 0, \]
where \( c(k) \) is a constant depending only on \( k \).

If \( k = 2 \), we can prove more.

**Theorem 2.**
\[ \sum_{a=1}^{p-1} \chi(a) \sin 3\theta(a) = \frac{1}{p} \chi(-1) g(\chi)^2 J(\chi, \chi^2), \]
where \( \chi \) is a nontrivial Dirichlet character to the modulus \( p \),
\[ g(\chi) = \sum_{x=1}^{p-1} \chi(x) e\left( \frac{x}{p} \right) \]
is a Gauss sum and
\[ J(\chi, \chi^2) = \sum_{x=1}^{p-1} \chi_1(x) \chi_2(1 - x) \]
is a Jacobi sum.
Another form of Theorem 2 is
\[ \sum_{a=1}^{p-1} \chi(a)S(a)^2 = \chi(-1)g(\chi)^2J(\chi, \overline{\chi}^2), \]
where \( \chi \) is a nontrivial Dirichlet character to the modulus \( p \). It is equivalent to
\[ (p - 1)S(a)^2 = p^2 - p - 1 + \sum_{\chi} \chi(-1)g(\chi)^2J(\chi, \overline{\chi}^2)\overline{\chi}(a), \]
where \( \chi \) runs over all nontrivial Dirichlet characters to the modulus \( p \). So
\[ (p - 1)S(a)^2 = p^2 - p - 1 + \sum_{\chi} \chi(-1)g(\chi)^2J(\chi, \overline{\chi}^2)\overline{\chi}(a), \]
where \( \chi \) runs over all nontrivial Dirichlet characters to the modulus \( p \). It implies
\[ \sum_{a=1}^{p-1} \chi(a)S(a)^2 | \leq 2p^{3/2}, \]
where \( \chi \) is a nontrivial Dirichlet character to the modulus \( p \). That is a little sharper than
\[ \sum_{a=1}^{p-1} \chi(a)S(a)^2 | \leq 4p^{3/2}, \]
which was proved for quadratic Dirichlet character \( \chi \) to the modulus \( p \), and conjectured for general nontrivial Dirichlet character \( \chi \) in [1] by Conrey and Iwaniec.

**Remark.** Theorems 1 and 2 generalize to the finite field case.

### 2. THE TWISTED SQUARE MOMENT

We now prove Theorem 2. Opening \( S(a)^2 \), we get
\[ \sum_{a=1}^{p-1} \chi(a)S(a)^2 = \sum_{a=1}^{p-1} \chi(a) \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} e\left(\frac{ax + y}{p}\right) e\left(\frac{a(x-1) + y}{p}\right). \]
Summing over \( a \) first and applying
\[ \sum_{a=1}^{p-1} \chi(a)e\left(\frac{ax}{p}\right) = \overline{\chi}(x)g(\chi), \]
we get
\[ \sum_{a=1}^{p-1} \chi(a)S(a)^2 = g(\chi) \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} e\left(\frac{x + y}{p}\right) \overline{\chi}(x^{-1} + y^{-1}). \]
A change of variable yields
\[ \sum_{a=1}^{p-1} \chi(a)S(a)^2 = g(\chi) \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} e\left(\frac{x + xy}{p}\right) \overline{\chi}(x^{-1} + x^{-1}y^{-1}). \]
Summing over \( x \) first and applying
\[ \sum_{a=1}^{p-1} \chi(a)e\left(\frac{ax}{p}\right) = \overline{\chi}(x)g(\chi), \]
Once more, we get
\[ \sum_{a=1}^{p-1} \chi(a)S(a)^2 = g(\chi)^2 \sum_{y=1}^{p-1} \chi(1+y^{-1})\overline{\chi}(1+y) \]
\[ = g(\chi)^2 \sum_{y=1}^{p-1} \chi(y)\overline{\chi}(1+y) = \chi(-1)g(\chi)^2J(\chi, \chi^2). \]

3. The twisted higher moments

We now prove Theorem 1. For \( q = p^m \), write
\[ S(a; q) = \sum_x e\left( \frac{\text{tr}(x + ax^{-1})}{p} \right) = 2\sqrt{q} \cos(\theta(a; q)), \]
where \( x \) runs over all nonzero elements in \( \mathbb{F}_q \), and \( \text{tr} \) is the trace map from \( \mathbb{F}_q \) to \( \mathbb{F}_p \). Write
\[ T(\psi; q) = \sum_a \psi(a) \frac{\sin(k+1)\theta(a; q)}{\sin \theta(a; q)}, \]
where \( a \) runs over all nonzero elements in \( \mathbb{F}_q \), and \( \psi \) is a multiplicative character of \( \mathbb{F}_q \). Write
\[ L(t) = \exp \left( \sum_{n \geq 1} T(\chi_m; p^m)t^m / m \right), \]
where \( \chi_m \) is the lift of \( \chi \) from \( \mathbb{F}_p \) to \( \mathbb{F}_{p^m} \). According to B. Dwork ([Dw]), \( L(t) \) is a rational function
\[ L(t) = \prod_{v \in I} (1 - \alpha_v(\chi)t) \prod_{v \in J} (1 - \alpha_v(\chi)t)^{-1}. \]
Equivalently,
\[ T(\chi_m; p^m) = - \sum_{v \in I} \alpha_v(\chi)^m + \sum_{v \in J} \alpha_v(\chi)^m. \]
According to P. Deligne ([D]),
\[ |\alpha_v(\chi)| = p^{i_v}/2 \]
and the total number of \( \alpha_v(\chi) \) is bounded by a number \( c(k) \) depending only on \( k \). We conclude that
\[ |\alpha_v(\chi)| \leq \sqrt{p}, \]
from which Theorem 2 follows. Otherwise, according to E. Bombieri’s arguments ([B]),
\[ |T(\chi_m; p^m)| > (1 - \varepsilon)p^m \]
for infinitely many \( m \). Then
\[ |T(\chi_m; p^m)|^2 + |T(\chi_m; p^m)|^2 > 2(1 - \varepsilon)^2 p^{2m} \]
for infinitely many \( m \), contradicting the following lemma.
Lemma 3. 

\[ \frac{1}{q-1} \sum_{\psi} |T(\psi; q)|^2 = q + O(\sqrt{q}), \]

where \( \psi \) runs over all multiplicative characters of \( \mathbb{F}_q \).

Indeed, we have

\[ \frac{1}{q-1} \sum_{\psi} |T(\psi; q)|^2 = \sum_{a} \left( \frac{\sin(k + 1)\theta(a; q)}{\sin\theta(a; q)} \right)^2. \]

As

\[ \left( \frac{\sin(k + 1)\theta}{\sin\theta} \right)^2 = 1 + \sum_{1 \leq i \leq 2k} \frac{\sin(l + 1)\theta}{\sin\theta}, \]

we have

\[ \frac{1}{q-1} \sum_{\psi} |T(\psi; q)|^2 = q - 1 + \sum_{1 \leq i \leq 2k} c_i \sum_{a} \frac{\sin(l + 1)\theta(a; q)}{\sin\theta(a; q)}. \]

The lemma now follows from N. Katz’s equidistribution result ([K])

\[ \left| \sum_{a} \frac{\sin(l + 1)\theta(a; q)}{\sin\theta(a; q)} \right| \leq \frac{l + 1}{2} \sqrt{q}. \]

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ADDED IN PROOF

From Theorem 1 one can deduce that, for every odd integer \( m \),

\[ \left| \sum_{a=1}^{p-1} \frac{\sin(k + 1)\theta(a^m)}{\sin\theta(a^m)} \right| \leq (m, p - 1)c(k)\sqrt{p}, \quad k > 0, \]

where \( c(k) \) is the constant in Theorem 1. That is, for every fixed odd integer \( m \), the angles \( \theta(a^m) \), for \( 1 \leq a \leq p - 1 \), are equidistributed with respect to the Sato-Tate measure as \( p \) goes to infinity.

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