PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 7, Pages 1961–1973
S 0002-9939(02)06568-1
Article electronically published on February 27, 2002

ON THE STABILITY OF THE STANDARD RIEMANN SEMIGROUP

STEFANO BIANCHINI AND RINALDO M. COLOMBO

(Communicated by Suncica Canic)

Abstract. We consider the dependence of the entropic solution of a hyperbolic system of conservation laws

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,\cdot) = u_0 \end{cases}$$

on the flux function f. We prove that the solution is Lipschitz continuous w.r.t. the C^0 norm of the derivative of the perturbation of f. We apply this result to prove the convergence of the solution of the relativistic Euler equation to the classical limit.

1. Introduction

Under suitable assumptions on the function $f\colon \Omega \to \mathbb{R}^n$ (with $\Omega \subseteq \mathbb{R}^n$), the system

$$u_t + [f(u)]_x = 0 \tag{1.1}$$

generates a *Standard Riemann Semigroup* (SRS) $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$; see [6]. The aim of this paper is to investigate the dependence of S on the flow function f.

Several papers in the current literature are concerned with the existence of an SRS; see for example [8] and the references in [6]. On the contrary, in the present paper the existence of an SRS is assumed as a starting point and the focus is on the correspondence $f \mapsto S$. In fact, the results in this paper imply that the SRS S is a Lipschitz function of the flow f, with respect to the C^0 norm of Df. An immediate consequence is the following. Assume that f depends on the parameters $(p_1, \dots, p_m)$ that may vary in a compact subset of $\mathbb{R}^m$. Given a continuous functional J defined on the solution u at time t to the Cauchy problem for (1.1), this result ensures the continuity of the map $(p_1, \dots, p_m) \mapsto J(u(t))$. Hence, by the Weierstrass Theorem, the optimization problem admits a solution.

For the sake of completeness, we only recall here that the existence of the SRS for the $n \times n$ system (1.1) was first proved in [8]. The main assumptions there are that Df is strictly hyperbolic with every characteristic field either linearly degenerate or genuinely nonlinear, and that the initial data has sufficiently small total variation. More recently, the existence of the SRS was extended also to the not genuinely nonlinear setting in the 2×2 case; see [1].

Received by the editors July 1, 2000.
2000 *Mathematics Subject Classification.* Primary 35L65, 76N10.
Key words and phrases. Hyperbolic systems, conservation laws, well posedness.
We thank Alberto Bressan for useful discussions.

Below we shall restrict our attention to *standard* solutions to Riemann problems and, hence, to general Cauchy problems for conservation laws. Here, by standard solutions we refer to those introduced by Lax [12] and then generalized by Liu [13]. Various extensions of the present work to other types of solutions are straightforward.

The present paper is organized as follows. In the next section we state the main results. The proofs are given in Section 3. The following Sections 4 and 5 are devoted to two applications: the classical limit of the relativistic Euler equations, and scalar conservation laws with L^∞ initial data.

2. Notation and main results

Consider the following hyperbolic system of conservation laws in one space dimension:

$$u_t + [f(u)]_x = 0 \tag{2.1}$$

where $f\colon \Omega \to \mathbb{R}^n$ is in $\mathrm{Hyp}(\Omega)$, i.e. f is a sufficiently smooth function that generates an SRS $S^f\colon [0,+\infty[\, \times \mathcal{D}^f \to \mathcal{D}^f$. Recall that by an SRS generated by f (see [4]) we mean a map $S^f\colon [0,+\infty[\, \times \mathcal{D}^f \to \mathcal{D}^f$ with the following properties:

(i) S^f is a semigroup: $S^f_0 = \mathrm{Id}$ and $S^f_t \circ S^f_s = S^f_{t+s}$;
(ii) S^f is Lipschitz continuous: there exists a positive L_f such that for all positive t, s and for all $u, w \in \mathcal{D}^f$, $\|S^f_t u - S^f_s w\|_{L^1} \le L_f \cdot (|t-s| + \|u-w\|_{L^1})$;
(iii) if u is piecewise constant, then for t small, $S^f_t u$ coincides with the gluing of standard solutions to Riemann problems.

For all $u \in \mathcal{D}^f$, it is well known (see [4]) that the map $t \mapsto S^f_t u$ is a weak entropic solution to (2.1).

Given $f \in \mathrm{Hyp}(\Omega)$, let $\mathcal{R}(\mathcal{D}^f)$ be the set of all piecewise constant functions in $\mathcal{D}^f$ having a single jump at the origin. In other words, $\mathcal{R}(\mathcal{D}^f)$ is the set of initial data to the Riemann problems

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0. \end{cases} \end{cases} \tag{2.2}$$

Below, by a *solution* to (2.2) we always mean the standard Lax (see [12]) self–similar entropic solutions.

Let $f, g \in \mathrm{Hyp}(\Omega)$ with

$$\mathcal{D}^g \subseteq \mathcal{D}^f, \tag{2.3}$$

and define the *"distance"* between f and g as (cf. [3])

$$\hat{d}(f,g) = \sup_{u \in \mathcal{R}(\mathcal{D}^g)} \frac{1}{|u^+ - u^-|} \cdot \left\| S^f_1 u - S^g_1 u \right\|_{L^1}. \tag{2.4}$$

The distance $\hat{d}(f,g)$ is well defined due to (2.3). The main result of this paper is the following theorem.

Theorem 2.1. *Let $f \in \mathrm{Hyp}(\Omega)$. Then, for all $g \in \mathrm{Hyp}(\Omega)$ with $\mathcal{D}^g \subseteq \mathcal{D}^f$ and for all $u \in \mathcal{D}^g$,*

$$\left\| S^f_t u - S^g_t u \right\|_{L^1} \le L_f \cdot \hat{d}(f,g) \cdot \int_0^t \mathrm{Tot.\,Var.}\,(S^g_t u)\; dt\ . \tag{2.5}$$

Recall that L_f is the Lipschitz constant of the semigroup S^f, see (ii) above. The proof of Theorem 2.1 is deferred to Section 3.

Observe that $\hat{d}$ generalizes the analogous quantity $\hat{d}_{\rm lin}$ defined in [3] with reference to the linear case. Let $\mathbb{M}_d^{n\times n}$ denote the set of $n\times n$ diagonalizable matrices with real eigenvalues. Note that $\mathbb{M}_d^{n\times n}\subseteq \mathrm{Hyp}(\mathbb{R}^n)$. Fix a $v\in\mathbb{R}^n$, $v\neq 0$. Denote by $A^t\star v$ the solution evaluated at time t of the linear system

$$\begin{cases} u_t + Au_x = 0, \\ u(0,x) = v\cdot\chi_{[0,+\infty)}(x) \end{cases}$$

(here, χ_I is the characteristic function of the interval I). Theorem 2.3 in [3] shows that

$$\hat{d}_{\rm lin}(A,B) \doteq \sup_{v\colon |v|=1} \left\| A^1\star v - B^1\star v\right\|_{L^1} \tag{2.6}$$

is a distance on $\mathbb{M}_d^{n\times n}$ such that for all $A,B\in\mathbb{M}^{n\times n}$

$$\|B-A\| \le \hat{d}_{\rm lin}(A,B)\ , \tag{2.7}$$

$\|\cdot\|$ being the usual operator norm. Moreover, $\left(\mathbb{M}_d^{n\times n},\hat{d}_{\rm lin}\right)$ is a complete metric space. Clearly, if f and g are linear, then $\hat{d}_{\rm lin}(f,g)=\hat{d}(f,g)$.

Furthermore, $\hat{d}$ is related to $\hat{d}_{\rm lin}$ computed on the derivatives of the flow functions, as shown by the following proposition.

Proposition 2.2. *Let $f,g\in\mathrm{Hyp}(\Omega)$ with $\mathcal{D}^g\subseteq\mathcal{D}^f$. Then*

$$\hat{d}(f,g) \ge \sup_{u\in\Omega}\ \hat{d}_{\rm lin}(Df(u),Dg(u))\ . \tag{2.8}$$

Thus, $\hat{d}(f,g)$ seems stronger than the C^0 distance between Df and Dg, in the sense of (2.7). Nonetheless, Corollary 2.5 below shows that once the flow functions and the domains $\mathcal{D}^f$, $\mathcal{D}^g$ are fixed, i.e. the total variation of the solutions $S_t^f u$, $S_t^g u$ are uniformly bounded, we can estimate the r.h.s. in (2.5) by means of $\|Df-Dg\|_{C^0}$.

Theorem 2.1 shows that the stability of the solution to Riemann problems w.r.t. the flow is the key point for the stability of the whole SRS w.r.t. the flow. From the more abstract point of view of quasidifferential equations in metric spaces (see [5, 15]) this is equivalent to relating the distance between semigroups to the distance between the vector fields generated by the semigroups.

As in [5] (see also [15]), in a metric space (E,d) one can define an equivalence relation on all the Lipschitz curves $\gamma\colon[0,1]\to E$ exiting a fixed point u as

$$\gamma\sim\gamma' \quad \text{iff} \quad \lim_{\theta\mapsto 0^+}\frac{d\left(\gamma(\theta),\gamma'(\theta)\right)}{\theta}=0. \tag{2.9}$$

The quotient space T_u so obtained is naturally equipped with the metric

$$\hat{d}(v_1,v_2) \doteq \limsup_{\theta\to 0^+}\frac{d\left(\gamma_1(\theta),\gamma_2(\theta)\right)}{\theta}, \tag{2.10}$$

where γ_i is a representative of the equivalence class v_i. By (2.9), $\hat{d}$ does not depend on the particular representatives chosen. A map $v\colon E\to\bigcup_{u\in E}T_u$ is a vector field, provided $v(u)\in T_u$ for all u.

Let $S\colon E \times [0,+\infty[\to E$ be a Lipschitz semigroup, i.e. S satisfies

$$\begin{array}{c} S_0 = \mathbb{I}_E, \\ S_s \circ S_t = S_{s+t}, \end{array} \qquad \text{and} \qquad \begin{array}{c} \exists\, L > 0 \text{ such that} \\ \forall\, t, s > 0 \text{ and } \forall\, u, w \in \mathcal{D} \\ d\left(S_t u, S_s w\right) \leq L \cdot \left(|t-s| + d(u,w)\right) \ . \end{array}$$

Then, S naturally defines a vector field v_S on E by

$$v_S(u) \quad \text{is the equivalence class of the orbit} \quad \theta \mapsto S_\theta u \ . \tag{2.11}$$

Theorem 2.1 has a natural abstract counterpart, namely

Proposition 2.3. *Let S, S' be two Lipschitz semigroups on E generating the vector fields v and v', respectively. Denote by L the Lipschitz constant of, say, S. Then*

$$d\left(S_t u, S'_t u\right) \leq L \cdot \int_0^t \hat{d}\left(v(S'_t u), v'(S'_t u)\right)\, dt \ . \tag{2.12}$$

The above proposition is an immediate corollary of the following widely used (see [1, 4, 5, 7, 8] and the references in [6]) error estimate:

Lemma 2.4. *Given a Lipschitz semigroup $S\colon E \times [0,+\infty[\to E$ with Lipschitz constant L, for every Lipschitz continuous map $w\colon [0,T] \to E$ one has*

$$d\left(w(T), S_t w(0)\right) \leq L \cdot \int_0^T \liminf_{h \mapsto 0^+} \frac{d\left(w(t+h), S_h w(t)\right)}{h}\, dt \ . \tag{2.13}$$

The proof of Theorem 2.1 consists of the following steps.

(1) Find an explicit definition of the vector field v_S generated by the SRS S.
(2) Compute the r.h.s. in (2.12).
(3) Apply Lemma 2.4.

Note that this procedure requires just the existence of the SRS. In several cases (see for instance [1, 2, 7]) the existence of the SRS is achieved through the construction of a sequence S^n of uniformly Lipschitz approximate semigroups defined on piecewise constant functions. In all these cases, the vector field v_n generated by S^n on the set of piecewise constant functions simply consists in the gluing of solutions to Riemann problems. Thus, under the further assumption that such an approximating sequence S^n exists, step (1) could be avoided.

The quantity $\hat{d}$ in (2.4) is thus the natural tool to estimate the dependence of the SRS upon the flow function (note also that no constant is involved in (2.5)). However, in view of possible applications of Theorem 2.1, we provide an estimate of the r.h.s. in (2.5) in terms of handier quantities.

Corollary 2.5. *Let $f \in \mathrm{Hyp}(\Omega)$ and assume that*

$$\mathcal{D}^f \subseteq \left\{u \in L^1(\mathbb{R}, K)\colon \ \mathrm{Tot.\,Var.}(u) \leq M\right\}$$

for suitable positive M and compact $K \subseteq \mathbb{R}^n$. Then, there exists a positive constant C such that for all $g \in \mathrm{Hyp}(\Omega)$ with $\mathcal{D}^g \subseteq \mathcal{D}^f$ and for all $u \in \mathcal{D}^g$

$$\left\|S^f_t u - S^g_t u\right\|_{L^1} \leq C \cdot \left\|Df - Dg\right\|_{C^0(\Omega)} \cdot t \ . \tag{2.14}$$

The above is the counterpart of the well known estimate for solutions of bounded variation in the scalar case given in [9]. Note that, unlike the linear case, the "distance" $\hat{d}(f,g)$ is equivalent to $\|Df - Dg\|_{C^0}$ because the total variation of both solutions $S^f u$ and $S^g u$ is fixed by the domain $\mathcal{D}^f$: thus the case of the example of Remark 3.3 in [3] is not valid here.

For scalar equations, assuming that the flow functions f and g are strictly convex, we are able to extend the estimate in [9] to L^∞ initial data.

Theorem 2.6. *Assume that the scalar flow functions f and g are uniformly strictly convex in a compact interval K; i.e. $f''(u), g''(u) \geq \kappa > 0$ for all $u \in K$. Let u (resp. w) denote the solution to*

$$\begin{cases} u_t + [f(u)]_x = 0, \\ u(0,x) = u_o(x), \end{cases} \qquad \textit{resp.} \qquad \begin{cases} w_t + [g(w)]_x = 0, \\ w(0,x) = u_o(x), \end{cases}$$

with the same initial data $u_o \in L^\infty(\mathbb{R}, K)$. Denote by $\hat{\lambda}$ an upper bound for the characteristic speeds, i.e. $\hat{\lambda} \geq \max_{u\in K}\{|f'(u)|, |g'(u)|\}$. Then

(2.15)
$$\int_a^b \big|u(t,x) - w(t,x)\big|\, dx \leq 2 \cdot \mathrm{diam}(K) \cdot t \cdot \frac{b - a + 4\hat{\lambda}t}{\kappa t} \cdot \max_{u\in K}\big|f'(u) - g'(u)\big| \ .$$

3. Proof of the main results

This section is devoted to the proofs of Theorem 2.1, Proposition 2.2 and Corollary 2.5. The first step consists in the explicit computation of the vector field $v(u)$ generated by an SRS in the sense of (2.11). This procedure follows [5]. Observe that, given an $f \in \mathrm{Hyp}(\Omega)$, the construction is accomplished for all $u \in \mathcal{D}^f$.

Fix $f \in \mathrm{Hyp}(\Omega)$ and $u \in \mathcal{D}^f$. Du stands for the total variation of the weak derivative of u. Let $\hat{\lambda}$ be a constant strictly greater than all the characteristic speeds induced by f on $\mathcal{D}^f$. Let $\xi \in \mathbb{R}$ be given. We denote by ω the self-similar solution of the Riemann problem

$$\begin{cases} \omega_t + f(\omega)_x = 0, \\ \omega(0,x) = \begin{cases} u(\xi-) & \text{if } x < 0, \\ u(\xi+) & \text{if } x > 0, \end{cases} \end{cases}$$

and let $U^\sharp_\xi$ be the function

$$U^\sharp_\xi(\theta, x) \doteq \begin{cases} \omega(\theta, x - \xi) & \text{if } |x - \xi| \leq \hat{\lambda}\theta, \\ u(x) & \text{if } |x - \xi| > \hat{\lambda}\theta. \end{cases} \tag{3.1}$$

Moreover, define $U^\flat_\xi$ as the solution to the linear hyperbolic problem with constant coefficients

$$\begin{cases} \omega_\theta + Df(\xi)\omega_x = 0, \\ \omega(0,x) = u(x). \end{cases} \tag{3.2}$$

Given $\epsilon > 0$, by an ϵ–covering of the real line we mean a family

$$\mathcal{F} \doteq \{I_1, \ldots, I_N, I'_1, \ldots, I'_M\} \tag{3.3}$$

of open intervals which cover $\mathbb{R}$ such that:

1) the intervals I_α are mutually disjoint, and no point $x \in \mathbb{R}$ lies inside more than two distinct intervals I'_β;
2) for every α there exists $\xi_\alpha \in I_\alpha$ such that $Du(I_\alpha \setminus \{\xi_\alpha\}) < \epsilon/N$;
3) for every β, $Du(I'_\beta) < \epsilon$.

Now, let ϵ_n be a sequence strictly decreasing to 0. If $\mathcal{F}_n = \{I_1, \dots, I_{N(n)}, I'_1, \dots, I'_{M(n)}\}$ is an ϵ_n–covering of $\mathbb{R}$, then also $\mathcal{F}_{n,\theta} = \{I_{1,\theta}, \dots, I_{N(n),\theta}, I'_{1,\theta}, \dots, I'_{M(n),\theta}\}$ is an ϵ_n–covering, where $I_{j,\theta} \doteq I_j + \lambda(-\theta, \theta)$ and $\theta \in [0, \theta_n]$, for θ_n sufficiently small and such that the sequence θ_n strictly decreases to 0. Define

$$u_n^\theta(x) \doteq \begin{cases} U^\sharp_{\xi_\alpha}(\theta, x) & \text{if } x \in I_{\alpha,\theta}, \\ U^\flat_{\xi_\beta}(\theta, x) & \text{if } x \in I'_{\beta,\theta}, x \notin \bigcup_\alpha I_{\alpha,\theta}, x \notin \bigcup_{\beta'<\beta} I'_{\beta',\theta}. \end{cases}$$

We finally obtain the vector field v_{S^f}, letting $v_{S^f}(u)$ be the equivalence class (w.r.t. (2.9)) of the curve $\theta \mapsto u^\theta$, where

$$u^\theta \doteq s u_n^{\theta_n} + (1-s) u_{n-1}^{\theta_{n-1}} \quad \text{if} \quad \theta = s\theta_n + (1-s)\theta_{n-1},\ s \in [0,1]\ . \tag{3.4}$$

In [5] it is shown that the trajectories of the SRS S^f generated by the hyperbolic system of conservation laws $u_t + f(u)_x = 0$ are the solution of the quasilinear equation $\dot{u} = v_{S^f}(u)$, where the vector field $v_{S^f}(u)$ is generated by the curve (3.4).

Let f, g be as in Theorem 2.1, and denote by u^θ, w^θ the two curves generating the vector fields v_{S^f} and v_{S^g} induced by the SRSs S^f and S^g. In view of (2.12), we can now compute the distance (2.10) between $v_{S^f}(u)$ and $v_{S^g}(u)$. A simple computation shows that by Proposition 2.2

$$\begin{aligned} \frac{d(u^{\theta_n}, w^{\theta_n})}{\theta_n} &= \frac{1}{\theta_n} \cdot \left\| w^{\theta_n} - u^{\theta_n} \right\|_{L^1} \\ &\le \sum_\alpha Du(\xi_\alpha) \cdot \hat{d}_{\text{lin}}(f,g) + \sum_\beta \text{Tot. Var.}(u, I'_\beta) \cdot \hat{d}\left(Df(\xi_\beta), Dg(\xi_\beta)\right) \\ &\le \text{Tot. Var.}(u) \cdot \hat{d}(f,g)\ . \end{aligned} \tag{3.5}$$

As a consequence, if s is as in (3.4), then

$$\begin{aligned} \frac{d(u^\theta, w^\theta)}{\theta} &= \frac{\left\| s \cdot (u_n^{\theta_n} - w_n^{\theta_n}) + (1-s) \cdot (u_{n-1}^{\theta_{n-1}} - w_{n-1}^{\theta_{n-1}}) \right\|_{L^1}}{\theta} \\ &\le \frac{s \cdot \theta_n + (1-s) \cdot \theta_{n-1}}{\theta} \cdot \text{Tot. Var.}(u) \cdot \hat{d}(f,g) \\ &= \text{Tot. Var.}(u) \cdot \hat{d}(f,g); \end{aligned} \tag{3.6}$$

hence

$$\hat{d}\left(v_{S^f}(u), v_{S^g}(u)\right) = \limsup_{n \to +\infty} \frac{d(u^\theta, w^\theta)}{\theta} \le \text{Tot. Var.}(u) \cdot \hat{d}(f,g)\ . \tag{3.7}$$

By Proposition 2.3, applying (3.7) to $S_t^g u$, the proof of Theorem 2.1 is completed.

Next we give a proof of Corollary 2.5. We only need to prove that there exists a constant C such that $\hat{d}(f,g) \le C \cdot \|Df - Dg\|_{C^0}$: this means that for all Riemann problems (2.2) we have

$$\left\| S_1^f u - S_1^g u \right\|_{L^1} \le C \cdot \left\| Df - Dg \right\|_{C^0} \cdot \left| u^+ - u^- \right|. \tag{3.8}$$

We recall that $S_t^g u$ is a self-similar solution, obtained by piecing together centered rarefaction waves and jump discontinuities. By the L^1_{loc} dependence of S^f, formula (3.8) will be proved if we can verify it for Riemann problems generating a single wave in the solution $S_t^g u$. We then have to consider two cases.

If $S_t^g u$ is a centered rarefaction wave, then by the Lipschitz continuity of the solution for $t > 0$, the functions $S_{t+h}^g u$ and $S_h^f \circ S_t^g u$ solve in the broad sense the quasilinear versions of (2.1):

$$\left[S_h^f \circ S_t^g u\right]_h + Df\left[S_h^f \circ S_t^g u\right]_x = 0 \quad \text{and} \quad \left[S_{t+h}^g u\right]_h + Dg\left[S_{t+h}^g u\right]_x = 0. \tag{3.9}$$

Applying Lemma 2.4, we obtain

$$\left\|S_1^f u - S_1^g u\right\|_{L^1} \le L_f \cdot \int_0^1 \liminf_{h\to 0} \frac{\left\|S_h^f \circ S_t^g u - S_{t+h}^g u\right\|_{L^1}}{h} dt \tag{3.10}$$

$$\begin{aligned} &= L_f \cdot \int_0^1 \left\|Df\left[S_t^g u\right]_x - Dg\left[S_t^g u\right]_x\right\|_{L^1} \\ &\le C \cdot \left\|Df - Dg\right\|_{C^0} \cdot |u^+ - u^-|, \end{aligned} \tag{3.11}$$

since in this case the L^1 limits as $h \to 0$ of $(S_h^f \circ S_t^g u)/h$ and $(S_{t+h}^g u)/h$ exist by (3.9) and are equal to $(Df - Dg)[S_t^g u]_x$.

Now we consider the case in which the jump u^-, u^+ is solved by a shock travelling with speed σ, where σ is given by the Rankine–Hugoniot condition

$$g(u^+) - g(u^-) = \sigma \cdot \left(u^+ - u^-\right). \tag{3.12}$$

To prove (3.8), we approximate the solution $S_t^g u$ by the Lipschitz continuous function $\tilde{u}(t)$, defined as

$$\tilde{u}(t,x) = u^- \cdot \chi_{(-\infty,0]}\left(x - \sigma t\right) + \left(u^+ - u^-\right) \cdot \min\left(\frac{x - \sigma t}{\delta}, 1\right) \cdot \chi_{(0,+\infty)}\left(x - \sigma t\right).$$

Roughly speaking, $\tilde{u}$ is obtained from $S_t^g u$ by replacing the jump u^-, u^+ at $x - \sigma t$ by a linear function. Since S^g is Lipschitz, we can write

$$\begin{aligned} \left\|S_h^f \circ S_t^g u - S_{t+h}^g u\right\|_{L^1} &\le \left\|S_h^f \circ S_t^g u - S_h^f \tilde{u}(t)\right\|_{L^1} + \left\|S_h^f \tilde{u}(t) - \tilde{u}(t+h)\right\|_{L^1} \\ &\quad + \left\|\tilde{u}(t+h) - S_{t+h}^g u\right\|_{L^1} \\ &\le \frac{\delta}{2} \cdot (1 + L^f) \cdot \left|u^+ - u^-\right| + \left\|\tilde{u}(t+h) - S_h^f \tilde{u}(t)\right\|_{L^1}. \end{aligned} \tag{3.13}$$

The last term above can be evaluated again by means of Lemma 2.4:

$$\begin{aligned} \left\|\tilde{u}(t+h) - S_h^f \tilde{u}(t)\right\|_{L^1} &\le L^f \cdot \int_t^{t+h} \liminf_{\xi\to 0^+} \frac{\left\|\tilde{u}(\eta+\xi) - S_\xi^f \tilde{u}(\eta)\right\|_{L^1}}{\xi} d\eta \\ &= h \cdot L^f \cdot \int_0^\delta \left|\left(\dot{\sigma}_\alpha \mathbb{I} - Dg\left(u_\alpha^- + (u_\alpha^+ - u_\alpha^-)\frac{y}{\delta}\right)\right)\left(\frac{u_\alpha^- + (u_\alpha^+ - u_\alpha^-)}{y\delta}\right)\right| dy \\ &\le h \cdot L^f \cdot \sup_{x\in K}\left\|Df(x) - Dg(x)\right\| \cdot \left|u_\alpha^+ - u_\alpha^-\right|, \end{aligned} \tag{3.14}$$

where we use the fact that $\dot{x}_\alpha \in [\sigma - \epsilon, \sigma + \epsilon]$ and the relation

$$\sigma \cdot (u_\alpha^+ - u_\alpha^-) = f(u_\alpha^+) - f(u_\alpha^-) = \int_0^1 Df\left((1-s)u_\alpha^- + s u_\alpha^+\right) ds \cdot (u_\alpha^+ - u_\alpha^-).$$

In fact, since $u_{\alpha,\delta}$ is Lipschitz continuous, we can use Lebesgue's dominated convergence theorem as in the previous case. Letting δ tend to 0, we obtain, finally,

$$\frac{1}{h}\left\|S_h^f \circ S_t^g u - S_{t+h}^g\right\|_{L^1} \leq L_f \cdot \sup_{x\in K} \|Df(x) - Dg(x)\| \cdot \left|u_\alpha^+ - u_\alpha^-\right|. \tag{3.15}$$

This concludes the proof of Corollary 2.5: in fact, an application of Lemma 2.4 immediately gives (3.8).

To end this section, we prove Proposition 2.2. Fix $u_o \in \Omega$ and $v \in \mathbb{R}^n$ with $|v| = 1$. Assume that for all positive and sufficiently small h the function $u_h = u_o + h \cdot \chi_{[0,+\infty)} \cdot v$ is in $\mathcal{D}^g$. Let

$$\tilde{f}_h(u) = \frac{1}{h} \cdot \big(f(u) - f(u_o)\big)\ , \qquad \tilde{g}_h(u) = \frac{1}{h} \cdot \big(g(u) - g(u_o)\big),$$

and note that, using a simple rescaling, we can write

$$\left\|S_1^{\tilde{f}_h}\left(\frac{1}{h}(u_h - u_o)\right) - S_1^{\tilde{g}_h}\left(\frac{1}{h}(u_h - u_o)\right)\right\|_{L^1} = \frac{1}{h} \cdot \left\|S_1^f u_h - S_1^g u_h\right\|_{L^1}$$

(recall that $(1/h) \cdot (u_h - u_o) = \chi_{[0,+\infty)} \cdot v$ is independent of h). Hence, passing to the limit as $h \to 0$, we get

$$\begin{aligned} \left\|Df(u_o)^1 \star v - Dg(u_o)^1 \star v\right\|_{L^1} &= \lim_{h\to 0} \frac{1}{h} \cdot \left\|S_1^f u_h - S_1^g u_h\right\|_{L^1}, \\ \hat{d}_{\text{lin}}\left(Df(u_o), Dg(u_o)\right) &= \sup_v \lim_{h\to 0} \frac{1}{h} \cdot \left\|S_1^f u_h - S_1^g u_h\right\|_{L^1}, \end{aligned} \tag{3.16}$$

and the proof is completed.

4. The classical limit of the relativistic Euler equations

In this section we apply Corollary 2.5 to the classical limit of the relativistic Euler equations, generalizing what was obtained in [14].

The relativistic p–system (see [16, 17]) is

$$\begin{cases} \left[\rho + \left(\rho + \dfrac{1}{c^2}p\right)\dfrac{(v/c)^2}{1-(v/c)^2}\right]_t + \left[\left(\rho + \dfrac{1}{c^2}p\right)\dfrac{v}{1-(v/c)^2}\right]_x = 0, \\ \left[\left(\rho + \dfrac{1}{c^2}p\right)\dfrac{v}{1-(v/c)^2}\right]_t + \left[\left(\rho + \dfrac{1}{c^2}p\right)\dfrac{v^2}{1-(v/c)^2} + p\right]_x = 0. \end{cases} \tag{4.1}$$

Here, ρ is the mass–energy density of the fluid, v the classical coordinate velocity, p the pressure and c the speed of light. We show below that as $c \to +\infty$, the problem (4.1) approaches its classical counterpart

$$\begin{cases} [\rho]_t + [\rho v]_x = 0, \\ [\rho v]_t + \left[\rho v^2 + p\right] = 0, \end{cases} \tag{4.2}$$

in the sense that the SRS S^c generated by (4.1) converges to the SRS S generated by (4.2) on a domain containing all physically reasonable data. In particular, the total variation of the data need not be small.

In (4.1) a standard choice (see [16] and the references therein) for the pressure law is

$$p = \sigma^2 \cdot \rho\ , \tag{4.3}$$

σ being the speed of sound.

Fix a positive lower bound for the density $\rho_{\min}$ and for the light speed c_o. Without any loss in generality, we may assume $\sigma < c_o$.

Let $\mathcal{V}_M = \{(\rho, \rho v) \in \mathrm{BV}\,(\mathbb{R}, (\rho_{\min}, +\infty) \times \mathbb{R}) : \text{ Tot. Var.}(\rho) + \text{Tot. Var.}(\rho v) \leq M\}$. In [10] it is proved that for any $M > 0$, (4.1) generates an SRS $S^{c,M}$ defined on a domain $\mathcal{D}^{c,M}$ containing $\mathcal{V}_M$ and consisting of functions of total variation bounded by, say, M^3 (provided M is sufficiently large). Similarly, (4.2) generates an SRS S^M on a domain $\mathcal{D}^M$ containing $\mathcal{V}_{M^3}$ and contained, say, in $\mathcal{V}_{M^9}$.

Thus, for all sufficiently large c and M, there exist domains $\mathcal{D}^{c,M}$, $\mathcal{D}^M$ such that

$$\mathcal{V}_M \subset \mathcal{D}^{c,M} \subset \mathcal{V}_{M^3} \subset \mathcal{D}^M \subset \mathcal{V}_{M^9} \tag{4.4}$$

and, moreover, the problems (4.1) and (4.2) generate the SRSs

$$S^{c,M} : \mathcal{D}^{c,M} \times (0, +\infty) \to \mathcal{D}^{c,M} \quad \text{and} \quad S^M : \mathcal{D}^M \times (0, +\infty) \to \mathcal{D}^M \ .$$

We are now ready to state and prove the following application of Corollary 2.5.

Theorem 4.1. *Fix a positive $\rho_{\min}$ and sufficiently large M, c_o. Let $\mathcal{D}^{c,M}$ and $\mathcal{D}^M$ satisfy* (4.4) *for $c > c_o$. Then there exists a constant C such that for all $c \geq c_0$ and for all $u \in \mathcal{D}^{M,c}$*

$$\left\| S_t^{c,M} u - S_t^M u \right\|_{L^1} \leq C \cdot \frac{1}{c^2} \cdot t. \tag{4.5}$$

In particular, by (4.4), the bound (4.5) holds for all initial data u with Tot. Var.(u) $< M$.

Proof. Note that (4.1) and (4.2) in conservation form become, respectively,

$$\left\{ \begin{array}{c} \rho_t + q_x = 0, \\ q_t + \left(\phi_c(\rho, q) \cdot \dfrac{q^2}{\rho} + p \right)_x = 0, \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{c} \rho_t + q_x = 0, \\ q_t + \left(\dfrac{q^2}{\rho} + p \right)_x = 0, \end{array} \right. \tag{4.6}$$

where

$$\phi_c(\rho, q) = 1 + \frac{1}{c^2} \cdot \left(1 - \frac{v^2(\rho, q)}{c^2} \right) \cdot \frac{p}{\rho + \dfrac{v^2(\rho, q)}{c^2} \cdot \dfrac{p}{c^2}} \ .$$

Call f_c and, respectively, f the fluxes in the two systems (4.6). Then, the estimate $\|Df_c - Df\|_{C^0} \leq C \cdot (1/c^2)$ follows from straightforward computations and completes the proof. □

We remark that the particular pressure law (4.3) is necessary only to ensure the existence of the SRS $S^{c,M}$ in the large. The above procedure remains true under much milder assumptions on the equation of state. Note, moreover, that the rate of convergence $\mathcal{O}(1/c^2)$ is exactly the rate expected by the convergence of relativistic to classical mechanics.

It is of interest that (4.5) also proves the uniform Lipschitz continuity of all semigroups $S^{c,M}$ for all sufficiently large c.

Remark 4.2. As is well known, linearizing (4.2) around $\rho = \rho_o$, $v = 0$ at constant entropy leads to the wave equation. Corollary 2.5 ensures that the solutions to (4.2) converge to the linearized equation in L^1 over finite time intervals.

5. Stability of a scalar equation w.r.t. flux

In this section we prove Theorem 2.6. We consider two scalar equations,

$$u_t + f(u)_x = 0, \tag{5.1}$$

$$v_t + g(u)_x = 0, \tag{5.2}$$

with the same initial condition: $u(0,x) = v(0,x) = u_0$. We assume that f, g are strictly convex C^2 functions: precisely, there exists a constant κ such that

$$\min_{u\in K}\{|f''(u)|, |g''(u)|\} \geq \kappa\,, \tag{5.3}$$

where K is a compact interval of $\mathbb{R}$ such that $u_0(x) \in K$ for all $x \in \mathbb{R}$. We recall that by the maximum principle also the entropic solutions of (5.1), (5.2) will satisfy the same bound.

We recall that, by [11], given a point (t,x), we can consider the set of characteristics $\xi(t,x)$ passing through (t,x). If we denote by $\xi^-(t,x)$ and $\xi^+(t,x)$ the minimal and maximal backward characteristics, then either $\xi^-(t,x) = \xi^+(t,x)$ and the solution u is continuous in (t,x), or we have an admissible shock and the jump is exactly given by the (constant) values of u on the characteristics ξ^-, ξ^+. By condition (5.3) we have that if at time t two characteristics $\xi^+(t)$ and $\xi^-(t)$ meet, then

$$\frac{d}{dt}(\xi^+ - \xi^-) \leq -\kappa(u^+ - u^-)\,. \tag{5.4}$$

Suppose that $\xi^-(0) < \xi^+(0)$, and consider now an initial datum $\tilde{u}_0$ defined as

$$\widetilde{u}_0(x) = \begin{cases} u_0(x), & x \leq \xi^-(0), \\ u_0^-, & \xi^-(0) < x \leq \Xi, \\ u_0^+, & \Xi < x \leq \xi^+(0), \\ u_0(x), & x > \xi^+(0), \end{cases} \tag{5.5}$$

where $\Xi \in \mathbb{R}$ is chosen such that

$$\Xi \doteq \frac{1}{u_0^- - u_0^+}\int_{\xi^-}^{\xi^+} u_0(x)dx + \frac{u_0^-\xi^- - u_0^+\xi^+}{u_0^- - u_0^+}\,. \tag{5.6}$$

Using the conservation of mass, it is easy to conclude that the solution in unchanged at time t. In fact, consider the triangle $\mathcal{T}$ whose vertices are $(0,\xi^-(0))$, $(0,\xi^+(0))$ and (t,x). Since the equation (5.1) can be written as

$$\operatorname{div}\begin{pmatrix} u \\ f(u) \end{pmatrix} = 0\,,$$

and u is constant along the lines $\xi(t)$, we have

$$\begin{aligned}
&\iint_{\mathcal{T}} (u_t + f(u)_x)dtdx \\
&\quad = -\int_{\xi^-}^{\xi^+} u_0(x)dx + \int_0^t (f(u^+) - f'(u^+)u^+)dt + \int_0^t (f'(u^-)u^- - f(u^-))dt \\
&\quad = -\int_{\xi^-}^{\xi^+} u_0(x)dx + (f'(u^-)u^- - f'(u^+)u^+ + f(u^+) - f(u^-))t = 0.
\end{aligned}$$

Using the relation

$$x = \xi^- + f'(u^-)t = \Xi + \frac{f(u^+) - f(u^-)}{u^+ - u^-}t = \xi^+ + f'(u^+)t\,,$$

we obtain (5.6). Using (5.6), we can change the initial of (5.1) or (5.2) so that $u(t)$ or $v(t)$ is unchanged. Since with this procedure we collect all the interactions at time 0, the total variation of $\tilde{u}_0$ has the same value of $\mathrm{Tot.\,Var.}(u(t))$.

To change the initial datum in such a way that both $u(t)$ and $v(t)$ are the same, consider now the test system

$$w_t + \left(\frac{\kappa}{2}w^2\right)_x = 0\,. \tag{5.7}$$

By (5.3) and (5.4), if two characteristics meet at $u(t)$ and $v(t)$, then they also meet in (5.7). Let us denote by $\tilde{u}_0$ the new initial condition, obtained by the above procedure using equation (5.7).

Consider now an interval $[a, b]$. By the definition of $\hat{\lambda}$ we have that the values of $u(t)$ and $v(t)$ in $[a, b]$ depend only on $\tilde{u}_0$ in $[a-\hat{\lambda}t, b+\hat{\lambda}t]$. Using standard estimates, we have

$$\begin{aligned}\int_a^b |u(t,x) - v(t,x)|dx \\ \leq t \cdot \max_{u\in K} |f'(u) - g'(u)| \cdot \mathrm{Tot.\,Var.}\Big(\widetilde{u}_0; [a - \widehat{\lambda}t, b + \widehat{\lambda}t]\Big).\end{aligned} \tag{5.8}$$

To estimate the total variation, an easy computation gives

$$\begin{aligned}\mathrm{Tot.\,Var.}\Big(\widetilde{u}_0; [a - \widehat{\lambda}t, b + \widehat{\lambda}t]\Big) &\leq \mathrm{Tot.\,Var.}\Big(w(t); [a - 2\widehat{\lambda}t, b + 2\widehat{\lambda}t]\Big) \\ &\leq 2 \cdot \mathrm{diam}(K) \cdot \frac{b - a + 4\widehat{\lambda}t}{\kappa t}\,.\end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9), we get

$$\begin{aligned}\int_a^b |u(t,x) - v(t,x)|dx \\ \leq \max_{u\in K}\big|f'(u) - g'(u)\big| \cdot 2 \cdot \mathrm{diam}(K) \cdot \frac{b - a + 4\widehat{\lambda}t}{\kappa}.\end{aligned} \tag{5.10}$$

Remark 5.1. When $t \to 0$ the integral does not converge to 0. This is clear since the initial datum is in L^∞, and then the semigroup is continuous but not Lipschitz continuous in time, since the amount of interaction at $t = 0$ is infinite. Consider for example the two equations

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

and

$$v_t + \left(-v + \frac{v^2}{2}\right)_x = 0\ ,$$

with the periodic initial condition

$$u_{0,n}(x) = \begin{cases} 1 & \text{if } x \in \big[k2^{-n+1}, k2^{-n+1} + 2^{-n}\big], k \in \mathbb{Z}, \\ -1 & \text{otherwise.}\end{cases}$$

At time $t_n = 2^{-n}$ the two solutions are

$$\begin{cases} u_n(t_n, x) = 2^n(x - k2^{-n+1}) & \text{if } x \in \big(k2^{-n+1} - 2^{-n}, k2^{-n+1} + 2^{-n}\big], \\ v_n(t_n, x) = 2^n(x - k2^{-n+1} - 2^{-n}) & \text{if } x \in \big(k2^{-n+1}, (k+1)2^{-n+1}\big], \end{cases}$$

so that

$$\int_0^1 \big|v_n(t_n, x) - u_n(t_n, x)\big|\, dx = 1 \ .$$

This depends on the fact that the modulus of continuity of the semigroup can be arbitrarily large.

Note, moreover, that since the solutions u, v are limits of wave front tracking approximations, the continuous dependence of the solution on the flux function f can be stated also if f non-convex. However, in general one cannot prove any uniform continuous dependence.

Acknowledgments

The present work was partly accomplished while the authors visited the Max Planck Institute in Leipzig, supported by the European TMR Network *"Hyperbolic Conservation Laws"* ERBMRXCT960033.

References

[1] F. Ancona, A. Marson, *Well posedness for general 2×2 systems of conservation laws*, preprint.

[2] P. Baiti, A. Bressan, *The semigroup generated by a Temple class system with large data*, Differential Integral Equations **10** (1997), no. 3, p. 401–418. MR **2000k:**35180

[3] S. Bianchini, *On the shift differentiability of the flow generated by a hyperbolic system of conservation laws*, Discr. Cont. Dyn. Sys. **6-2** (2000), pag. 329-350. MR **2000m:**35120

[4] A. Bressan, *The unique limit of the Glimm scheme*, Arch. Rat. Mech. Anal. **130** (1995), 205-230. MR **96d:**65143

[5] A. Bressan, *On the Cauchy problem for nonlinear hyperbolic systems*, preprint S.I.S.S.A., 1998.

[6] A. Bressan, "Hyperbolic Systems of Conservation Laws: the One Dimensional Cauchy Problem", Oxford Univ. Press (2000).

[7] A. Bressan, R.M. Colombo, *The semigroup generated by 2×2 conservation laws*, Arch. Ration. Mech. Anal. 133, No.1, 1-75 (1995). MR **96m:**35198

[8] A. Bressan, G. Crasta, B. Piccoli, *Well posedness of the Cauchy problem for $n \times n$ systems of conservation laws*, Mem. Amer. Math. Soc. **694** (2000). MR **2000m:**35122

[9] F. Bouchut, B. Perthame, *Kruzkov's estimates for scalar conservation laws revisited*, Trans. Amer. Math. Soc. 350, 7 (1998), 2847–2870. MR **98m:**65156

[10] R.M. Colombo, N.H. Risebro, *Continuous Dependence in the Large for some Equations of Gas Dynamics*, Comm. in P.D.E., **23l**, 9&10 (1998), 1693–1718. MR **99h:**35124

[11] C.M. Dafermos, *Generalized characteristics in hyperbolic systems of conservation laws*, Arch. Rational Mech. Anal., **107**, 2 (1989), 127–155. MR **90h:**35150

[12] P.D. Lax, *Hyperbolic systems of conservation laws, II*, Comm. Pure Appl. Math., **10** (1957), 537–566. MR **20:**176

[13] T.P. Liu, *The Riemann problem for general systems of conservation laws*, J. Differential Equations, **18** (1975), 218–234. MR **51:**6168

[14] L. Min, S. Ukai, *Non–relativistic global limit of weak solutions of the relativistic Euler equations*, J. Math. Kyoto Univ., **38–3** (1998), 525–537. MR **99i:**76134

[15] A.I. Panasyuk, *Quasidifferential equations in a metric space*, Differentsialnye Uravneniya, **21** (1985), no 8. MR **87b:**34078

[16] J. Smoller, B. Temple, *Global Solutions of the Relativistic Euler Equations*, Comm. Math. Phys. **156** (1993), 67–99. MR **94h:**35216

[17] A.H. Taub, *Relativistic Rankine–Hugoniot Equations*, Physical Review **74** (1948), 328–334. MR **10:**72f

Istituto per le Applicazioni del Calcolo, Viale del Policlinico 137, 00161 Roma, Italy

E-mail address: bianchin@iac.rm.cnr.it

URL: http:/www.iac.rm.cnr.it/~bianchin/

Department of Mathematics, University of Brescia, Via Valotti 9, 25133 Brescia, Italy

E-mail address: rinaldo@ing.unibs.it

URL: http://bsing.ing.unibs.it/~rinaldo/