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AN ANALYTICITY CRITERION FOR REGULARIZED SEMIGROUPS

QUAN ZHENG AND YONGSHENG ZHAO

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Abstract. A generalization of Kato's analyticity criterion for C_0-semigroups to exponentially bounded regularized semigroups is given by using the method of Laplace transforms.

The motivation for this note is that Liu [4] and Kantorovitz [2] proved an analyticity criterion for semigroups and contraction semigroups, respectively. In fact, Liu's result was known earlier by Kato [3, p. 492]. On the other hand, regularized semigroups have received much attention since 1987 (see, e.g. [1] and the references therein). In this note, we will generalize Kato's result to exponentially bounded regularized semigroups.

In Kato [3], the analyticity criterion for semigroups was derived from the resolvent growth characterization of generators of analytic semigroups. Other proofs were given by Liu [4] and Kantorovitz [2]. The latter, for example, is based on an exponential formula of semigroups and the use of normal families, while the present proof is based on a characterization of Laplace transforms of abstract analytic functions with growth restrictions [5].

Let $B(X)$ be the set of all bounded linear operators from a Banach space X into itself. By $\mathcal{D}(A)$ and $\mathcal{R}(A)$ we denote the domain and range of a linear operator A, respectively. For an injective operator C in $B(X)$, we denote by $\rho_C(A) := \{\lambda \in \mathbf{C} : \lambda - A \text{ is injective and } \mathcal{R}(C) \subset \mathcal{R}(\lambda - A)\}$ its C-resolvent set and by $R_C(\lambda, A) := (\lambda - A)^{-1}C$ $(\lambda \in \rho_C(A))$ its C-resolvent. Set

$$\Delta_\alpha := \{\lambda \in \mathbf{C} : |\arg \lambda| < \alpha\} \setminus \{0\}$$

and

$$\Delta'_\alpha = \{\lambda \in \mathbf{C} : |\arg \lambda| \le \alpha\},$$

where $0 < \alpha \le \pi$. Moreover, M_β denotes a constant that depends only on β.

Definition 1. Let C be an injective operator in $B(X)$. A strongly continuous family $T : [0, \infty) \to B(X)$ is called an (exponentially bounded) C-regularized semigroup if $T(0) = C$, $T(t+s)C = T(t)T(s)$ $(t, s \ge 0)$, and $\|T(t)\| \le Me^{\omega t}$

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$(t \geq 0)$ for some constants $M \geq 0, \omega \in \mathbf{R}$. Its generator, A, is defined by

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{t \downarrow 0} (T(t)x - Cx)/t \text{ exists and is in } \mathcal{R}(C) \right\},$$
$$Ax = C^{-1} \lim_{t \downarrow 0} (T(t)x - Cx)/t \text{ for } x \in \mathcal{D}(A).$$

We refer to [1, 6] for the following lemma, which will be used to conclude that A generates a C-regularized semigroup.

Lemma 1. *A linear operator A is the generator of a C-regularized semigroup if and only if $A = C^{-1}AC$, $(\omega, \infty) \subset \rho_C(A)$, and there exists a strongly continuous family $\{T(t)\}_{t \geq 0}$ with $\|T(t)\| \leq Me^{\omega t}$ $(t \geq 0)$ for some constants $M \geq 0, \omega \in \mathbf{R}$ such that*

$$R_C(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)x \, dt \quad \textit{for } \lambda > \omega, \ x \in X.$$

Definition 2. A C-regularized semigroup $\{T(t)\}_{t \geq 0}$ is called an analytic C-regularized semigroup if

(a) $t \mapsto T(t)$ can be extended analytically to Δ_α for some $\alpha \in (0, \pi/2]$;
(b) for every $\beta \in (0, \alpha)$, there exist constants $M_\beta \geq 0$, $\omega \in \mathbf{R}$ such that $\|T(t)\| \leq M_\beta e^{\omega \operatorname{Re} t}$ for $t \in \Delta_\beta$;
(c) $t \mapsto T(t)$ is strongly continuous in Δ'_β for every $\beta \in (0, \alpha)$.

In this case, we write $(A, T(\cdot)) \in H_C(\omega, \alpha)$, where A is the generator of $\{T(t)\}_{t \geq 0}$.

The following lemma can be found in [7], which is a modification of Neubrander's result [5].

Lemma 2. *Let $\omega \in \mathbf{R}, \alpha \in (0, \pi/2]$ and $F : (\omega, \infty) \to X$. Then the following statements are equivalent*:

(a) *F is analytic in $\omega + \Delta_{\alpha + \pi/2}$, and $\|(\lambda - \omega)F(\lambda)\| \leq M_\beta$ $(\lambda \in \Delta_{\beta + \pi/2})$ for every $\beta \in (0, \alpha)$.*
(b) *There exists an analytic function $h : \Delta_\alpha \to X$ with $\|h(t)\| \leq M_\beta e^{\omega \operatorname{Re} t}$ $(t \in \Delta_\beta)$ for every $\beta \in (0, \alpha)$ such that*

$$F(\lambda) = \int_0^\infty e^{-\lambda t} h(t) \, dt \quad \textit{for } \lambda > \omega.$$

The main result of this note is the following.

Theorem. *Let $\omega \in \mathbf{R}$ and $\alpha \in (0, \pi/2]$. Then $(A, T(\cdot)) \in H_C(\omega, \alpha)$ if and only if the following statements hold*:

(a) *For every $\theta \in (-\alpha, \alpha)$, $e^{i\theta}A$ generates a C-regularized semigroup $\{T_\theta(t)\}_{t \geq 0}$.*
(b) *For every $\beta \in (0, \alpha)$, there exists a constant $M_\beta \geq 0$ such that $\|T_\theta(t)\| \leq M_\beta e^{\omega t \cos\theta}$ for $t \geq 0$ and $|\theta| \leq \beta$.*
(c) *For every $\beta \in (0, \alpha), x \in X$, $\lim_{t \downarrow 0} \sup_{|\theta| \leq \beta} \|T_\theta(t)x - Cx\| = 0$.*

In the case $\overline{\mathcal{D}(A)} = X$, $(A, T(\cdot)) \in H_C(\omega, \alpha)$ if and only if conditions (a) *and* (b) *are satisfied.*

Proof. We assume without loss of generality that $\omega = 0$. Otherwise we will replace $(A, T(t)) \in H_C(\omega, \alpha)$ by $(A_1, T_1(t)) \in H_C(0, \alpha)$, where $A_1 = A - \omega$ and $T_1(t) = e^{-\omega t}T(t)$ $(t \in \Delta_\alpha)$.

"$\Rightarrow$" For every $\theta \in (-\alpha,\alpha)$, let $T_\theta(t) = T(e^{i\theta}t)$ $(t \geq 0)$, then $\{T_\theta(t)\}_{t\geq 0}$ is a C-regularized semigroup and satisfies (b) and (c). It remains to show that $e^{i\theta}A$ is the generator of $\{T_\theta(t)\}_{t\geq 0}$. By Lemma 1 and the properties of Laplace transforms we have that $\Delta_{\pi/2} \subset \rho_C(A)$ and

$$R_C(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt \quad \text{for } \operatorname{Re}\lambda > 0,\ x \in X.$$

In particular, for $\lambda > 0$, we have $\lambda e^{-i\theta} \in \rho_C(A)$ and

$$R_C(\lambda e^{-i\theta}, A)x = \int_0^\infty \exp(-\lambda e^{-i\theta} t) T(t)x\,dt \quad \text{for } x \in X.$$

It therefore follows from Definition 2 that $(0,\infty) \subset \rho_C(e^{i\theta}A)$ and

$$(*) \qquad \begin{aligned} R_C(\lambda, e^{i\theta}A)x &= \int_{\Gamma_\theta} e^{-\lambda z} T(e^{i\theta}z)x\,dz \\ &= \int_0^\infty e^{-\lambda t} T_\theta(t)x\,dt \quad \text{for } \lambda > 0,\ x \in X, \end{aligned}$$

where $\Gamma_\theta = \{te^{-i\theta} : t \geq 0\}$. Also, by Lemma 1 we conclude that $e^{i\theta}A$ is the generator of $\{T_\theta(t)\}_{t\geq 0}$.

"$\Leftarrow$" For every $\theta \in (-\alpha,\alpha)$, by Lemma 1 we have that $\Delta_{\pi/2} \subset \rho_C(e^{i\theta}A)$ and

$$(**) \qquad R_C(\lambda, e^{i\theta}A)x = \int_0^\infty e^{-\lambda t} T_\theta(t)x\,dt \quad \text{for } \operatorname{Re}\lambda > 0,\ x \in X.$$

Consequently

$$\rho_C(A) \supset \bigcup_{|\theta|<\alpha} \{\lambda \in \mathbf{C} \setminus \{0\} : -\theta - \pi/2 < \arg\lambda < -\theta + \pi/2\} = \Delta_{\alpha+\pi/2}$$

and, by $(**)$, $R_C(\cdot, A) : \Delta_{\alpha+\pi/2} \to B(X)$ is analytic. Also, for $\lambda \in \Delta_{\beta+\pi/2}$ $(0 < \beta < \alpha)$, we can choose $|\theta| < \beta$ such that $e^{i\theta}\lambda \in \Delta_{\pi/2}$ and thus, by $(**)$ and (b),

$$\|R_C(\lambda, A)\| = \|R_C(e^{i\theta}\lambda, e^{i\theta}A)\| \leq M_\beta/|\lambda|.$$

Now, by Lemma 2, there exists an analytic function $T : \Delta_\alpha \to B(X)$ with $\|T(t)\| \leq M_\beta$ $(t \in \Delta_\beta)$ such that

$$R_C(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)\,dt \quad \text{for } \operatorname{Re}\lambda > 0.$$

Similarly to the proof of $(*)$, we obtain that

$$(***) \qquad R_C(\lambda, e^{i\theta}A)x = \int_0^\infty e^{-\lambda t} T(te^{i\theta})x\,dt \quad \text{for } \lambda > 0.$$

Combining $(**)$, $(***)$ and the uniqueness of Laplace transforms we find that $T(te^{i\theta}) = T_\theta(t)$ for $t \geq 0$ and $|\theta| < \alpha$, and therefore $(A, T(\cdot)) \in H_C(\omega,\alpha)$ follows from conditions (a)–(c).

Finally, if $\overline{\mathcal{D}(A)} = X$, then we only need to show that statements (a) and (b) imply statement (c). In fact, from the proof of the implication "$\Leftarrow$" and a property of regularized semigroups [1, Theorem 3.4(d)] we deduce that

$$\begin{aligned}\lim_{\Delta'_\beta \ni t \to 0} \|T(t)x - Cx\| &= \lim_{\Delta'_\beta \ni t \to 0} \left\| \int_0^{|t|} T_\theta(s)(e^{i\theta}A)x\, ds \right\| \\ &\leq M_\beta \lim_{\Delta'_\beta \ni t \to 0} |t|\, \|Ax\| \\ &= 0 \quad \text{for } x \in \mathcal{D}(A),\end{aligned}$$

where $\theta = \arg t$. Since $\overline{\mathcal{D}(A)} = X$, statement (c) follows now from statement (b). □

When $\overline{\mathcal{D}(A)} = X$, from Lemma 2 and the proof of the Theorem we have the following Corollary, in which the equivalence of statements (a) and (c) is due to [7, Corollary 3].

Corollary. *Let $\overline{\mathcal{D}(A)} = X, \omega \in \mathbf{R}$ and $\alpha \in (0, \pi/2]$. Then the following statements are equivalent*:

(a) $(A, T(\cdot)) \in H_C(\alpha, \omega)$.

(b) $(\omega, \infty) \subset \rho_C(A), A = C^{-1}AC$, *and there exists an analytic function* $T(\cdot) : \Delta_\alpha \to B(X)$ *such that* $\|T(t)\| \leq M_\beta e^{\omega \operatorname{Re} t}$ $(t \in \Delta_\beta)$ *for every* $\beta \in (0, \alpha)$, *and* $R_C(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)\, dt$ *for* $\lambda > \omega$.

(c) $\omega + \Delta_{\alpha + \pi/2} \subset \rho_C(A), A = C^{-1}AC$, *and* $R_C(\lambda, A)$ *is analytic in* $\omega + \Delta_{\alpha+\pi/2}$ *and satisfies* $\|(\lambda - \omega)R_C(\lambda, A)\| \leq M_\beta$ $(\lambda \in \omega + \Delta_{\beta + \pi/2})$ *for every* $\beta \in (0, \alpha)$.

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Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China